The Annals of Probability
1986, Vol. 14, No. 3, 996–1004

THE CONTINUOUS AND DIFFERENTIABLE DOMAINS OF ATTRACTION OF THE EXTREME VALUE DISTRIBUTIONS[1]

BY JAMES PICKANDS III

University of Pennsylvania

The domains of attraction of the univariate extreme value distributions are characterized using inverse cumulative hazard functions. The results are much simpler than those using cumulative distribution functions. We also characterize the differentiable domains of attraction. A particularly simple characterization is given for the twice differentiable domain of attraction.

1. Introduction. Let $\{X_i\}_{i=1}^n$ be mutually independent with common CDF (cumulative distribution function) $F(x) := p\{X \le x\}$. Now let $Z_n := \bigvee_{i=1}^n X_i$, the maximum among the first n observations. Clearly $Z_n \to x_\infty := \text{lub}\{x|F(x) < 1\}$ in probability as $n \to \infty$. We say that $\Lambda(x)$ is an "extreme value CDF" and that $F(x)$ lies in its "domain of attraction" if and only if $\Lambda(x)$ and $F(x)$ are nondegenerate and there exist nonrandom sequences $(a_n, b_n)_{n=1}^\infty$ with $a_n > 0$, such that

$$\lim_{n\to\infty} p\{Z_n \le a_n x + b_n\} = \lim_{n\to\infty} F^n(a_n x + b_n) = \Lambda(x) \tag{1.1}$$

for all x at which $\Lambda(x)$ is continuous. We say that $F(x)$ lies in the "L times differentiable domain of attraction" if and only if $F(x)$ and $\Lambda(x)$ are nondegenerate and L times differentiable for all sufficiently large $x < x_\infty$ and

$$\lim_{n\to\infty} a_n^l (F^n)^{(l)}(a_n x + b_n) = \Lambda^{(l)}(x),$$

uniformly for all x in any finite interval for all $l = 0, 1, 2, \dots, L$, where (l) denotes the lth derivative of the function with respect to its argument. From here on we will assume nondegeneracy without mentioning it.

In particular, $F(x)$ lies in the (once) differentiable domain of attraction of $\Lambda(x)$ if and only if

$$\begin{aligned}\lim_{n\to\infty} a_n (F^n)^{(1)}(a_n x + b_n) &= \lim_{n\to\infty} n a_n F^{n-1}(a_n x + b_n) f(a_n x + b_n) \\ &= \lambda(x) = \Lambda^{(1)}(x),\end{aligned} \tag{1.2}$$

uniformly for all x in any finite interval where the density function $f(x) := F^{(1)}(x)$ exists for all sufficiently large $x < x_\infty$. Notice that (1.2) implies (1.1).

Received March 1982; revised November 1985.

[1]This research was supported in part by the Analysis Center, The Wharton School, University of Pennsylvania, through a contract with the United States Department of Energy, Contract # DE-AC01-10494.

AMS 1980 *subject classification*. Primary 60F05.

Key words and phrases. Extreme order statistics, limits, domains of attraction.

Gnedenko (1943) proved that a cumulative distribution function is an extreme value one if and only if it belongs to one of three families. See Galambos (1978), pages 51–57. See also Leadbetter, Lindgren, and Rootzen (1983), page 10. These three families can actually be written as a single three-parameter family. This was shown by von Mises (1936). Statistical applications are discussed in the book by Gumbel (1958). The domains of attraction are characterized in the monograph by de Haan (1970). For a thorough study of characterizations involving exponential distributions see the book by Galambos and Kotz (1978). In this work we characterize the domains of attraction in terms of the ICHF (inverse cumulative hazard function) and we use the von Mises one-family parametrization. The results seem to be much simpler and more unified. The preliminaries are in Section 2. The limiting ICHF's are derived in Section 3. It is shown there that the results for extreme order statistics are equivalent to the known ones for maxima. In Section 4 we characterize the differentiable domains of attraction. We show by counterexamples that the differentiable domains of attraction are strictly smaller than the domains of attraction. Falk (1984) discusses uniform convergence to the limiting joint distribution of the extreme order statistics. Weissman (1984) gives sufficient conditions for uniform convergence when the number of extreme order statistics may be increasing as $n \to \infty$. Resnick (1985) gives exact uniform bounds. In Section 5 we characterize the twice differentiable domains of attraction. We show, again using counterexamples, that these are still smaller. We give a simple characterization in Theorem 5.2. Most textbook continuous distributions satisfy this condition.

de Haan and Resnick (1982) studied the continuous and once differentiable domains of attraction in depth. They gave sufficient conditions and rates of convergence. They also gave results concerning L_p convergence. See also the more recent paper by Sweeting (1985).

2. Preliminaries. Any random variable can be represented as a function of a standard uniform random variable. That is, $X := F^{-1}(U)$ where $U \sim U(0,1)$ and F^{-1} is an inverse function for $F(x) := p\{X \le x\}$. Let

$$F^{-1}(u) := \text{lub}\{x | F(x) < u\} = \text{glb}\{x | F(x) \ge u\}.$$

Let $\{X_{(k)}\}_{k=1}^{n}$ be the descending order statistics from a sample of size n. Clearly,

$$X_{(k)} = F^{-1}(U_{(k)}), \tag{2.1}$$

where $\{U_{(k)}\}_{k=1}^{n}$ are the descending order statistics from a standard uniform sample.

We call the function

$$H(x) := -\log(1 - F(x))$$

the CHF or "cumulative hazard function" for the random variable X. Its inverse function is

$$H^{-1}(u) = F^{-1}(1 - e^{-u}).$$

Recall that a random variable Z has a standard negative exponential distribution if and only if $1 - e^{-Z}$ has a standard uniform one. Thus any scalar variable can be written

$$X = H^{-1}(Z),$$

where $Z \sim e(1)$. That is, Z has a standard negative exponential distribution. The descending order statistics are $\{X_{(k)} = H^{-1}(Z_{(k)})\}_{k=1}^{n}$, where $Z_{(k)}$ are the descending order statistics for an $e(1)$ sample. Suppose $\{X_k\}_{k=1}^{n}$ are mutually independent random variables with common CDF $F(x)$. If F is continuous then $\{F(X_k)\}_{k=1}^{n}$ constitute a standard uniform random sample. It follows that the limit distribution as $n \to \infty$ of $n(1 - F(X_{(1)}))$ and $\{(n-k)(F(X_{(k)}) - F(X_{(k+1)}))\}_{k=1}^{K}$, is that of $K + 1$ mutually independent standard negative exponential random variables. Consequently, the same is true for $\{n(F(X_{(k)}) - F(X_{(k+1)}))\}_{k=0}^{K}$ where $F(X_{(0)}) \equiv 1$. But $Z_{(k)}$ are the order statistics from an $e(1)$ sample and so $F(x) = 1 - e^{-x}$. It follows that $\{n(e^{-Z_{(k+1)}} - e^{-Z_{(k)}})\}_{k=0}^{K}$ are mutually independent and $e(1)$, asymptotically, where $Z_{(0)} := \infty$ and so $e^{-Z_{(0)}} = 0$. It follows that $\{T_k\}_{k=1}^{K}$ have as $n \to \infty$ the same limiting joint distribution as that of the first K event times of a homogeneous one-dimensional Poisson process where

$$T_k \equiv ne^{-Z_{(k)}} \tag{2.2}$$

or, equivalently,

$$Z_{(k)} \equiv -\log(T_k/n). \tag{2.3}$$

Since $F(x) = 1 - e^{-x}$ has continuous derivatives of all orders, it follows that the limits of derivatives of all orders of the joint distribution function are the corresponding derivatives of the limiting joint distribution. The normalized extreme order statistics are $\{(X_{(k)} - b_n)/a_n\}_{k=1}^{K}$, where

$$\begin{aligned}(X_{(k)} - b_n)/a_n &= \left(H^{-1}(\log n - \log T_k) - b_n\right)/a_n \\ &= \left(H^{-1}(-\log(T_k/n)) - b_n\right)/a_n.\end{aligned}$$

A limiting joint distribution exists as $n \to \infty$ if and only if

$$\lim_{n \to \infty} \left(H^{-1}(\log n - \log t) - b_n\right)/a_n = \psi^{-1}(-\log t) \tag{2.4}$$

for all t at which $\psi^{-1}(-\log t)$ is continuous. Because H^{-1} and ψ^{-1} are both monotone, convergence is uniform on finite intervals. The limiting joint distribution of the K largest order statistics then is that of $\{\psi^{-1}(-\log T_k)\}_{k=1}^{K}$, where $\{T_k\}_{k=1}^{K}$ are the first K event times of a standard one-dimensional Poisson process.

The function $H^{-1}(\cdot)$ is nondecreasing by definition. Thus the limit $\psi^{-1}(-\log t)$, if it exists [that is, if it is finite for all $t \in (0, \infty)$], is also nondecreasing as a function of $-\log t$. Thus its discontinuity set is at most countable and so that set has measure 0. Suppose we replace n in (2.4) by $n + 1$. Notice that $\log(n+1) - \log n = \log(1 + 1/n) \sim 1/n \to 0$ as $n \to \infty$. The limit is the same for any t at which $\psi^{-1}(-\log t)$ is continuous, that is, for almost all t. We can, without loss of generality, replace n; or $\log n$, in (2.4) by a real argument.

If (2.4) holds (with n not necessarily integer valued) and if $\psi^{-1}(u)$ is nondegenerate, we say that the ICHF $\psi^{-1}(u)$ is an extreme value one and that the ICHF $H^{-1}(u)$ lies in its domain of attraction. By "nondegenerate," here, we mean that $\psi^{-1}(u)$ is finite for all $u \in (0, \infty)$ and not everywhere constant. If, in addition, the first L derivatives of the expression on the left side of (2.4) converge to the corresponding derivatives of the limiting expression on the right side of (2.4), uniformly on all finite intervals, we say that $H^{-1}(u)$ lies in the L times differentiable domain of attraction of $\psi^{-1}(u)$. It is well known that the limiting extreme value distributions are continuous everywhere. Furthermore, between $x_{-\infty} := \operatorname{glb}\{x|\Lambda(x) > 0\}$ and x_{∞}, all derivatives exist and they are finite. See, for example, Galambos (1978), pages 51–57. Derivatives of inverse functions are continuously related to the corresponding derivatives of the original functions. The following theorem results.

THEOREM 2.1. *A distribution function $F(x)$ lies in the L times differentiable domain of attraction of $\Lambda(x)$ if and only if the corresponding ICHF $H^{-1}(u)$ lies in the L times differentiable domain of attraction of the corresponding function $\psi^{-1}(u)$.*

REMARK. Since a CHF increases from 0 to ∞ an ICHF is a function from R^1_+ into R^1. The referee has pointed out that $\psi^{-1}(u) := \Lambda^{-1}(\exp - e^{-u})$. Since $\psi^{-1}(u)$ is defined even for $u < 0$ it is not, strictly speaking, an ICHF.

3. Domains of attraction. In this section we characterize the limiting ICHF's and their domains of attraction.

LEMMA 3.1. *A nonconstant finite function ψ^{-1} is a limiting extreme value ICHF if and only if*

$$\begin{aligned}(3.1)\qquad &\big(\psi^{-1}(y+a) - \psi^{-1}(y)\big)/\big(\psi^{-1}(y+b) - \psi^{-1}(y)\big)\\ &\qquad \equiv \big(\psi^{-1}(a) - \psi^{-1}(0)\big)/\big(\psi^{-1}(b) - \psi^{-1}(0)\big)\end{aligned}$$

for all $a, b, y \in (-\infty, \infty)$. An ICHF $H^{-1}(u)$ lies in its domain of attraction if and only if

$$\begin{aligned}(3.2)\qquad &\lim_{z\to\infty} \big(H^{-1}(z+a) - H^{-1}(z)\big)/\big(H^{-1}(z+b) - H^{-1}(z)\big)\\ &\qquad = \big(\psi^{-1}(a) - \psi^{-1}(0)\big)/\big(\psi^{-1}(b) - \psi^{-1}(0)\big)\end{aligned}$$

for all $a, b \in (-\infty, \infty)$.

PROOF. Recall (2.4). Replace $H^{-1}(u)$ in (3.2) by $(H^{-1}(u) - b(z))/a(z)$. Notice that the scale and location terms, $a(z)$ and $b(z)$, respectively, cancel in (3.2). By continuity the result (3.2) follows if and only if (2.4) does. For all

a, b, $y \in (-\infty, \infty)$,

$$
\begin{aligned}
&(\psi^{-1}(a) - \psi^{-1}(0))/(\psi^{-1}(b) - \psi^{-1}(0)) \\
&\quad = \lim_{z \to \infty} \big(H^{-1}(z + y + a) - H^{-1}(z + y)\big)/ \\
&\qquad\qquad \big(H^{-1}(z + y + b) - H^{-1}(z + y)\big) \\
&\quad = \big(\psi^{-1}(y + a) - \psi^{-1}(y)\big)/\big(\psi^{-1}(y + b) - \psi^{-1}(y)\big),
\end{aligned} \tag{3.3}
$$

substituting $z + y$ for z in the limit. Thus (3.1) is true and the necessity part of the theorem is proved. To prove sufficiency we need only observe that, by (3.1) and (3.2), $\psi^{-1}(u)$ lies in its own domain of attraction. □

DEFINITION. A "type" is any equivalence class of nondecreasing functions connected by the affine group. That is $\psi_0^{-1}(u)$ and $\psi_1^{-1}(u)$ are equivalent if and only if

$$\psi_1^{-1}(u) = a\psi_0^{-1}(u) + b$$

for some $a > 0$ and $b \in (-\infty, \infty)$.

LEMMA 3.2. *If $\psi_0^{-1}(u)$ is a limiting ICHF $\psi^{-1}(u)$ satisfying* (2.4), *then $\psi_1^{-1}(u)$ is such, for the same ICHF $H^{-1}(u)$, if and only if ψ_0^{-1} and ψ_1^{-1} are of the same type. Furthermore, for any $\psi^{-1}(u)$, if a function $H^{-1}(u)$ lies in its domain of attraction then so does $aH^{-1}(u) + b$ for any $a > 0$ and $b \in (-\infty, \infty)$.*

REMARK. By Lemma 3.2 we can speak of limiting types and the domains of attraction as disjoint families of types. By Lemma 3.1 each limiting type is a member of its own domain of attraction.

PROOF OF LEMMA 3.2. This result is fairly well known and intuitively appealing. See the proof of Lemma 1 in de Haan (1976). □

LEMMA 3.3. *A nonconstant function $\psi^{-1}(u)$ is such that* (3.1) *holds for all y if and only if*

$$\psi^{-1}(u) = g + f\int_0^u e^{cs}\, ds, \tag{3.4}$$

where $0 < f < \infty$, $-\infty < g$, $c < \infty$.

PROOF. Let $\Delta \in (0, \infty)$. Let $\{q_k := \psi^{-1}(k\Delta)\}_{k=0}^{\infty}$. By (3.1)

$$(q_k - q_{k-1})/(q_{k-1} - q_{k-2}) \equiv p$$

for all k where $0 < p < \infty$ independently of k. It follows that for all k, $q_k - q_{k-1} = p^{k-1}f_*$ where $f_* \equiv q_1 - q_0$ and so $q_k = q_0 + \sum_{j=1}^{k}(q_j - q_{j-1}) = g + f_*\sum_{j=1}^{k}p^{j-1}$ where $g \equiv q_0$. In closed form

$$
\begin{aligned}
q_k &= g + f_*(p^k - 1)/(p - 1), \qquad p \neq 1 \\
&= g + f_*k, \qquad p = 1.
\end{aligned}
$$

Whether or not $p = 1$ we can write

$$q_k := \psi^{-1}(k\Delta) = g + f_* \int_0^{k\Delta} e^{cs}\,ds \Big/ \int_0^{\Delta} e^{cs}\,ds = g + f\int_0^u e^{cs}\,ds$$

for all $u = k\Delta$, where $c = \Delta^{-1}\log p$, and $f \equiv f_*/\int_0^\Delta e^{cs}\,ds$. So $\psi^{-1}(u)$ is of the form (3.4) with the same values of f, g, c for all u of the form $k/2^n$. That is, (3.4) holds throughout the dyadic set. By continuity of (3.4) and the requirement that $\psi^{-1}(u)$ be nondecreasing, the function $\psi^{-1}(u)$ is of the form (3.4) for all real u. The necessity part of the lemma is proved.

Now suppose $\psi^{-1}(u)$ is of the form (3.4). Notice that in (3.1) g and f will cancel. It is easy to verify that (3.1) holds. The sufficiency part of the lemma is proved. □

Lemmas 3.1, 3.2, and 3.3 combine to prove the following:

THEOREM 3.1. *A nonconstant finite function $\psi^{-1}(u)$ is a limiting (extreme value) ICHF if and only if*

$$\psi^{-1}(u) = b + a\int_0^u e^{cs}\,ds \tag{3.5}$$

for some $a \in (0, \infty)$, $b, c \in (-\infty, \infty)$. The ICHF $H^{-1}(u)$ lies in its domain of attraction if and only if for all $\Delta \in (0, \infty)$,

$$\lim_{z\to\infty} \left(H^{-1}(z + 2\Delta) - H^{-1}(z + \Delta)\right)/\left(H^{-1}(z + \Delta) - H^{-1}(z)\right) = e^{c\Delta}. \tag{3.6}$$

REMARK. We could say, in place of (3.5), that the limiting type is the one that contains $\int_0^u e^{cs}\,ds$.

This characterization of the domains of attraction is equivalent, through a logarithmic transformation, to de Haan (1970), Theorem 2.4.1, page 76. This result is due to Mejzler (1949).

Now the limiting joint distribution of the K largest order statistics is that of $\{b + a\int_0^{-\log T_k} e^{cs}\,ds\}_{k=1}^K$ where T_k are the first K event times of a homogeneous one-dimensional Poisson process. Consider the limiting distribution of largest values. Now T_1 has the standard negative exponential distribution. We can easily verify that $b + a\int_0^{-\log T_1} e^{cs}\,ds$ has an extreme value distribution with shape parameter c in the von Mises (1936) parametrization.

The author showed (1968), implicitly, that convergence is uniform to the right.

4. Differentiable domains of attraction. Notice that (3.2) in Lemma 3.1 can be written, equivalently,

$$\lim_{z\to\infty} \left(H^{-1}(z + y + \Delta) - H^{-1}(z + y)\right)/\left(H^{-1}(z + \Delta) - H^{-1}(z)\right)$$
$$\equiv \left(\psi^{-1}(y + \Delta) - \psi^{-1}(y)\right)/\left(\psi^{-1}(\Delta) - \psi^{-1}(0)\right)$$

for all $y \in (-\infty, \infty)$, $\Delta > 0$. Now H^{-1} lies in the differentiable domain of attraction of $\psi^{-1}(u)$ if and only if

$$\lim_{z\to\infty} (H^{-1})^{(1)}(z+y)/(H^{-1})^{(1)}(z) = (\psi^{-1})^{(1)}(y)/(\psi^{-1})^{(1)}(0),$$

where (1) denotes the first derivative with respect to the argument. By Theorems 2.1 and 3.1 the following is immediate.

THEOREM 4.1. *In order that $H^{-1}(u)$ lie in the differentiable domain of attraction of $\psi^{-1}(u)$, given by* (3.4), *it is necessary and sufficient that $H^{-1}(u)$ be differentiable for all sufficiently large u and that*

$$\lim_{z\to\infty} (H^{-1})^{(1)}(z+y)/(H^{-1})^{(1)}(z) = e^{cy} \tag{4.1}$$

for all $y \in (-\infty, \infty)$.

REMARK. A referee has pointed out that (4.1) is equivalent to the condition that $H^{-1}(\log t)$ varies regularly with exponent c as $t \to \infty$. See Sweeting (1985) for the case $c = 0$.

We show that some distributions lie in a domain of attraction but not in the corresponding differentiable one. Let

$$H^{-1}(u) \equiv \int_0^u (e^{cs} + \rho(s))\,ds = c^{-1}(e^{cu} - 1) + \int_0^u \rho(s)\,ds, \qquad c \neq 0. \tag{4.2}$$

Notice that $H^{-1}(u)$ is an ICHF if $\rho(s) \geq 0$ for all s. Now

$$\begin{aligned}(H^{-1}(u+y) - H^{-1}(u))/e^{cu} &= \int_0^y (e^{cs} + \rho(s+u)e^{-cu})\,ds \\ &\to \int_0^y e^{cs}\,ds = (e^{cy} - 1)/c\end{aligned} \tag{4.3}$$

as $u \to \infty$ if and only if

$$e^{-cu}\int_0^y \rho(s+u)\,ds \to 0 \quad \text{as } u \to \infty \text{ for all } y \in (-\infty, \infty). \tag{4.4}$$

For differentiable convergence it is necessary and sufficient that the derivative of the left side of (4.3) converge as $u \to \infty$ to the derivative of the right side. But it does if and only if

$$e^{-cu}\rho(u+y) \to 0$$

as $u \to \infty$ for all $y \in (-\infty, \infty)$. Equivalently,

$$\lim_{u\to\infty} e^{-cu}\rho(u) = 0. \tag{4.5}$$

Let $\{d_n\}_{n=1}^{\infty}$ be such that $d_n \in (0,1)$ and $\Sigma_{n=1}^{\infty} d_n < \infty$ and let

$$\rho(u) := e^{cu}, \qquad n \leq u \leq n + d_n, \tag{4.6}$$

for each integer n and let it $:= 0$, otherwise. Now (4.4) is true but (4.5) is not. Consequently, the convergence (4.3) takes place but not differentiably.

Sweeting (1985) showed, in effect, that differentiable convergence is uniform to the right.

5. Twice differentiable domains of attraction. By Theorem 4.1, if and only if $H^{-1}(u)$ lies in the differentiable domain of attraction of $\psi^{-1}(u)$, given by (3.5), then

$$\lim_{z\to\infty}(H^{-1})^{(1)}(z+\Delta)/(H^{-1})^{(1)}(z)=e^{c\Delta}$$

and so

$$\lim_{z\to\infty}\big((H^{-1})^{(1)}(z+\Delta)-(H^{-1})^{(1)}(z)\big)/(H^{-1})^{(1)}(z)=(e^{c\Delta}-1).$$

The following theorem is immediate:

THEOREM 5.1. *In order that $H^{-1}(u)$ lie in the twice differentiable domain of attraction of $\psi^{-1}(u)$ it is necessary and sufficient that*

$$(5.1)\qquad \lim_{z\to\infty}(H^{-1})^{(2)}(z)/(H^{-1})^{(1)}(z)=\lim_{z\to\infty}\big(\log(H^{-1})^{(1)}\big)^{(1)}(z)=c.$$

THEOREM 5.2. *In order that a distribution function $F(x)$ lie in the twice differentiable domain of attraction of the extreme value distribution $\Lambda(x|c)$, $-\infty<c<\infty$, it is necessary and sufficient that $F(x)$ be twice differentiable for all sufficiently large $x<x_\infty\equiv\operatorname{lub}\{x|F(x)<1\}$ and that*

$$(5.2)\qquad \lim_{x\uparrow x_\infty}d((1-F(x))/f(x))/dx=c,$$

where the density

$$f(x):=F^{(1)}(x).$$

REMARKS. It is known that the condition (5.2) is equivalent to the von Mises condition, which is sufficient for the domain of attraction. For the case $c=0$ see Marcus and Pinsky (1969). More generally, for all values of c see de Haan (1970), pages 108–113. The author (1984) considered nonparametric estimation of the "Pareto function" (5.2). Notice that convergence here is uniform to the right.

PROOF OF THEOREM 5.2. Recall Theorem 2.1. We show that the term whose limit is c in (5.1) is the same as the term in (5.2) with the same limit. Let $z=H(x):=-\log(1-F(x))$ or, equivalently, $x:=H^{-1}(z)$. Now

$$\begin{aligned}\big(\log(H^{-1})^{(1)}\big)^{(1)}(z)&=(H^{-1})^{(2)}(z)/(H^{-1})^{(1)}(z)=d(H^{-1})^{(1)}(z)/dH^{-1}(z)\\&=d\big(dH^{-1}(z)/dz\big)/dx\\&=d(1/H'(x))/dx=d((1-F(x))/f(x))/dx.\end{aligned}$$

The theorem is proved. □

It is easily verified that all "textbook" continuous distributions satisfy this condition.

Some distributions lie in a once differentiable domain of attraction but not in the corresponding twice differentiable one. Recall $H^{-1}(u)$ given by (4.2). It converges twice differentiably if and only if $\rho(u)$ is differentiable for all sufficiently large u and

$$e^{-cu}\rho'(u+y) \to 0$$

as $u \to \infty$ for all y. Equivalently,

$$\lim_{u\to\infty} e^{-cu}\rho'(u) = 0. \tag{5.3}$$

Let $\rho'(u)$ be defined as is $\rho(u)$ in (4.6). Then (4.5) is true but (5.3) is not. So the convergence is differentiable but not twice differentiable.

Acknowledgments. The author is grateful to a referee for several helpful comments leading to both simplification and clarification and for pointing out the paper by de Haan (1976). The author thanks Ishay Weissman for helpful comments.

REFERENCES

DE HAAN, L. (1970). *On Regular Variation and Its Application to the Weak Convergence of Sample Extremes.* Mathematical Centre Tracts. Mathematisch Centrum, Amsterdam.

DE HAAN, L. (1976). Sample extremes: An elementary introduction. *Statist. Neerlandica* **30** 161–172.

DE HAAN, L. and RESNICK, S. I. (1982). Local limit theorems for sample extremes. *Ann. Probab.* **10** 396–413.

FALK, M. (1984). Uniform convergence of extreme order statistics. Technical report, University of Siegen, West Germany.

GALAMBOS, J. (1978). *The Asymptotic Theory of Extreme Order Statistics.* Wiley, New York.

GALAMBOS, J. and KOTZ, S. (1978). *Characterizations of Probability Distributions.* Springer, New York.

GNEDENKO, B. V. (1943). Sur la distribution limite du terme maximum d'une séries aléatoire. *Ann. of Math.* **44** 423–453.

GUMBEL, E. J. (1958). *Statistics of Extremes.* Columbia Univ. Press, New York.

LEADBETTER, M. R., LINDGREN, G. and ROOTZEN, H. (1983). *Extremes and Related Properties of Random Sequences and Processes.* Springer, New York.

MARCUS, M. B. and PINSKY, M. (1969). On the domain of attraction of $\exp(-e^{-x})$. *J. Math. Anal. Appl.* **28** 440–449.

MEJZLER, D. G. (1949). On a theorem of B. V. Gnedenko. *Sb. Trudov Inst. Mat. Akad. Nauk. Ukrain. SSR* **12** 31–35.

PICKANDS, J. (1968). Moment convergence of sample extremes. *Ann. Math. Statist.* **39** 881–889.

PICKANDS, J. (1984). Spline and isotonic estimation of the Pareto function. In *Statistical Extremes and Applications* (J. Tiago de Oliveira, ed.) 285–296. Reidel, Dordrecht.

RESNICK, S. I. (1985). Uniform rates of convergence to extreme value distributions. Technical report, Colorado State Univ.

SWEETING, T. J. (1985). On domains of uniform local attraction in extreme value theory. *Ann. Probab.* **13** 196–205.

VON MISES, R. (1936). La distribution de la plus grande de n valeurs. Reprinted in *Selected Papers II.* Amer. Math. Soc., Providence R.I. (1954) 271–294.

WEISSMAN, I. (1984). On the asymptotic joint distribution of an unbounded number of sample extremes. Technical report, Technion, Haifa, Israel.

THE WHARTON SCHOOL
DEPARTMENT OF STATISTICS
UNIVERSITY OF PENNSYLVANIA
PHILADELPHIA, PENNSYLVANIA 19104