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A CLASS OF MARKOV PROCESSES WHICH ADMIT LOCAL TIMES

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A class of standard processes which admit local times at each point is considered. The following regularity properties are assumed: $T_x \to T_a = 0$ (as $x \to a$) in P^a-probability and $P^a(T_b < \infty) > 0$ for all pairs of points a, b ($T_x = \inf\{t > 0\colon X_t = x\}$). The class under consideration turns out to be very large. It is already known that a wide class of processes with independent increments fulfill our hypothesis. We also observe that the class is left invariant by the usual transformations: time change, subprocess and u-process (h-path) transformations.

The first important result of the paper is that every continuous additive functional may be represented as a mixture (integral) of local times. This theorem is used to prove two further results. The first one asserts that every process in the class has a dual process which remains in the class. Particularly Hunt's hypothesis (F) is satisfied. The second one generalises the occupation time and downcrossing approximating models. Such approximation theorems are proved for a C.A.F. whose representing measure is given.

Introduction. In this paper we study a class of standard processes for which each point admits a local time. Mild regularity properties, expressed by (1.1a–b) in the text, are assumed. These properties are equivalent to the following: $T_a = 0$, P^a-a.s., $T_x \to 0$ (as $x \to a$) in P^a-probability and $P^a(T_b < \infty) > 0$ for any a, b in the state space. Interestingly enough, these conditions turn out to be also equivalent to the following: the resolvent admits a density (Green) function which is finite, strictly positive and continuous, with respect to a certain Radon measure.

The class under consideration is rather large. By a result of Kesten (see [4]) every process with independent increments which admits a local time at the origin fulfills our hypotheses. Moreover, in Section 4 we show that the usual transformations (time change, subprocess and u-process (h-path) transformations) leave the class invariant.

A central result of the paper is Theorem 2.2, which asserts that every continuous additive functional may be represented as a mixture (integral) of local times by means of a Radon measure. The problem of representing additive functionals was previously treated in the literature by several authors. For example, Griego [8] proved under Hunt's hypothesis (F) a statement identical with Theorem 2.2. However, in our work hypothesis (F) will be obtained "a posteriori" in Section 5 by using the representation. Further results on measures

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associated with additive functionals were proved by Azema, Kaplan-Duflo and Revuz [1], Revuz [14], Nevison [12] and Rao [13]. Though our subject is similar, the assumptions, methods of proof and applications are different.

The representation theorem is used in this paper to get two further results. The first one is about duality. It is proved in Section 5 that for every process in the class there exists a dual process which remains in the class (see Theorem 5.4). The works of Hervé [9], Taylor [17], Smith and Walsh [15] and Chung and Rao [5] deal also with the construction of duality. Although the papers [15] and [5] are more probabilistic, their general results cannot be directly applied. On the other hand, though some of their ideas would work in our frame, major difficulties still remain to be solved. So we do not adopt the methods from these papers and prefer a direct intrinsic approach in the spirit of the potentially more theoretical papers [9] and particularly [17].

In another line, the representation theorem is used for getting approximation theorems for continuous additive functionals. It is well known that the local time at a point can be approximated by means of various processes constructed pathwise from the "geometry of the trajectory" (e.g., occupation time and downcrossing processes). A natural question would be to generalise this result to a functional whose representing measure is known.

In Theorem 6.4 it is shown that the correspondence between additive functionals and their representing measures is bicontinuous, under certain circumstances, so that the vague convergence of measures is equivalent to the convergence of the functionals in some sense. Thus if we have a measure μ, the associated additive functional can be approximated as follows: first approximate the measure by discrete measures of the form $\Sigma\alpha_i\varepsilon_{x_i}$, then approximate the local times at points x_i, by known methods with processes A^i, finally the process $\Sigma\alpha_i A^i$ should approximate the functional corresponding to μ. This is proved in Theorems 7.1 and 7.2. We point out that Sections 6 and 7 are completely independent of the results in Sections 3, 4 and 5. Therefore the reader interested only in convergence theorems may read them immediately after Section 2.

1. Preliminary results. Let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ be a standard process with l.c.c.b. state space $(E, \mathscr{E})$. We refer to Blumenthal and Getoor [3] for terminology and notation not explicitly introduced in this paper. Sometimes we will work with a metric d compatible with the topology of E. The Alexandrov compactification point will be denoted δ and the lifetime of the process is denoted by $\zeta = \inf\{t > 0: X_t = \delta\}$. (U_p) will be the resolvent of the process. For a continuous additive functional (C.A.F.) A, the fine support will be denoted by $\operatorname{supp} A$, and the p-potential kernel and 1-potential will be denoted by

$$U_A^p f(x) = E^x\left(\int_0^\infty e^{-pt} f(X_t)\, dA_t\right) \quad \text{and} \quad u_A(x) = U_A^1 1(x).$$

For $f \in \mathscr{E}_+$ we will denote by $f \cdot A$ the C.A.F. defined by $(f \cdot A)_t = \int_0^\infty f(X_s)\, dA_s$. The space of continuous functions on E will be denoted by $C(E)$, while $C_c(E)$, $C_0(E)$ and $C_b(E)$ will be the subspaces of functions with compact support, resp. vanishing to δ, resp. bounded. The uniform norm of a function will be denoted by $\|f\| = \sup\{f(x): x \in E\}$. The support of a measure μ will be supp μ.

A major role in this paper is played by the functions:

$$\varphi_x^p(y) = E^y(\exp(-pT_x); T_x < \infty), \qquad p \geq 0,$$

where $T_x = \inf\{t > 0: X(t) = x\}$. For $p = 1$ we shall write simply $\varphi_x(y)$ instead of $\varphi_x^1(y)$. Note also that $\varphi_x^0(y) = P^y(T_x < \infty)$.

Our hypotheses on the process X are

$$\text{(1.1a)} \qquad \lim_{x \to a} \varphi_x(a) = 1 = \varphi_a(a) \quad \text{for every } a \in E,$$

$$\text{(1.1b)} \qquad \varphi_x(y) > 0 \quad \text{for every } x, y \in E.$$

The first equality in (1.1a) is equivalent to each of the following conditions:

$$\text{(1.2)} \qquad \lim_{x \to a} T_x = 0 \quad \text{in } P^a\text{-probability for every } a \in E,$$

$$\text{(1.3)} \qquad \lim_{x \to a} \varphi_x^p(a) = 1 \quad \text{for every } a \in E \text{ and } p > 0.$$

To see this we write for some $\varepsilon > 0$,

$$\varphi_x(a) \leq e^{-\varepsilon} P^a(T_x > \varepsilon) + P^a(T_x \leq \varepsilon),$$

and hence

$$(1 - e^{-\varepsilon}) P^a(T_x > \varepsilon) \leq 1 - \varphi_x(a),$$

which yields (1.2) if we assume the first equality in (1.1a). The other implications are obvious. It is also clear that under our hypotheses condition (1.3) holds also for $p = 0$.

Condition (1.1b) is equivalent to

$$\text{(1.4)} \qquad \varphi_x^p(y) > 0 \quad \text{for every } x, y \in E$$

for some $p \geq 0$.

Let us now list the properties following directly from our hypotheses:

(1.5) *The fine topology coincides with the initial topology of E. Every p-excessive function is continuous for any $p \geq 0$. The resolvent of the process is strong Feller. There exists a reference measure. X is a Hunt process.*

PROOF. Let D be a finely open set and $x \in D$. Then $E^x(\exp(-T_{D^c})) < 1$, and so, under (1.1a) we may choose a neighbourhood V of x such that $\varphi_y(x) > E^x(\exp(-T_{D^c}))$ for every $y \in V$. For $y \in D^c$ we have $T_y \geq T_{D^c}$ and so $\varphi_y(x) \leq E^x(\exp(-T_{D^c}))$. We conclude that $V \subset D$, which proves the first assertion. The next two assertions are immediate consequences of the first one. The existence of a reference measure for a process with strong Feller resolvent is well known, and the last assertion follows from Theorem 4.3 in [16]. □

REMARK. If the space E is connected, then condition (1.1a) implies (1.1b). To show this we fix a point $a \in E$ and put $C = (\varphi_a > 0)$, $D = (\varphi_a = 0)$. Since

$T_a \le T_y + T_a \circ \theta_{T_y}$ for every $a, y \in E$, it follows that

$$(1.5') \qquad \varphi_a(x) \ge \varphi_a(y)\varphi_y(x).$$

Let $x \in D$ and $V = \{y\colon \varphi_y(x) > 0\}$. By (1.1a) V is a neighbourhood of x. By the above inequality $V \subset D$. So we have proved that D is an open set. Since relation (1.1b) is not used in the proof of (1.5), it follows that φ_a is continuous and so C is also open. We have $E = C \cup D$, $C \cap D = \varnothing$ and $a \in C$. If E is connected, then D should be empty, which implies (1.1b).

Next we give some useful properties of φ^p:

(1.6) *For any* $p \ge 0$

(a) $|\varphi_x^p - \varphi_y^p| \le \varphi_x^p \cdot \gamma_p(x, y)/\varphi_x^p(y)$ *for every* $x, y \in E$, *with* $\gamma_p(x, y) = 1 - \varphi_x^p(y)\varphi_y^p(x)$.

(b) *For every p-excessive function s and every* $x, y \in E$ $|s(x) - s(y)| \le s(x)\gamma_p(x, y)/\varphi_y^p(x)$.

Except for the constant functions 0 *and* ∞ *every p-excessive function is strictly positive and finite.*

(c) *The function* $(x, y) \to \varphi_x(y)$ *is continuous.*

PROOF. Since the proof is analogous for any $p \ge 0$, we shall give it only for $p = 1$. In order that, for some $c > 0$,

$$|\varphi_x(z) - \varphi_y(z)| \le \varphi_x(z)c \quad \text{for any } z \in E,$$

it suffices that

$$(*) \qquad \varphi_x(z)(1 - c) \le \varphi_y(z) \quad \text{and} \quad \varphi_y(z) \le (1 + c)\varphi_x(z).$$

By (1.5′) we have $\varphi_y(z) \ge \varphi_y(x)\varphi_x(z)$ and so, in order to get the first inequality in $(*)$ we may take c such that $(1 - c) \le \varphi_y(x)$, that is $c \ge 1 - \varphi_y(x)$. Similarly the second inequality in $(*)$ holds provided $c \ge \varphi_x(y)^{-1} - 1$. Since $c = \gamma(x, y)/\varphi_x(y)$ fulfills these conditions, (a) is proved.

To prove (b) one writes

$$(1.7) \qquad s(x) \ge E^x\big(\exp(-T_y)s\big(X(T_y)\big)\big) = s(y)\varphi_y(x),$$

which yields the inequality in (b). The statement in (b) follows immediately from the inequality.

For the assertion in (c) we first write

$$|\varphi_a(b) - \varphi_x(y)| \le |\varphi_a(b) - \varphi_a(y)| + |\varphi_a(y) - \varphi_x(y)|.$$

Since φ_a is 1-excessive, it is continuous, and hence the first term in the right-hand side of the inequality vanishes when $y \to b$. The second term is dominated by $\gamma(a, x)/\varphi_a(x)$, which also vanishes when $x \to a$. □

We shall now discuss the problem of recurrence and transience. We recall one of their equivalent definitions [Corollary (2.3) and Proposition (2.4) in [6]]:

Definition. A standard process is called transient if for every compact set $K, U(\cdot, K) < \infty$ (U is the potential kernel of X).

The process is called recurrent if for each finely open set $B, U(\cdot, B) = \infty$.

(1.8) *The process X [which fulfills (1.1a) and (1.1b)] is either transient or recurrent.*

To prove the above assertion we need the following lemma:

Lemma. *Let X be a standard process such that $U_p(C_b(E)) \subset C(E)$ for some $p > 0$. Assume that for each $x \in E$ there is a finite continuous excessive function ψ such that $\lim_{t\to\infty} P_t\psi = 0$ and $\psi(x) > 0$. Then X is transient.*

Proof. For any $q > 0$

$$\infty > p^{-1}\psi \geq U_p\psi = U_q\psi - (p-q)U_qU_p\psi$$
$$\geq U_q(\psi - pU_p\psi) \to U(\psi - pU_p\psi) \quad \text{as } q \to 0.$$

Since $\lim_{t\to 0} P_t\psi = 0$, we have $pU_p\psi(x) < \psi(x)$ and so $\psi - pU_p\psi$ is continuous and strictly positive in a neighborhood V of x. Hence $U(\cdot, V) < \infty$, which ends the proof of the lemma. □

We go on and prove (1.8): If X is not recurrent then there exists a finely open set V such that $U1_V(x) < \infty$ for some $x \in E$. Since the function $\psi = U1_V$ is excessive, by (1.6)(b) ψ is strictly positive and finite everywhere. Now, by using the above lemma, one gets (1.8).

Remark. In the recurrent case each excessive function is constant and $P^x(T_y < \infty) = 1$ for every $x, y \in E$. This follows from Proposition (2.4)(v) in [6].

(1.9) *The process X admits a σ-finite excessive reference measure. In the recurrent case this measure is even invariant.*

In the transient case the excessive reference measure may be easily constructed in the following way: one chooses a sequence (x_n) such that $\{x_n: n \in N\}$ is dense in E and put $\mu = \Sigma 2^{-n}\varepsilon_{x_n}$. Then the needed measure is μU. The existence of an invariant reference measure in the recurrent case is discussed in [1], [10] and [18]. In our framework the invariant measure is defined by

$$\mu(f) = E^x\left(\int_0^{T_y} f(X_s)\,ds\right) + E^y\left(\int_0^{T_x} f(X_s)\,ds\right), \qquad f \in \mathscr{E}_+,$$

where x and y are two arbitrary fixed points in E. The proof can be done as in [10] and is even simpler.

In Section 5 we will see that each excessive measure is a Radon measure (see Remark 5.7).

2. Representation of C.A.F.'s. Under our hypothesis every point $a \in E$ is regular for itself and consequently it has a local time. We denote by L^a a version of the local time which is normalised by

$$E^a\left(\int_0^\infty e^{-s}\, dL_s^a\right) = 1. \tag{2.1}$$

It is known from Theorem 1 in [7] that, under this normalisation $(s, x, \omega) \to L_s(\omega)$ is measurable. By the strong Markov property one obtains

$$E^y\left(\int_0^\infty e^{-s}\, dL_s^x\right) = \varphi_x(y). \tag{2.1'}$$

From (1.5) it follows that each natural potential is regular and so each natural additive functional with finite 1-potential is continuous. The aim of this section is to show that such a functional may be represented as the integral of L^x with respect to a Radon measure $\mu(dx)$. We recall that a continuous additive functional (C.A.F.) A is uniquely determined by its 1-potential

$$u_A(x) = E^x\left(\int_0^\infty e^{-s}\, dA_s\right).$$

Thus the representation of A is equivalent to the representation of u_A by means of the extremal 1-potentials φ_x, $x \in E$. So Choquet's representation theorem would work here. Revuz's measure can be also used to produce the representing measure. However, we prefer a direct proof for the sake of completeness. In the end of this section we will discuss the connection with Revuz's measure.

Now we give without proof the simple part of the result.

PROPOSITION 2.1. *Let μ be a Radon measure on E such that the function defined by*

$$x \to \int \varphi_y(x)\mu(dy), \qquad x \in E, \tag{2.2}$$

is finite. Then $A_t^\mu = \int L_t^x \mu(dx)$, $t \geq 0$, defines a C.A.F. $A^\mu = (A_t^\mu)$, whose 1*-potential is the function from* (2.2).

The main result of this section is the following theorem.

THEOREM 2.2. *For every C.A.F. A, with finite* 1*-potential, there exists a unique Radon measure μ such that the function in* (2.2) *is finite and*

$$A_t = \int L_t^x \mu(dx) \qquad \textit{for every } t \geq 0 \textit{ a.s.} \tag{2.3}$$

Proof. We note first that by restriction we may assume that the fine support of A is included in some compact set K. For a fixed $\varepsilon > 0$ one may choose a finite family of points $\{x_i\colon\ i \le n\}$ and a measurable partition of $K\{V_i\colon\ i \le n\}$ (n depending on ε), such that $x_i \in V_i \subset \{x\colon\ d(x, x_i) \le \varepsilon\}$ and $K \subset \bigcup V_i$. Then we define $A_i = 1_{V_i}A$, $u_i = u_{A_i}$, $\alpha_i = u_i(x_i)$ and $\mu_\varepsilon = \sum \alpha_i \varepsilon_{x_i}$. Then for every $x \in E$ we have

$$\left|u_A(x) - \sum \alpha_i \varphi_{x_i}(x)\right| \le \sum \alpha_i \left|\alpha_i^{-1} u_i(x) - \varphi_{x_i}(x)\right|.$$

Lemma 2.3 from below implies

$$\|\alpha_i^{-1} u_i - \varphi_{x_i}\| \le 2 \sup\left\{\gamma(x_i, x)\varphi_x(x_i)^{-1}\colon\ x \in \overline{V}_i \cap K\right\},$$

and hence

$$\text{(2.4)} \quad \left\|u_A - \sum \alpha_i \varphi_{x_i}\right\| \le 2\left(\sum \alpha_i\right) \sup\left\{\gamma(y, x)\varphi_x(y)^{-1}\colon\ x, y \in K,\ d(x, y) \le \varepsilon\right\}.$$

Let us fix a point $a \in E$. By using (1.7) one gets

$$\mu_\varepsilon(E) = \sum \alpha_i \le \Theta^{-1} \sum u_i(a) = \Theta^{-1} u_A(a) < \infty,$$

where $\Theta = \inf\{\varphi_x(a)\colon\ x \in K\} > 0$. It follows that the measures μ_ε, $\varepsilon > 0$ are uniformly bounded. By a compactness argument one may find a sequence $\varepsilon_p \to 0$ and a measure μ on K such that $\mu_{\varepsilon_p} \to \mu$ weakly. Then

$$\lim_{p \to \infty} \int \varphi_y(x) \mu_{\varepsilon_p}(dy) = \int \varphi_y(x) \mu(dy).$$

Since $\int \varphi_y(x) \mu_\varepsilon(dy) = \sum \alpha_i \varphi_{x_i}(x)$, by (2.4) one gets $u_A = u_{A^\mu}$, with the notation from the preceding proposition. Therefore we obtain $A = A^\mu$. The unicity of μ results in a more precise form from Proposition 2.4 below. □

Lemma 2.3. *Let K be a compact set, $a \in K$ and u, v two 1-excessive functions such that $u(a) = v(a) = 1$ and $P_K u = u$, $P_K v = v$. Then for any $x \in E$ we have*

$$|u(x) - v(x)| \le 2 \sup\left\{\gamma(a, y)\varphi_y(a)^{-1}\colon\ y \in K\right\}.$$

Proof. For $x \in K$ the inequality follows from (1.6)(b). Then by using the relation $P_K(u - v) = u - v$ one gets the inequality for every $x \in E$. □

Proposition 2.4. *If A is a C.A.F. and μ is a measure satisfying* (2.3), *then*

$$U_A^1 f(x) = \int f(y) \varphi_y(x) \mu(dy), \qquad f \in \mathscr{E}_+,\ x \in E,$$

$$\mu(dy) = \varphi_y(x)^{-1} U_A^1(x, dy), \qquad x \in E,$$

$$U_A^1(x, dy) = \varphi_y(x) \varphi_y(z)^{-1} U_A^1(z, dy), \qquad x, z \in E.$$

Proof. By (2.1′) one gets

$$U_{L^y}^1 f(x) = f(y) \varphi_y(x), \qquad y \in E.$$

Then the first stated relation follows by integration with $\mu(dy)$. The other relations are easy consequences of the first one. □

REMARK 2.5. If A and μ satisfy (2.3) then the fine support of A, which is a closed set in the usual topology, coincides with the support of μ.

Griego [8] has a result similar to Theorem 2.2. under Hunt's hypothesis (F). The representing measure of the potential associated to the additive functional is used to represent the additive functional as in (2.3). This is possible because in that paper the local time is normalised by

$$E^a\left(\int_0^\infty e^{-ps}\, dL_s^a\right) = g_p(a,a), \qquad a \in E,$$

with g_p the Green density function given by hypothesis (F). Revuz's measure introduced for processes with a reference excessive measure, generalises the representing measure of the potential associated to the additive functional, which is defined under duality hypothesis. In the remainder of this section we shall discuss the relation between the representing measure obtained in Theorem 2.2 and Revuz's measure.

Let ξ be a fixed σ-finite reference excessive measure (as we observed in (1.9) such a measure exists). If A is a C.A.F. with finite 1-potential, we denote by μ_A the measure obtained in Theorem 2.2, while Revuz's measure will be denoted by ν_A. It is defined by the following increasing limit (see [14]):

$$\nu_A(f) = \lim_{n\to\infty} 2^n E^{\xi}(f \cdot A(2^{-n})), \qquad f \in \mathscr{E}_+. \tag{2.5}$$

Since $\nu_{f\cdot A} = f \cdot \nu_A$, it follows that ν_A is carried by the fine support of A. Hence $\nu_L x = h(x)\varepsilon_x$, where $0 < h(x) < \infty$. The function $x \to h(x)$ is measurable because

$$h(x) = \nu_L x(E) = \lim 2^n E^{\xi}(L^x(2^{-n})).$$

From (2.3) and (2.5) one deduces that $\nu_A(dx) = h(x)\mu_A(dx)$. In Remark 5.7 we will show that h is continuous by using duality results. It seems difficult to prove this directly.

3. The Green function. Let us denote by η the representing measure for the usual time of the process. This measure is defined by $t \wedge \zeta = \int L_t^y \eta(dy)$ and satisfies:

$$U_1 f(x) = E^x\left(\int_0^\zeta e^{-s} f(X_s)\, ds\right) = \int f(y)\varphi_y(x)\, d\eta(y) \quad \text{for } f \in \mathscr{E}_+,$$

where ζ is the lifetime of the process.

Obviously η charges any open set. The above relation and the uniqueness of the representing measure ensure that

$$\int_0^t f(X_s)\, ds = \int L_t^y f(y)\eta(dy) \quad \text{for } f \in \mathscr{E}_+.$$

This yields

$$
\begin{aligned}
U_p f(x) &= E^x\left(\int_0^\zeta e^{-ps} f(X_s)\,ds\right) \\
&= \int E^x\left(\int_0^\zeta e^{-ps}\,dL_s^y\right) f(y)\eta(dy) \\
&= \int g_p(x, y) f(y)\eta(dy)
\end{aligned}
$$

and so, the Green function with respect to the reference measure η is

$$
g_p(x, y) = E^x\left(\int_0^\zeta e^{-ps}\,dL_s^y\right), \qquad p \geq 0. \tag{3.1}
$$

By the strong Markov property

$$
g_p(x, y) = \varphi_y^p(x) g_p(y, y), \qquad p \geq 0. \tag{3.2}
$$

In particular, on account of (2.1), $g_1(x, y) = \varphi_y(x)$. We also mention the following straightforward relation (see Proposition 2.3, Chapter IV in [3]):

$$
\begin{aligned}
&\big(g_p(x, y) - g_q(x, y)\big)/(q - p) \\
&= \int g_p(x, z) g_q(z, y)\,d\eta(z) \\
&= \int g_q(x, z) g_p(z, y)\,d\eta(y) \quad \text{for } p \geq 0,\ q > 0.
\end{aligned} \tag{3.3}
$$

We give now some properties of g_p.

PROPOSITION 3.1. *For* $p > 0$

(a) $0 < g_p \leq \max(p^{-1}, 1)$,
(b) $x \to g_p(x, y)$ *is p-excessive for every* $y \in E$,
(c) $(x, y) \to g_p(x, y)$ *is continuous.*

PROOF. The first inequality in (a) is a consequence of (3.2). To prove the second inequality let us write (3.3) in the form

$$
g_p(x, y) = g_1(x, y) + (1 - p) U_p g_1(\cdot, y)(x). \tag{3.4}
$$

Since (U_p) is sub-Markovian and $g_1 = \varphi \leq 1$ it follows that $U_p g_1(\cdot, y) \leq 1/p$ and so $g_p \leq 1 + (1 - p)/p = 1/p$ for $p < 1$. If $p > 1$, (3.4) implies $g_p \leq g_1 \leq 1$. Assertion (b) follows from (3.1).

By (1.6)(b) the functions $g_p(\cdot, y)$, $y \in K$, are equicontinuous for every compact set K. Since $g_1(x, y) = \varphi_y(x)$, which is continuous, the continuity in y follows from (3.4), and (c) is proved. □

PROPOSITION 3.2. *In the transient case g_0 is finite and assertions* (b) *and* (c) *in Proposition* 3.1 *hold also for* $p = 0$.

REMARK. Using the results of Section 5 one can show that in the recurrent case $g_0 \equiv \infty$.

PROOF. We first prove that g_0 is finite. Let $y \in E$ be fixed. Clearly $\varphi_y^0(x) = E^x(T_y < \infty)$ is 0-excessive. Let us check that in fact φ_y^0 is a regular potential. Since X is transient one may apply Hunt's theorem and get a sequence f_n, $n \in N$ of bounded measurable functions such that $Uf_n \uparrow \varphi_y^0$. For any $\varepsilon > 0$ one may choose n_ε such that $Uf_n(y) \geq \varphi_y^0(y) - \varepsilon = 1 - \varepsilon$ for $n \geq n_\varepsilon$. It follows that

$$Uf_n(x) + \varepsilon \geq E^x\big(Uf_n\big(X_{T_y}\big);\ T_y < \infty\big) + \varepsilon \geq \big(Uf_n(y) + \varepsilon\big)P^x\big(T_y < \infty\big) \geq \varphi_y^0(x),$$

for every $x \in E$ and $n \in N$. We may conclude that φ_y^0 is a regular potential, as a uniform limit of regular potentials. Let A be the C.A.F. with 0-potential equal to φ_y^0, that is,

$$E^x(A_\infty) = \varphi_y^0(x).$$

Since $\varphi_y^0(y) = 1$ we have

$$E^x\big(\varphi_y\big(X_{T_y}\big);\ T_y < \infty\big) = E^x\big(T_y < \infty\big) = \varphi_y^0(x),$$

and so $E^x(A(T_y)) = 0$. It follows that the fine support of A is exactly $\{y\}$. This implies that A is a version of the local time. Since two versions of the local time differ by a multiplicative constant, $E^x(A_\infty) < \infty$ implies that $g_0(x, y) = E^x(L_\infty^y) < \infty$.

Let us prove the continuity of g_0. As $g_0(\cdot, y)$ is 0-excessive, it is continuous for each fixed y. It remains to prove that for each compact set K, the functions in the family $g_0(x, \cdot)$, $x \in K$ are equally continuous. One uses (3.4) to get

$$g_0(x, y) = g_1(x, y) + (U_0\varphi_y)(x).$$

Since $(x, y) \to g_1(x, y)$ is continuous it remains to study the function $(x, y) \to (U_0\varphi_y)(x)$. Let $a \in E$ be fixed. By (1.6)(a) we have

$$\big|(U_0\varphi_y)(x) - (U_0\varphi_a)(x)\big| \leq U_0(|\varphi_y - \varphi_a|)(x) \leq \gamma(a, y)\varphi_a(y)^{-1}(U_0\varphi_a)(x),$$

which shows that the functions of the family $y \to (U_0\varphi_y)(x)$, $x \in K$, are equally continuous at a. This completes the proof. □

We finish the section by giving a proposition which formulates conditions (1.1a–b) in terms of resolvents.

PROPOSITION 3.3. *Let Y be a Hunt process and (W_p) its resolvent. We assume that for some fixed $p \geq 0$*

$$W_p f(x) = \int f(y)h_p(x, y)\, d\mu(y) \quad for\ f \in \mathscr{E}_+,$$

where μ is a Radon measure and $h \in \mathscr{E} \otimes \mathscr{E}_+$.

(i) *If h_p is locally bounded and continuous in each argument, then*

$$h_p(x, y) = \varphi_y^p(x)h_p(y, y).$$

(ii) *If* $(x, y) \to h_p(x, y)$ *is finite, strictly positive and continuous, then* φ^p *is continuous and* Y *fulfills* (1.1a–b).

PROOF. Let y be fixed, $(V_n)_{n\in N}$ a sequence of open relatively compact sets such that $\overline{V}_{n+1} \subset V_n$, and $\bigcap_n V_n = \{y\}$, and $f_n = \mu(V_n)^{-1}1_{V_n}$. Since h_p is continuous in y, $\lim_n W_p f_n(x) = h_p(x, y)$. By the strong Markov property one gets

$$(3.5)\qquad E^x\big(\exp(-pT_{V_n})W_p f_m\big(X(T_{V_n})\big)\big) = W_p f_m(x) \quad \text{for } m \geq n.$$

Obviously $W_p f_m(x) \leq \sup\{h_p(x, z)\colon z \in \overline{V}_1\}$ and so, by the above relation one gets

$$W_p f_m \leq \sup\{h_p(x, z)\colon x, z \in \overline{V}_1\}.$$

Now we may take limits over m in (3.5) and get

$$E^x\big(\exp(-T_{V_n})h_p\big(X(T_{V_n}), y\big)\big) = h_p(x, y), \quad \text{for every } n \in N.$$

The first term in the above relation converges (as $n \to \infty$) to

$$E^x\big(\exp(-pT_y)\big)h_p(y, y) = \varphi_y^p(x)h_p(y, y),$$

which proves (i). Assertion (ii) follows immediately.

REMARK 3.4. For a Hunt process conditions (1.1a), (1.1b) are equivalent to the existence of a finite, strictly positive, continuous Green function.

4. Transformations of the process. In this section we shall prove that the main transformations considered in the literature leave invariant the class of processes defined by (1.1a–b).

Let us begin with the time change transformation. For a C.A.F. A which is finite on $[0, \zeta]$ the process $\tilde{X}(t) = X(\tau(t))$ with $\tau(t) = \inf\{s > 0\colon A(s) > t\}$ is the so called time changed process. Let $\tilde{E}$ be the fine support of A. Since $\tilde{E}$ is closed the remark from page 233 in [3] ensures that $\tilde{X}$ is a standard process with state space E.

PROPOSITION 4.1. *Conditions* (1.1a–b) *hold for* $\tilde{X}$.

PROOF. Let $\tilde{T}_x = \inf\{s > 0\colon X(s) = x\}$. A simple look shows that $\tilde{T}_x = A(T_x)$ a.s. for $x \in \tilde{E}$. Then

$$\tilde{\varphi}_y(x) = \tilde{E}^x\big(\exp(-\tilde{T}_y)\big) = E^x\big(\exp(-A(T_y))\big).$$

By (1.2) $\lim_{y\to x} T_y = 0$ in P^x-probability, hence from every sequence $y_n \to x$ one may extract a subsequence such that $T_y \to 0$, P^x a.s. on this subsequence. An analysis argument shows that $\lim_{y\to x}\tilde{\varphi}_y(x) = 1$, which proves (1.1a). Condition (1.1b) obviously holds. □

Let us now look at the subprocess transformation. Consider a strong multiplicative functional M for which every point is permanent. Let $\overline{X}$ be the canonical subprocess associated to the functional. This is a standard process. For notations and properties related to $\overline{X}$ we refer to [3]. For $(\omega, \lambda) \in \Omega \times R_+$ and $x \in E$, $\overline{T}_x(\omega, \lambda) = \inf\{s > 0;\ \overline{X}_s(\omega, \lambda) = x\}$ is equal to $T_x(\omega)$ if $T_x(\omega) < \lambda$ and ∞ if $T_x(\omega) \geq \lambda$. Then

$$\overline{\varphi}_y(x) = \overline{E}^y(\exp(-\overline{T}_x)) = E^y(M(T_x)\exp(-T_x)).$$

The same argument as in Proposition 4.1 shows that $\lim_{y \to x}\overline{\varphi}_y(x) = 1$, which proves:

PROPOSITION 4.2. *Conditions* (1.1a–b) *hold for* $\overline{X}$.

To finish we shall verify that the u-process transformation (or h-path transformation) conserves properties (1.1a) and (1.1b). Concerning u-processes we refer to Chapter 1, Section 4 in [11]. Let u be a finite strictly positive 0-excessive function. The corresponding u-process is Hunt and its resolvent is

$$W_p(x, dy) = u(x)^{-1}U_p(x, u(y)\,dy).$$

Since u is continuous, (W_p) fulfills the conditions in Proposition 3.4, and so (1.1a–b) hold for the u-process. Thus we have:

PROPOSITION 4.3. *The u-process transformation conserves* (1.1a–b).

5. Duality. The aim of this section is to prove that under conditions (1.1a–b) the process X admits a dual process $\hat{X}$ fulfilling also this hypothesis. The key result of the section is the following:

PROPOSITION 5.1. *Let X be a standard process satisfying* (1.1a–b). *Assume that X is transient. Then there exists a Feller process $\hat{X}$ which is in duality with X and satisfies* (1.1a–b).

PROOF. Let us consider the Green function g_p, $p \geq 0$, and the reference measure η defined in Section 3, and define

$$f\hat{U}_p(y) = \int g_p(x, y)f(x)\eta(dx) \quad \text{for } f \in \mathscr{E}_+,\ p \geq 0,\ y \in E.$$

By (3.3) $(\hat{U}_p)$ is a resolvent. Clearly $(\hat{U}_p)$ is in duality with (U_p) with respect to η, that is

$$\langle U_p f, g\rangle = \langle f, g\hat{U}_p\rangle \quad \text{for } f, g \in \mathscr{E}_+ \text{ and } p \geq 0$$

(co-kernel and duality notations are those from [3] and [11]).

Since $(\hat{U}_p)$ is not sub-Markovian, it could not be associated to a Markov process. Therefore we shall modify it by means of an appropriate $(\hat{U})_p$-excessive function, in order to get a resolvent which is sub-Markovian and even Feller. The needed function is produced by the following lemma.

LEMMA 5.2. *There exists a finite, strictly positive, continuous $(\hat{U}_p)$-excessive function w such that*

$$\lim_{y\to\delta} g_0(x, y)/w(y) = 0 \quad \textit{for every } x \in E. \tag{5.1}$$

PROOF. If E is compact we may take $w = g_0(a, \cdot)$ with $a \in E$ and (5.1) is trivial. If E is not compact, let us consider a sequence (K_n) of compact sets such that $K_n \subset \mathring{K}_{n+1}$ and $\bigcup_n K_n = E$. We denote by $T_n = T_{K_n^c}$ and note that $\sup_n T_n \geq \zeta$. Since $K_n^c \neq \varnothing$ one deduces from (1.1b) that $P^x(T_n < \infty) > 0$ for any $x \in E$. Let us fix a point $a \in E$ and define

$$w_n(y) = E^a(g_0(X(T_n), y), T_n < \infty) > 0.$$

By (3.3) one has

$$q(g_0(x, \cdot)\hat{U}_q)(y) = qU_q g_0(\cdot, y)(x) \quad \text{for } x, y \in E.$$

Since $g_0(\cdot, y)$ is excessive, it follows that $g_0(x, \cdot)$ is $(\hat{U}_q)$-excessive. Hence w_n is also $(\hat{U}_q)$-excessive.

By (3.2) $g_0(x, y) \leq g_0(y, y)$ for any $x, y \in E$. Since $y \to g_0(y, y)$ is finite and continuous, it is bounded on compacts and hence one may apply Lebesgue's theorem to check that w_n is continuous. By the definition of g_0 and the strong Markov property

$$w_n(y) = E^y(L_\infty^y - L_{T_n}^y) \to 0.$$

We consider now another fixed point $z \in E$. By taking a subsequence we may assume that $w_n(z) \leq 2^{-n}$. Since $T_z \leq T_y + T_z \circ \theta_{T_y}$, P^x a.s., it follows that $\varphi_z^0(x) \geq \varphi_z^0(y)\varphi_y^0(x)$, and so

$$g_0(x, z)g_0(y, y) \geq g_0(y, z)g_0(x, y) \quad \text{for any } x, y \in E. \tag{5.2}$$

One concludes that

$$w_n(y) \leq w_n(z)g_0(y, y)/g_0(y, z) \leq 2^{-n}g_0(y, y)/g_0(y, z),$$

which ensures that $w = \Sigma_n w_n$ is a continuous, strictly positive and finite $(\hat{U}_q)$-excessive function. Next we note that $w_n(y) = g_0(a, y)$ for $y \in K_n^c$ and so $ng_0(a, y) \leq w(y)$ for $y \in K_n^c$, which yields (5.1) for $x = a$. By using (5.2) one gets (5.1) for every $x \in E$, which completes the proof of the lemma. □

Now we use w in the above lemma to define the resolvent

$$W_p(dx, y) = w(y)^{-1}\hat{U}_p(w(x)\,dx, u); \qquad p \geq 0.$$

LEMMA 5.3. *The resolvent* (W_p) *is of Feller type.*

PROOF. Clearly this resolvent is sub-Markovian. Let us consider $p > 0$. Since g_p and w are continuous it follows that for $f \in C_c(E)$, $W_p f$ is continuous. By (3.3) $g_0 \geq g_p$ and so, from (5.1) it follows that $W_p f \in C_0(E)$. Since $C_c(E)$ is dense in $C_0(E)$ and the operator W_p is bounded, one has $W_p(C_0(E)) \subset C_0(E)$. Next, to show that $W_p(C_0(E))$ is dense in $C_0(E)$ we shall use Hahn and Banach's theorem (this kind of argument was used for the first time in connection with duality in the framework of axiomatic potential theory; see Lemma 29.4 in Hervé [9]). By using this theorem it will suffice to show that any signed bounded measure ν such that $\nu(W_p f) = 0$ for any $f \in C_0(E)$, is null.

Let us write such a measure in the form $\nu = \nu_+ - \nu_-$ where ν_+ and ν_- are bounded nonnegative measures. We define

$$s_+(x) = \int g_p(x, y) w(y)^{-1} \nu_+(dy), \qquad s_-(x) = \int g_p(x, y) w(y)^{-1} \nu_-(dy).$$

Since $g_p(x, \cdot) w(\cdot)^{-1} \in C_0(E)$, s_+ and s_- are finite. They are the p-potentials of the following C.A.F.'s:

$$A_t^+ = \int L_t^x w(x)^{-1} \nu_+(dx), \qquad A_t^- = \int L_t^x w(x)^{-1} \nu_-(dx).$$

Next, for any $f \in C_0(E)$, one has

$$\begin{aligned} 0 &= \int f W_p(y) \nu(dy) \\ &= \iint f(z) w(z) g_p(z, y) w(y)^{-1} \eta(dz) \nu(dy) \\ &= \int f(z) w(z) (s_+(z) - s_-(z)) \eta(dz). \end{aligned}$$

It follows that $s_+ = s_-$, η a.s., and hence, s_+ and s_- being excessives, they are equal everywhere. Since a C.A.F. is uniquely determined by its p-potential, this yields $A^+ = A^-$. by the unicity of the representing measure, $w^{-1}\nu_+ = w^{-1}\nu_-$, which implies $\nu = 0$.

Now let $\hat{X}$ be the process associated to (W_p). Clearly this process is in duality with X with respect to the measure $w \cdot \eta$. The Green function with respect to this duality measure is $w(y)^{-1} g_p(x, y)$, $p \geq 0$. Since this function satisfies the hypothesis of Proposition 3.3(ii), $\hat{X}$ fulfills (1.1a–b). This completes the proof of Proposition 5.1. □

Now we are able to prove the main result of this section.

THEOREM 5.4. *Let* X *be a standard process satisfying* (1.1a–b). *Then for every excessive measure* ξ, *there exists a standard process satisfying* (1.1a–b), *which is in duality with* X *with respect to the measure* ξ. *The resolvent of this process is uniquely determined.*

PROOF. Let us assume first that X is transient. Let $\hat{X}$ be the Feller process produced in Proposition 5.1. The excessive measure ξ has a density v with respect to the duality measure $w \cdot \eta$, which is excessive with respect to $\hat{X}$ (see Proposition 1.11 of Chapter VI in [3]). Since $\hat{X}$ satisfies (1.1a–b), v is strictly positive, finite and continuous. Then by Proposition 4.3, the u-process associated to $\hat{X}$ and v satisfies (1.1a–b). Clearly this process is in duality with X with respect to ξ. The unicity is an immediate consequence of the duality relation. □

Let us now prove the theorem in the recurrent case. For any $p > 0$ the p-subprocess satisfies (1.1a–b) and is transient. Since the measure ξ is excessive with respect to the p-subprocess one may associate to it a dual process with a resolvent $(V_r^p)_{r>0}$. From the unicity property one deduces that $V_r^p = V_{r+p-q}^q$, for every $0 < q < p$. This allows us to define a resolvent $(V_r)_{r>0}$, with $V_r = V_{r-p}^p$, for $p < r$. Since the resolvents $(V_r^p)_{r>0}$ are associated to standard processes, the hypotheses of Theorem 4.3 in [16] are verified, and hence the resolvent $(V_r)_{r>0}$ produces a Hunt process. Then by Remark 3.4 this process fulfills (1.1a–b). This completes the proof. □

REMARK 5.5. We note that the duality obtained here has better regularity properties than Hunt's hypothesis (F). In our case the Green function is continuous, strictly positive and finite. The excessive and co-excessive functions are also continuous.

REMARK 5.6. If X is transient, then any dual is transient. This is an easy consequence of the duality relation and of the fact that under (1.1a–b) an excessive function is either identical infinite or finite everywhere. If the process X is recurrent, then there exists only one dual process. This follows from the fact that in this case the co-excessive functions are constant.

REMARK 5.7. Each $(\hat{U}_p)$-excessive function is finite, strictly positive and continuous. If ξ is an excessive measure then there exists an $(\hat{U}_p)$-excessive function u such that $\xi = u \cdot \eta$, and consequently, every such measure is a Radon measure. If ν_x is the Revuz measure associated to L^x with respect to ξ, then $u(x) = \nu_x(1)$.

In proving the first two assertions one may consider only the transient case, because p-subprocesses, $p > 0$, can be used in the recurrent case. If w and (W_p) are those constructed in the proof of Proposition 5.1, then a function v is $(\hat{U}_p)$-excessive iff vw^{-1} is (W_p)-excessive, which implies the two assertions. For the third assertion one should use a formula from [14] in order to get:

$$\begin{aligned}\nu_x(1) &= \lim_{\alpha\to\infty} \alpha \int U^{\alpha}_{L^x}(1)(y)\xi(dy) \\ &= \lim_{\alpha\to\infty} \alpha \int g_\alpha(y,x)\xi(dy) = \lim_{\alpha\to\infty} \alpha u \hat{U}_\alpha(x) = u(x).\end{aligned}$$

REMARK 5.8. In constructing a dual process we could not use the resolvent $(\hat{U}_p)$, because the sub-Markov property would be lacking. An example of this may be obtained by modifying the Brownian motion on the real line by a random time change associated to an additive functional with density function which is strictly positive, bounded and continuous.

It would appear natural to try to work from the beginning with an excessive reference measure ξ instead of η and with its associated dual resolvent, which would be sub-Markovian. However, it seems to be difficult to show directly that the corresponding Green function is continuous.

6. Convergence to a C.A.F. In [2] the convergence of increasing processes to the local time is studied. By using the representation theorem in Section 2 and the methods introduced in [2], we shall give a convergence theorem to a C.A.F. When speaking about an "increasing process" A we shall assume without other mention that A is a nondecreasing, adapted, "càdlàg" process with $A(0) = 0$ and $A_\infty = \sup_t A_t$. Information about such processes will be expressed in terms of the modified processes:

$$\overline{A}(t) = \int_{[0,t]} e^{-s}\,dA_s.$$

The distance between two increasing processes will be defined by

$$d_2(A,B) = \sup_x E^x\Big(\sup(\overline{A}_t - \overline{B}_t)^2\Big)^{1/2}.$$

We note that for every $\varepsilon > 0$

$$\begin{aligned}(1+2\varepsilon^{-1})\sup_x E^x&\Big(\sup_t e^{-(2-\varepsilon)t}(A_t - B_t)^2\Big)^{1/2}\\ &\geq d_2(A,B)\\ &\geq \tfrac{1}{2}\sup E^x\Big(\sup e^{-2t}(A_t - B_t)^2\Big)^{1/2}.\end{aligned}$$

In order to evaluate the distance between two increasing processes we shall introduce three parameters. The first one is the 1-potential of A, that is,

$$u_A(x) = E^x(\overline{A}_\infty) = E^x\Big(\int_0^\infty e^{-s}\,dA_s\Big).$$

The second one is the "additivity error." Assuming that $\overline{A}_\infty < \infty$ a.s. one may define

$$\Gamma_t = \overline{A}_\infty - \overline{A}_t - e^{-t}\overline{A}_\infty \circ \theta_t. \tag{6.1}$$

Then, for each $x \in E$, the P^x-optional projection of Γ is

$$\Gamma_t^x = E^x(\overline{A}_\infty | \mathscr{F}_t) - \overline{A}_t - e^{-t}u_A(X_t),$$

where the first term in the right-hand side of the above equality is the "càdlàg" version of that martingale.

Now $\overline{\Gamma}$ is defined by

$$\overline{\Gamma} = \sup_x E^x\Big(\sup_t (\Gamma_t^x)^2\Big)^{1/2}. \tag{6.2}$$

The last parameter refers to discontinuities:

$$\Delta(t) = \overline{A}(t) - \overline{A}(t-)$$

and

$$\overline{\Delta} = \sup_x E^x\Big(\sup_t \Delta_t^2\Big)^{1/2}.$$

In [2] the following result is proved:

LEMMA 6.1. *Let A be a C.A.F. and B an increasing process with parameters $\overline{\Gamma}$ and $\overline{\Delta}$. Let us denote by $c = \sup_x E^x(\overline{B}^2)^{1/2} \vee 1$. Then*

$$d_2(A, B) \le cK_1\big(\|u_A - u_B\|^{1/2} + \overline{\Gamma} + \overline{\Delta}\big) \le cK_2 d_2(A, B)^{1/2},$$

where K_1 and K_2 are universal constants.

We shall also need the following lemma:

LEMMA 6.2. *For any increasing process B*

$$|u_B(x) - u_B(y)| \le (\gamma(x, y)u_B(x) + 2\overline{\Gamma})/\varphi_y(x).$$

PROOF. By integrating in (6.1) one gets

$$u_B(x) = E^x\big(\overline{B}(T_y)\big) + E^x\big(\Gamma(T_y)\big) + \varphi_y(x)u_B(y),$$

which yields

$$u_B(x) \ge E^x\big(\Gamma(T_y)\big) + \varphi_y(x)u_B(y)$$

and

$$u_B(y) \ge E^y(\Gamma(T_x)) + \varphi_x(y)u_B(x).$$

From these two inequalities one deduces

$$E^x(\Gamma(T_x))/u_B(x) + \varphi_x(y) \le u_B(y)/u_B(x) \le \varphi_y(x)^{-1}\big(1 - E^x\big(\Gamma(T_y)\big)/u_B(x)\big),$$

and further

$$-\overline{\Gamma}/u_B(x) + \varphi_x(y) \le u_B(y)/u_B(x) \le \varphi_y(x)^{-1}(1 + \overline{\Gamma}/u_B(x)),$$

which yields the assertion of the lemma. □

Let us now consider a sequence of increasing processes A^n, $n \in N$, with parameters $u_n = u_{A^n}$, $\overline{\Gamma}_n$ and $\overline{\Delta}_n$. We assume that A^n are supported by a compact K, in the sense that $\int_0^\infty 1_{K^c}(X_s)\, dA_s^n = 0$ a.s. Let us also consider a C.A.F. A supported by the same compact K and set $u = u_A$.

THEOREM 6.3. *The following assertions are equivalent*:

(i) $u_n(x) \to u(x)$ *for every* $x \in E$, $\overline{\Gamma}_n \to 0$ *and* $\overline{\Delta}_n \to 0$.
(ii) $d_2(A^n, A) \to 0$.

PROOF. The only interesting implication is (i) $\Rightarrow$ (ii). To prove it we shall first show that

$$\|u_n - u\| \to_n 0. \tag{6.3}$$

Let us write

$$\begin{aligned} u_n(x) &= E^x(\bar{A}^n(\infty) - \bar{A}^n(T_K)) + E^x(\bar{A}^n(T_K) - \bar{A}^n(T_K -)) \\ &= E^x\big(\exp(-T_K)u_n(X_{T_K})\big) + E^x(\Gamma(T_K)) + E^x(\Delta(T_K)), \\ u(x) &= E^x(\exp(-T_K)u(X(T_K))), \end{aligned}$$

which yields $\|u_n - u\| \le \|u_n - u\|_K + \bar{\Gamma}_n + \bar{\Delta}_n$, where

$$\|u_n - u\|_K = \sup\{|u_n(x) - u(x)|: x \in K\}.$$

By using Lemma 6.2 one gets

$$|u_n(y) - u(y)| \le |u_n(x) - u(x)| + [2\Gamma_n + \gamma(x, y)(u_n(x) + u_n(y))]/\varphi_y(x).$$

Since $\bar{\Gamma}_n \to 0$, $u_n(x) \to u(x)$, the map $(x, y) \to \gamma(x, y)$ is continuous and $\gamma(x, x) = 0$, a compactness argument yields $\|u_n - u\|_K \to 0$, which ends the proof of (6.3).

Let us verify that

$$\sup_n \sup_x E^x\big((\bar{A}^n_\infty)^2\big) < \infty. \tag{6.4}$$

We have

$$\begin{aligned} &E^x\left(\left(\int_0^\infty e^{-t}\, dA^n_t\right)^2\right) \\ &\quad = 2E^x\left(\int_0^\infty e^{-t}(\bar{A}^n_\infty - \bar{A}^n_t)\, dA^n_t\right) + E^x\left(\int_0^\infty e^{-t}\Delta^n_t\, dA_t\right) \\ &\quad = 2E^x\left(\int_0^\infty e^{-t}\bar{A}^n_\infty \circ \theta_t\, dA^n_t\right) + 2E^x\left(\int_0^\infty e^{-t}\Gamma_n(t)\, dA^n_t\right) + E^x\left(\int_0^\infty e^{-t}\Delta^n_t\, dA^n_t\right). \end{aligned}$$

Let us denote by $a_n = \sup_x E^x((\bar{A}^n)^2)$. By passing to the optional projection under the integrals in the last member of the above equalities, some simple calculations yield

$$a_n \le 2\|u_n\| + (2\bar{\Gamma}_n + \bar{\Delta}_n)\sqrt{a_n}.$$

Since by (6.3), $\sup_n\|u_n\| < \infty$ and by the hypothesis in (i), $\sup_n\bar{\Gamma}_n < \infty$ and $\sup_n\bar{\Delta}_n < \infty$, the above inequality implies that $\sup_n a_n < \infty$, and (6.4) is proved.

Now, by using (6.3), (6.4) and Lemma 6.1, the proof ends. □

The topological properties of the correspondence between C.A.F.'s and representing measures are given by the following theorem:

THEOREM 6.4. *Let A^n, $n \in N$, and A be C.A.F.'s with the fine support included in the same compact set K. Put $u_n = u_{A^n}$, $u = u_A$ and denote by μ_n, $n \in N$, μ the measures associated to the C.A.F.'s via* (2.3). *Then the following* *assertions are equivalent*:

(a) $\mu_n \to \mu$ *in the weak topology*;
(b) $U^1_{A^n}(x, dy) \to U_A(x, dy)$ *in the weak topology for every* $x \in E$;
(c) $U^1_{A^n}(x, dy) \to U^1_A(x, dy)$ *in the weak topology for some* $x \in E$;
(d) $u_n(x) \to u(x)$ *for every* $x \in E$;
(e) $u_n \to u$, *uniformly on compacts*;
(f) $d_2(f \cdot A^n, f \cdot A) \to 0$ *for* $f \in C_b(E)$, $f \geq 0$;
(g) $d_2(A^n, A) \to 0$.

PROOF. The equivalences (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) follow from Proposition 2.4. The implication (b) $\Rightarrow$ (f) and the equivalence (d) $\Leftrightarrow$ (g) are consequences of Theorem 6.3.

The implication (e) $\Rightarrow$ (d) is obvious and the converse follows from (1.6)(b), which shows that (u_n) are equicontinuous. Implication (f) $\Rightarrow$ (g) is obvious. The last thing we have to prove is (d) $\Rightarrow$ (a). For $x \in E$ one has

$$u_n(x) \geq \Theta \cdot \mu_n(K), \quad \text{where } \Theta = \inf\{\varphi_y(x) : y \in K\} > 0.$$

This shows that the measures μ_n, $n \in N$ are uniformly bounded. If ν is a limit point of the sequence (μ_n), then the implication (a) $\Rightarrow$ (d) ensures that the 1-potential of $A^\nu = \int L^x \, d\nu(x)$ coincides with the 1-potential of A, and hence $A = A^\nu$. By the unicity of the representing measure it follows that $\mu = \nu$ and the proof ends. □

REMARK. If the local time has a version for which the map $(t, x) \to L^n_t$ is continuous, (a) implies that $\lim_n |A^n_t - A_t| = 0$ for every $t \geq 0$, a.s.

7. Approximation of a C.A.F. by occupation time and downcrossing models. In this section we shall use Theorem 6.3 in order to obtain approximation models for a C.A.F.; models which are analogous to those already known for the local time. Let us consider a C.A.F. A with compact fine support K, 1-potential u and representing measure μ. For a fixed $\varepsilon > 0$ we choose a finite measurable partition of K, $\{V_k, k \leq n_\varepsilon\}$, such that $K \subset \bigcup V_k$ and $\sup\{d(x, y): x, y \in V_k\} \leq \varepsilon$, for every $k \leq n_\varepsilon$. Let us also choose a system of points x_k, $k \leq n_\varepsilon$ such that $x_k \in V_k$. We denote by $h_K(\varepsilon) = \sup\{1 - \varphi_x(y)\varphi_y(x): x, y \in K, d(x, y) \leq \varepsilon\}$. This frame will be used in the following two applications.

a. Occupation time model. Let η be the representing measure of the usual time (see Section 3). We assume that the sets V_k are chosen such that $\eta(V_k) > 0$, for every $k \leq n_\varepsilon$. Let us define

$$O^\varepsilon_k(t) = \int_0^t 1_{V_k}(X_s) \, ds$$

and

$$A^\varepsilon(t) = \sum_k O^\varepsilon_k(t) \mu(V_k)/\eta(V_k).$$

THEOREM 7.1.

(a) $\lim_{\varepsilon\to 0} d_2(A^\varepsilon, A) = 0$.

(b) *If the local time has a version for which* $(t, x) \to L_t^x$ *is continuous, then* $\lim_{\varepsilon\to 0} A_t^\varepsilon = A_t$ *for every* $t \geq 0$ *a.s.*

(c) *In any case, if* (ε_n) *is a sequence such that* $\Sigma h_K(\varepsilon_n) < \infty$, *then* $\lim_n A_t^{\varepsilon_n} = A_t$ *for every* $t \geq 0$ *a.s.*

PROOF. Let us denote by μ_ε and u_ε the representing measure and the 1-potential of A^ε. It is easy to see that $\mu_\varepsilon(V_k) = \mu(V_k)$ and so one has

$$\begin{aligned}|u_\varepsilon(x) - u(x)| &\leq \left|\int \varphi_y(x)\mu_\varepsilon(dy) - \sum \varphi_{x_k}(x)\mu_\varepsilon(V_k)\right| \\ &\quad + \left|\sum \varphi_{x_k}(x)\mu(V_k) - \int \varphi_y(x)\mu(dy)\right| \\ &\leq \sum \int_{V_k} |\varphi_y(x) - \varphi_{x_k}(x)|\mu_\varepsilon(dy) + \sum \int_{V_k} |\varphi_y(x) - \varphi_{x_k}(x)|\mu(dy).\end{aligned}$$

It follows that

$$\|u_\varepsilon - u\| \leq 2h_K(\varepsilon)\mu(K). \tag{7.1}$$

Theorem 6.3 implies (a), and Theorem 6.4 implies (b). By Lemma 6.1 one gets $d_2(A^\varepsilon, A) \leq \mathrm{c}h_K(\varepsilon)\mu(K)$ and a Borel–Cantelli argument implies (c) [by (6.4) the constant c does not depend on ε]. □

b. Downcrossings. For this model we assume that x_k is an interior point of V_k. Let $S_k = T_{V_k^c}$, and $T_k = S_k + T_{x_k} \circ \theta_{S_k}$ (the crossing times). Define

$$T_k^0 = 0, \qquad T_k^1 = T_{x_k} \quad \text{and} \quad T_k^{n+1} = T_k^n + T_k \circ \theta_{T_k^n}, \quad \text{for } n \geq 1.$$

Since $T_k^n \uparrow \infty$ as $n \to \infty$, one may define

$$D_k^\varepsilon(t) = n \quad \text{on } T_k^n \leq t < T_k^{n+1},$$

$$d_k^\varepsilon = 1 - E^{x_k}(\exp(-T_k))$$

and

$$D_t^\varepsilon = \sum_k d_k^\varepsilon D_k^\varepsilon(t)\mu(V_k).$$

THEOREM 7.2.

(a) $\lim_{\varepsilon\to 0} d_2(D^\varepsilon, A) = 0$.

(b) *Let* (ε_n) *be a sequence such that* $\Sigma h_K(\varepsilon_n) < \infty$. *Then* $\lim D_t^{\varepsilon_n} = A_t$ *for every* $t \geq 0$ *a.s.*

PROOF. A simple calculation yields

$$E^x(\overline{D}_k^\varepsilon(\infty)) = \varphi_{x_k}(x)/d_k^\varepsilon.$$

Let $y \in K \cap V_k^c$ be such that $d(x_k, y) \le 2\varepsilon$. Then from $T_y + T_{x_k} \circ \theta_{T_y} \ge T_k$, P^{x_k}-a.s. one deduces

$$E^x(\exp(-T_k)) \ge \varphi_{x_k}(y)\varphi_y(x_k).$$

It follows that $d_k^\varepsilon \le 1 - \varphi_y(x_k)\varphi_{x_k}(y) \le h_K(2\varepsilon)$. We note that $\overline{\Gamma}(D_k^\varepsilon)$, $\overline{\Delta}(D_k) \le 1$ and so

$$\overline{\Gamma}(D^\varepsilon) \le \sum d_k^\varepsilon \overline{\Gamma}(D_k^\varepsilon)\mu(V_k) \le h_K(2\varepsilon)\mu(K),$$

$$\overline{\Delta}(D^\varepsilon) \le \sum d_k^\varepsilon \overline{\Delta}(D_k^\varepsilon)\mu(V_k) \le h_K(2\varepsilon)\mu(K),$$

$$u_{D^\varepsilon}(x) = \sum d_k^\varepsilon u_{D_k^\varepsilon}(x)\mu(V_k) = \varphi_{x_k}(x)\mu(V_k).$$

Further, as in the case of the above occupation time model we get

$$|u_{D^\varepsilon}(x) - u_A(x)| = \left|u_{D^\varepsilon}(x) - \int \varphi_y(x)\mu(dy)\right| \le h_K(\varepsilon)\mu(K).$$

By Lemma 6.1 this yields

$$d_2(D^\varepsilon, A) \le C\nu(K)h_K(\varepsilon),$$

where C is a constant which does not depend on ε [see (6.4)]. (a) and (b) follow. □

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