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EDGE FLUCTUATIONS FOR THE ONE DIMENSIONAL SUPERCRITICAL CONTACT PROCESS[1]

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We consider the one dimensional supercritical contact process with initial configurations having infinitely many particles to the left of the origin and only finitely many to its right. Starting from any such configuration, we first prove that in the limit as time goes to infinity the law of the process, as seen from the edge, converges to the invariant distribution constructed by Durrett [12]. We then prove a functional central limit theorem for the fluctuations of the edge around its average, showing that the corresponding diffusion coefficient is strictly positive. We finally characterize the space time structure of the system. In particular we prove that its distribution shifted in space by αt (t denotes the time and α the drift of the edge) converges when t goes to infinity to a $\frac{1}{2}-\frac{1}{2}$ mixture of the two extremal invariant measures for the contact process.

1. Introduction. In this article we study the one dimensional supercritical contact process, a stochastic process with infinitely many interacting particles which move on Z, the set of all the integers. We consider initial states with infinitely many particles to the left of the origin and finitely many to its right. We first prove that at large times the state of the system approaches some "definite structure" which, as time increases, "rigidly shifts to the right": A steady state propagates from the left to the right with constant speed through the formerly empty region. The system is a (very schematic and rough) microscopic model for the formation and propagation of one dimensional shock waves, as we shall see in some more detail in the sequel.

At large times we can distinguish three different space regions: The first two are semiinfinite intervals which extend to $-\infty$ and $+\infty$, respectively, while the third one is the interval which connects the first two.

In the first region the state looks like the nonzero density equilibrium state for the supercritical contact process, while, in the second it is the empty state. The third is the interesting region. It describes the "wave front." We find that the state there is (close to) a superposition of the two extremal equilibrium states. The weight of the decomposition ranges from 1 (all the weight to the nonzero density state) to 0, as one moves from left to right. As the time t increases, the previous picture shifts to the right and, at the same time, the size of the front wave increases (like $t^{1/2}$). The picture is very similar indeed to that found by Wick [25] in a zero range model. Our result is a consequence of a very detailed analysis of the motion of the first particle in the system, the edge, and of a precise knowledge of the structure of the state as seen from the edge. Such

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questions are interesting by themselves and have been extensively considered both in the contact process and in several other model systems.

We prove in this paper a functional central limit theorem for the edge process, thus extending the description given by Durrett [11], who proved a law of large numbers for the edge. Our result answers Liggett's problem 6 [21, Chapter 6].

With reference to the state seen from the edge, we answer a problem posed by Durrett [12]. We prove that the state constructed in [12] is the only stationary one for the process seen from the edge. We also characterize its domain of attraction as the set of all configurations with infinitely many particles (to the left of the edge).

Several questions arise naturally from the analysis of the space time structure of the contact process as compared to other model systems, as we briefly discuss in this section. As a first example consider independent asymmetric random walks having drift $\alpha > 0$. Assume the initial state has no particles to the right of the origin, while those to the left have Poisson distribution with density $\rho > 0$. Then, by Doob's theorem, at any later time the state is a product of Poisson distributions. As time increases, like in the contact process, we distinguish three regions. In the first two the state is (approximately) a Poisson point process with density ρ and 0, respectively. In the intermediate region, in contrast to what happens in the contact process, the state is again (locally) close to Poisson, its density varying from ρ to 0 as one moves from left to right. With time the picture shifts to the right with speed α (the drift of any of the random walks).

The same thing happens for the one dimensional symmetric simple exclusion process [9], [14]. The speed in such case is 0, but, again, in the intermediate region the state is locally (close to) an extremal equilibrium state (the equilibrium measures for the simple exclusion are Bernoulli). A reason for the different behavior from the contact process comes from the presence in the latter of just two extremal equilibrium states, so that it is "physically" impossible to connect them smoothly (with other extremal equilibrium states). This is not the only reason, as shown by the (already mentioned) example due to Wick [25], where the equilibrium states are parametrized by their densities which can take any nonnegative value. Yet, in the front wave, the state is a superposition of two extremal ones, just as in the contact process. We believe that this happens also in the asymmetric simple exclusion and zero range processes which exhibit shock wave phenomena (cf. [1], [2], [4] and [23] for the simple exclusion and [3] for the zero range; the result in the latter case has been recently extended by E. Andjel and M. E. Vares [in preparation]).

The analogy with physical models suggests that the occurrence of such phenomena in a model system where shocks are present should be connected to the following fact: There should be an invariant state seen from some suitably defined tagged particle, which, far from the tagged particle, to its right and left, is close, respectively, to two mutually distinct extremal invariant states. These latter states would then be the states before and after the shock. The drift of the tagged particle should be equal to the speed of the shock, while the fluctuations should determine the weights of the mixture.

The two drastically distinct behaviors observed in the previous models are evident also in apparently different situations. Consider the evolution of the symmetric exclusion process in a large but finite volume, with annihilation of particles at the boundaries. Start with all sites occupied. Then the state of the system changes in a smooth "deterministic" way toward the empty configuration, as easily can be seen. At the other extreme is the supercritical contact process. Assume the process is restricted to some finite interval F, so that its only invariant measure has support on the empty configuration. Assume, as before, that initially all sites in F are occupied. In the infinite volume case the process has a nonzero density invariant state, and actually also in the finite interval F it will remain for a very long time (long according to the size of F) in an apparently stable (in fact metastable) state until a sudden breakdown occurs. The process then moves abruptly toward the stable state and all particles die (cf. [5] and [24]).

The preceding phenomena are certainly interrelated, but a clear understanding of their precise connections still seems an open, though interesting, question.

This paper is organized as follows: Basic definitions and results can be found in Section 2, while the proofs are given in Sections 3 and 4.

2. Basic definitions and results. The contact process has been extensively studied since Harris' papers [16], [17], so we will go very rapidly through the basic definitions. We refer to [12], [15] and [21] for details.

The basic one dimensional contact process describes the time evolution of a system of particles in Z, the set of all integers. Each site in Z is either empty or occupied (by one single particle). Each particle either dies, after an exponential time of mean 1, or it tries to create a new particle at one of its two nearest neighbor sites, after the first out of two (one for each site) independent exponential times of mean λ^{-1}, $\lambda > 0$. If the position chosen for the new particle is already occupied, then the creation is not allowed. All the exponential random times so far introduced are mutually independent.

Graphical representation of the contact process. The contact process can be constructed with the help of a directed percolation structure on $Z \times R_+$ in the following way: For x in Z let $\{U_n^{(x,x+1)}: n \geq 1\}$, $\{U_n^{(x,x-1)}: n \geq 1\}$ and $\{\overset{+}{U}{}_n^x: n \geq 1\}$ be mutually independent Poisson point processes in R_+, with intensities λ, λ and 1, respectively. We suppose that the Poisson processes corresponding to different values of x are also mutually independent and we denote by $(\Omega, \mathscr{A}, P)$ the probability space where all such Poisson processes are defined. For $0 \leq s \leq t$, $\mathscr{A}_{[s,t]}$ denotes the σ-algebra of all events in the preceding Poisson processes which refer to the time interval $[s, t]$. All the versions of the contact process in this paper will always be defined in $(\Omega, \mathscr{A}, P)$.

We shall draw arrows and crosses on $Z \times R_+$. An arrow directed from x to $x \pm 1$ at time t indicates that there exists n such that $U_n^{(x,x\pm1)} = t$, while a cross at x corresponds to $\overset{+}{U}{}_n^x = t$, for some n. Arrows indicate attempts to create new particles, while crosses are death times for particles. Arrows are effective if they join an occupied site to an empty one, while a cross annihilates a particle if it appears at an occupied site.

Fix a configuration of particles at time 0 and a point in Ω. We shall then say that a particle is at x at time t if it is possible to connect (x, t) to some $(y, 0)$ in such a way that (1) the path from $(y, 0)$ to (x, t) moves vertically on $Z \times R_+$ and it can only turn following the arrows (such turns are allowed, not compulsory), (2) the path never meets a cross and (3) there is a particle at $(y, 0)$. More precisely we pose the following:

Definition. Given s and t in R, $s < t$, x and y in Z and ω in Ω we say that there is a ω-path from (x, s) to (y, t) and write $(x, s) \to_\omega (y, t)$ if there is a finite sequence of points $x_0, x_1, \dots, x_k$ with $x_0 = x$, $x_k = y$ and $|x_i - x_{i+1}| = 1$, for $i = 0, \dots, k-1$, and there are integers $n_1, \dots, n_k$ such that

$$s < U_{n_1}^{(x_0, x_1)}(\omega) < \cdots < U_{n_k}^{(x_{k-1}, x_k)}(\omega) < t$$

and for no j and m,

$$s \le \overset{+}{U}{}_m^x \le U_{n_1}^{(x, x_1)},$$

$$U_{n_{j-1}}^{(x_{j-1}, x_j)}(\omega) \le \overset{+}{U}{}_m^{x_j}(\omega) \le U_{n_j}^{(x_j, x_{j+1})}(\omega),$$

$$U_{n_k}^{(x_{k-1}, y)}(\omega) \le \overset{+}{U}{}_m^{y} \le t.$$

The contact process takes value on $\{0,1\}^{\mathbf{Z}}$ (the configuration space). Given a configuration η, two times $s < t$ and a point ω in Ω, we define the configuration $\xi_t^{(\eta, s)}(\omega)$ as follows: For any x in Z, $\xi_t^{(\eta, s)}(\omega, x) = 1$ if and only if there is y in Z such that $\eta(y) = 1$ and $(y, s) \to_\omega (x, t)$. For $s = 0$ we simply write ξ_t^η in place of $\xi_t^{(\eta, s)}$ and if η has only one particle which is at a [$\eta(x) = 1$ if and only if $x = a$], then we simply write $\xi_t^{(a, s)}$. The process $(\xi_t^\eta)_{t \ge 0}$ is the contact process starting at time zero from the configuration η. We will use the following:

Notation.

1. Given a configuration η,

$$|\eta| = \sum_{x \in \mathbb{Z}} \eta(x),$$

$$r(\eta) = \sup\{x \in \mathbb{Z}: \eta(x) = 1\},$$

$$l(\eta) = \inf\{x \in \mathbb{Z}: \eta(x) = 1\}.$$

2. $E = \{\eta \in \{0,1\}^{\mathbf{Z}}: r(\eta) < \infty, |\eta| = \infty\}$.
3. $\tilde{E} = \{\eta \in E: r(\eta) = 0\}$.
4. S is the map from E to $\tilde{E}$ such that $S\eta(x) = \eta(x + r(\eta))$.

There is a critical value of λ which we denote by λ^*. For $\lambda < \lambda^*$ there is only one invariant measure δ_0, i.e., the Dirac measure on the zero configuration. For

$\lambda > \lambda^*$ there are two extremal states, δ_ϕ and μ. The latter has support on configurations with some positive density which depends on λ. For this and most facts about the contact process we will use, see [21, Chapter 6].

Durrett [12] has recently proved that there is a probability $\tilde{\mu}$ on $\tilde{E}$ invariant for the process $S\xi_t^\eta$, i.e., the process seen by the edge. We extend this result by proving:

THEOREM 1. *For any η in E, the process $(S\xi_t^\eta)_{t\geq 0}$ converges in law, when $t \to +\infty$, to $\tilde{\mu}$.*

As a consequence $\tilde{\mu}$ is the unique invariant measure for the process seen from the edge and its domain of attraction is the whole $\tilde{E}$. Let $\bar{\eta}$ be such that $\bar{\eta}(x) = 1$ if and only if $x \leq 0$. Let $\bar{r}_t = r(\xi_t^{\bar{\eta}})$. Let $\alpha > 0$ be the a.s. limit of $t^{-1}\bar{r}_t$ as $t \to +\infty$ (the existence of such limit was proved by Durrett [11].

We have the following result:

THEOREM 2. *Let η be any configuration in E. Then the process $t \in R_+ \to \varepsilon[r(\xi^\eta_{\varepsilon^{-2}t}) - \alpha\varepsilon^{-2}t]$ converges in law, when ε goes to zero, to a Brownian motion B_t with strictly positive diffusion coefficient σ^2, and σ^2 is independent of η.*

The preceding result answers problem 6 of [21, Chapter 6]. To the left and to the right of the region where the edge is likely to be, the state is close to μ and δ_ϕ, respectively. The next result concerns the structure of the process in the region where the edge is likely to be. We first need the following:

NOTATION. Given x in Z and a configuration η, let $\eta - x$ be the configuration η shifted to the left by x, i.e., $(\eta - x)(y) = \eta(y + x)$.

We then prove the following theorem:

THEOREM 3. *Let T_ε be any monotone positive function such that $\lim_{\varepsilon\to 0} T_\varepsilon = +\infty$ and $\lim_{\varepsilon\to 0}\varepsilon T_\varepsilon = 0$. Let $t > 0$, $r \in R$ and define the empirical measure $A_\varepsilon(t, r)$ so that for any cylinder function f,*

$$A_\varepsilon(t, r)(f) = T_\varepsilon^{-1}\int_{\varepsilon^{-2}t}^{\varepsilon^{-2}t+T_\varepsilon} ds\, f\big(\xi_s^{\bar{\eta}} - [\alpha\varepsilon^{-2}t + \varepsilon^{-1}r]\big).$$

Then $A_\varepsilon(t, r)$ converges weakly when ε goes to zero to the measure

$$P(B_t < r)\mu + P(B_t > r)\delta_\phi,$$

where $(B_t)_{t\geq 0}$ is the Brownian motion starting at 0 and with diffusion coefficient $\sigma^2 > 0$ obtained in Theorem 2.

3. Proof of Theorem 1 and that the increments of the edge are α-mixing. The proofs of Theorems 1, 2 and 3 are based on the good "mixing" properties of the time evolution. Mixing originates from the following mechanism: Suppose that at least one of the descendants of the particle which is the edge at some time t^* remains alive for ever, namely assume that $|\xi_t^{(r_{t^*}, t^*)}| \neq 0$ for all $t \geq t^*$. Since we are considering the nearest neighbor one dimensional contact process, the increments of the edge after t^* will not (as we will show) depend on what happened before t^*. We then need to prove that $t^* < \infty$ a.s., starting from any configuration η in E.

The crucial point is that the event $\{|\xi_t^{(r_s, s)}| \neq 0\}$ for all $t \geq s$ is independent of $\{\xi_t^\eta, t \leq s\}$ for any η in E, due to the Poisson structure of the process. After such a remark we observe that since the contact process is supercritical, then the event $\{|\xi_t^0| \neq 0$, for all $t \geq 0\}$ has positive probability. Therefore, it will occur a.s. at some time. Such arguments can be made rigorous, as we shall see in this section, and they will enable us to prove that the increments of the edge are α-mixing, with rapidly decaying coefficient.

The invariance principle follows then from general theorems as discussed in Section 4. The proof that the diffusion coefficient is strictly positive requires some ad hoc considerations.

The independence property of the increments of the edge in the sense previously discussed is one of the crucial points in our proof. It comes out automatically in the following realization of the process, which, as usual, is framed in the basic $(\Omega, \mathscr{A}, P)$.

Given $u > 0$, let k be a positive integer and η a configuration in E. We define, for $t \geq ku$, the random variable $\hat{\xi}_t^{(\eta, ku)}$ with values in E as follows:

(i) $\hat{\xi}_{ku}^{(\eta, ku)} = \eta$.

(ii) If $nu \leq t < (n+1)u$, with $n \geq k$, then $\hat{\xi}_t^{(\eta, ku)} = \xi_t^{(S\zeta, nu)} + r(\zeta)$, where $\zeta = \hat{\xi}_{nu}^{(\eta, ku)}$.

When $k = 0$, we just write $\hat{\xi}_t^\eta$ instead of $\hat{\xi}_t^{(\eta, 0)}$. For configurations η such that $\eta(x) = 1$ if and only if $x = a$, we simply write $\hat{\xi}_t^{(a, ku)}$.

NOTATION. For any k and u as before let $\tilde{\xi}_t^{(\eta, ku)} = S\hat{\xi}_t^{(\eta, ku)}$. $(\tilde{\xi}_t)_{t \geq 0}$ is the process seen from the edge.

The fact that $(\hat{\xi}_t^\eta, t \geq 0)$ is really a version of the contact process starting from η, where η is in E, follows immediately from the translation invariance and independence in disjoint sets property of the Poisson point process. There are two quite obvious features in the construction which will be basic in the sequel.

1. Given any nonnegative integer k, the event $\{|\tilde{\xi}_t^{(0, ku)}| \neq 0\}$ is measurable with respect to the sub-σ-algebra $\mathscr{A}_{[ku, t]} \subset \mathscr{A}$ (generated by all the Poisson events in $Z \times [ku, t]$). As a consequence, it is independent of any event in $\mathscr{A}_{[0, ku]}$.
2. Assume that for some ω in Ω, $k \geq 0$, $t \geq ku$, it happens that $|\tilde{\xi}_t^{(0, ku)}(\omega)| \neq 0$; then, for any η in E, $\tilde{\xi}_t^\eta(\omega, x) = \tilde{\xi}_t^{(0, ku)}(\omega, x)$ for all x such that $l(\tilde{\xi}_t^{(0, ku)}) \leq x \leq 0$.

Notice that feature 1 is just the same as stating that for any η in E and $k \geq 0$, the event

$$\left\{|\tilde{\xi}_t^{(r_{ku}(\eta), ku)}| \neq 0\right\} \quad \text{with } t \geq ku$$

is measurable on the σ-algebra $\mathscr{A}_{[ku,t]}$. Observe that this is not true for the variable $S\xi_t^\eta$, i.e., for the usual "basic" realization of the edge process, as done in [15], [21] and in our Section 2.

Property 1 is a trivial consequence of the definition of the process $\tilde{\xi}_t$, while 2 is based on the nearest neighbor nature of the interaction. It is proven just like its analog for the basic contact process: At any time t, no particle to the left of $l(\tilde{\xi}_t^{(0,ku)})$ can be responsible for the birth of one to its right because (by definition) there is a particle in $l(\tilde{\xi}_t^{(0,ku)})$ [we are neglecting events of zero probability, like for instance the fact that the particle at $l(\tilde{\xi}_t^{(0,ku)})$ dies and just at the same time a particle at $l(\tilde{\xi}_t^{(0,ku)}) - 1$ creates a particle to its right...].

Next we introduce the set $\Lambda_{y,t}$ as follows. Given a negative integer y and $t \geq 0$ we set

$$\Lambda_{y,t} = \bigcup_{k \leq [u^{-1}t]} \left\{ |\tilde{\xi}_t^{(0,ku)}| \neq 0 \text{ and } l\left(\tilde{\xi}_t^{(0,ku)}\right) < y \right\},$$

where $[a]$ is the integer part of a. Then for any cylinder set B in $\tilde{E}$ defined by a subset of sites contained in $\{y, \dots, -1, 0\}$ and for any configuration η in E we have, by property 2,

$$\begin{aligned} |\tilde{\mu}(B) - P(\tilde{\xi}_t^\eta \in B)| &= \left| \int \tilde{\mu}(d\zeta) P(\tilde{\xi}_t^\zeta \in B) - P(\tilde{\xi}_t^\eta \in B) \right| \\ &\leq \int \tilde{\mu}(d\zeta) P\left(\tilde{\xi}_t^\zeta(x) \neq \tilde{\xi}^\eta(x) \text{ for some } x \text{ such that } y \leq x \leq 0\right) \\ &\leq P\left(\Lambda_{y,t}^c\right), \end{aligned}$$

where $\tilde{\mu}$ is the invariant probability measure whose existence was shown by Durrett [12]. Therefore, Theorem 1 will follow once we show that $P(\Lambda_{y,t})$ converges to 1 when t goes to infinity. We will actually prove something stronger, which implies the α-mixing property for the process r_t.

For y and t as before, let

$$\Gamma_{y,t} = \bigcup_{k \leq [u^{-1}t]} \left\{ |\tilde{\xi}_s^{(0,ku)}| \neq 0 \text{ for all } s \geq ku \text{ and } l\left(\tilde{\xi}_s^{(0,ku)}\right) < y \text{ for all } s \geq t \right\}.$$

Lemma 4. *Fix any $t > 0$, let $u = t^{1/2}$ and consider then the process $(\tilde{\xi}_s^{\cdot})$ for such value of u. Then*

$$P\left(\Gamma_{y,t}^c\right) \leq ae^{-bu},$$

where a and b are positive constants which only depend on y and λ.

Proof. In what follows a and b are positive constants whose values will change from line to line. Let

$$\tau^{(0,ku)} = \inf\left\{ s > ku \colon |\tilde{\xi}_s^{(0,ku)}| = 0 \right\},$$

$$K = \min\left\{ k = 0, 1, 2, \dots \colon \tau^{(0,ku)} > (k+1)u \right\}.$$

Then

$$
\begin{aligned}
P(\Gamma_{y,t}^c) &\le P(K \ge [u^{-1}t]) \\
(3.1) \qquad &+ \sum_{k<[u^{-1}t]} P(K=k)\big[P((k+1)u < \tau^{(0,ku)} < \infty) \\
&\qquad + P\big(\tau^{(0,ku)} = \infty,\ l\big(\xi_s^{(0,ku)}\big) > y \text{ for some } s > t\big)\big].
\end{aligned}
$$

Now, by the translation invariance and independence in disjoint sets property of the Poisson point process,

$$P(K \ge [u^{-1}t]) = P(\tau^{(0,0)} < u)^{[u^{-1}t]} \le P(\tau^{(0,0)} < \infty)^{[u^{-1}t]} = (1-\rho)^{[u^{-1}t]}.$$

A theorem by Durrett and Griffeath [13] provides the second upper bound,

$$P\big((k+1)u < \tau^{(0,ku)} < \infty\big) \le ae^{-bu},$$

where a and b are positive constants which depend only on λ (cf. [21, Chapter 6, Theorem 3.23]).

For the remaining terms in the r.h.s. of (3.1), we write

$$
\begin{aligned}
(3.2) \qquad & P\big(\tau^{(0,ku)} = \infty,\ l\big(\xi_s^{(0,ku)}\big) > y \text{ for some } s \ge t\big) \\
&\quad \le P\big(|\xi_t^0| \ge 1, \forall t \ge 0,\ r\big(\xi_s^0\big) - l\big(\xi_s^0\big) \le |y| \text{ for some } s \ge u\big).
\end{aligned}
$$

Notice that

$$\big\{r\big(\xi_s^0\big) - l\big(\xi_s^0\big) \le |y|\big\} \subset \big\{r\big(\xi_s^0\big) \le |y|\big\} \cup \big\{l\big(\xi_s^0\big) \ge y\big\},$$

so that

$$
\begin{aligned}
\text{r.h.s. of (3.2)} &\le 2P\big(|\xi_t^0| \ge 1, \forall t \ge 0;\ r\big(\xi_s^0\big) \le |y| \text{ for some } s \ge u\big) \\
&\le 2P(\bar{r}_s \le |y| \text{ for some } s \ge u),
\end{aligned}
$$

where $\bar{r}_t = r(\xi_t^{\bar{\eta}})$, $\bar{\eta}$ being the configuration where all sites $x \le 0$ are occupied while all the others are empty.

At this point we use an estimate due to Durrett [12, Section 12, page 1031], namely that

$$(3.3a) \qquad P(\bar{r}_t \le (\alpha - \delta)t) \le ae^{-bt},$$

where t and δ are any positive numbers and a and b depend only on δ and λ (cf. [21, Chapter 6, Corollary 3.22]). The same inequality (with different a and b) holds for all $s \ge t$, namely,

$$(3.3b) \qquad P\Big(\inf_{s \ge t} s^{-1}\bar{r}_s \le \alpha - 2\delta\Big) \le ae^{-bt}.$$

Proof of (3.3b). We first notice that the above inequality holds if s is restricted to be an integer ($\ge t$). Therefore,

$$P\Big(\inf_{s \ge t} s^{-1}\bar{r}_s \le \alpha - 2\delta\Big) \le ae^{-bt} + P\Big(\inf_{s \ge t} s^{-1}\bar{r}_s \le \alpha - 2\delta,\ \inf_{\substack{s \ge t\\ s \in \mathbf{Z}}} s^{-1}\bar{r}_s > \alpha - \delta\Big).$$

Since the last term is exponentially small in δt, (3.3b) is proven. □

We now choose $\delta = \alpha/4$. For t large enough $u = t^{1/2} > 2|y|/\alpha$, hence

$$P(\bar{r}_s \le |y| \text{ for some } s \ge u) \le P\Big(\inf_{s \ge u} s^{-1}\bar{r}_s < \alpha/2\Big) \le ae^{-bu}. \tag{3.4}$$

We have so far proven an upper bound of the form ae^{-bu} for each term appearing in the r.h.s. of (3.1). Lemma 4 is therefore proven. □

PROOF THAT r_t HAS α-MIXING INCREMENTS. For notational simplicity we restrict ourselves to discrete times. (It is easy to see that the CLT for discrete times implies the corresponding result for r_t.) Call

$$\tilde{\Omega} = \tilde{E} \times \Omega.$$

For every $(\eta, \omega) \in \tilde{\Omega}$ and $n \ge 0$, we define

$$X_n(\eta, \omega) = r\big(\xi_{n+1}^{\eta}(\omega)\big) - r\big(\xi_n^{\eta}(\omega)\big).$$

Let

$$\mathscr{F}_m = \sigma(X_n\colon n \le m), \qquad \mathscr{F}^m = \sigma(X_n\colon n \ge m),$$

be the σ-algebras of the past and the future at time m, respectively, of the process X_n. Finally, we denote by $P_{\tilde{\mu}}$ the direct product of P and $\tilde{\mu}$. $P_{\tilde{\mu}}$ is therefore the distribution of the process starting from $\tilde{\mu}$, the stationary distribution for the edge. We now have the following proposition:

PROPOSITION 5. *For every $m \ge 0$ and $n > m$,*

$$\sup{}^*|P_{\tilde{\mu}}(F \cap G) - P_{\tilde{\mu}}(F)P_{\tilde{\mu}}(G)| \le a\exp\big(-b(n-m-1)^{1/2}$$

where the sup* *is over all sets F and G which are, respectively, in $\mathscr{F}_m$ and $\mathscr{F}^n$. a and b are positive constants which depend only on λ.*

PROOF. Let f and g be two measurable functions from $\mathbb{Z}^{\mathbb{N}}$ to $[0,1]$ such that f depends only on the coordinates $x_0, \dots, x_m$ and g only on $x_n, x_{n+1}, \dots,$ n being larger than m. We need to show that there are a and b positive so that

$$\big|E_{\tilde{\mu}}[f(X.)g(X.)] - E_{\tilde{\mu}}[f(X.)]E_{\tilde{\mu}}[g(X.)]\big| \le a\exp\big(-b(n-m-1)^{1/2}\big)$$

for any f and g as before. For $n \ge ku$, in the set $\{\tau^{(0,ku)} = \infty\}$ we define

$$X_n^k = r\big(\xi_{n+1}^{(0,ku)}\big) - r\big(\xi_n^{(0,ku)}\big).$$

Notice that in the set $\{\tau^{(0,ku)} = \infty\}$, the equality $X_n(\eta, \omega) = X_n^k(\omega)$ holds for all $n \ge ku$ and for all η (cf. property 2 at the beginning of this section). On the other hand, $f(X.)$ only depends on the initial configuration and on the σ-algebra $\mathscr{A}_{[0,m]}$. Therefore, by property 1 (stated at the beginning of this section) we have, for $m \le ku \le n$,

$$E_{\tilde{\mu}}\big[f(X.)g(X.), \tau^{(0,ku)} = \infty\big] = E_{\tilde{\mu}}\big[f(X.)\big]E_{\tilde{\mu}}\big[g(X.), \tau^{(0,ku)} = \infty\big]. \tag{3.5}$$

We now define the random variable

$$K_m = \inf\{k > [u^{-1}(m+1)] : \tau^{(0,ku)} > (k+1)u\}.$$

From (3.5) we then get

$$\begin{aligned} &E_{\tilde\mu}[f(X.)g(X.)] \\ &\quad = E_{\tilde\mu}[f(X.)] \sum_{[u^{-1}(m+1)]\le k\le[u^{-1}n]} E_{\tilde\mu}[g(X.), K_m = k, \tau^{(0,ku)} = \infty] \\ (3.6)\qquad &\qquad + \sum_{[u^{-1}(m+1)]\le k\le[u^{-1}n]} E_{\tilde\mu}[f(X.)g(X.), K_m = k, \tau^{(0,ku)} < \infty] \\ &\qquad + E_{\tilde\mu}[f(X.)g(X.), K_m > [u^{-1}n]]. \end{aligned}$$

We rewrite (3.6) with $E_{\tilde\mu}[f(X.)]$ in place of $f(X.)$. We subtract one from the other and we get

$$\begin{aligned} &|E_{\tilde\mu}[f(X.)g(X.)] - E_{\tilde\mu}[f(X.)]E_{\tilde\mu}[g(X.)]| \\ &\quad \le 2P(K_m > [u^{-1}n]) + 2P(u < \tau^{(0,0)} < \infty) \\ &\quad \le 2(1-\rho)^{[u^{-1}(n-m-1)]} + ae^{-bu}. \end{aligned}$$

It is enough to take $u = (n-m-1)^{1/2}$ to conclude the proof. □

4. Proofs of Theorems 2 and 3. Hereafter in this section r_t will denote the random variable in $(\tilde\Omega, P_{\tilde\mu})$ defined as

$$r_t(\eta, \omega) = r(\hat\xi_t(\omega)), \qquad (\eta, \omega) \in \tilde\Omega := \tilde E \times \Omega,$$

where $\hat\xi_t$ is the contact process as realized in Section 3 (we will specify the value of u when needed).

THEOREM 6. *The process $\{\varepsilon[r_{\varepsilon^{-2}t} - E_{\tilde\mu}(r_{\varepsilon^{-2}t})], t \ge 0\}$ converges in law, as $\varepsilon \to 0$, to a Brownian motion with strictly positive diffusion coefficient σ^2.*

PROOF. Since $\tilde\mu$ is the invariant probability for the process as seen from the edge, then the increments of r_t are stationary. Moreover, Proposition 5 assures that its α-mixing coefficient decreases as $\exp(-bn^{1/2})$, $b > 0$. Therefore, classical results on the invariance principle will imply convergence to a Brownian motion once we show that for some $\delta > 2$

$$E_{\tilde\mu}(|r_1 - E_{\tilde\mu}(r_1)|^\delta) < \infty \tag{4.1}$$

(cf. [18], [6], [10]). This is done in the next lemma.

LEMMA 7. *For every $\delta > 0$ the inequality* (4.1) *holds.*

PROOF. We will prove that there are a and b positive so that for any $d > 0$,

$$P_{\tilde\mu}(|r_1| > d) \le ae^{-bd^{1/2}},$$

from which the lemma follows. For any d (large enough) set $t = d/2\lambda$. Divide the time interval $[0, t]$ into subintervals of length u, where u is the integer such that

$$u^2 \le t < (u+1)^2.$$

We consider for such u the realization $\hat{\xi}_t^\eta$ of the contact process defined in Section 3. Let K be the stopping time introduced in the proof of Lemma 4. Then

$$\begin{aligned} P_{\tilde{\mu}}(|r_1| > d) &= P_{\tilde{\mu}}(|r_t - r_{t-1}| > d) \\ &\le \sum_{k=0}^{u-2} P(K = k, |r_t - r_{t-1}| > d) + P_{\tilde{\mu}}(K > u-2). \end{aligned} \tag{4.2}$$

As in the proof of Lemma 4, we have that

$$P_{\tilde{\mu}}(K > u-2) \le (1-\rho)^{u-1} \le (1-\rho)^{t^{1/2}-2} \le ae^{-bd^{1/2}},$$

since $t = d/2\lambda$. For each of the other terms in the r.h.s. of (4.2) we write $(k \le u-2)$

$$\begin{aligned} &P_{\tilde{\mu}}(K = k, |r_t - r_{t-1}| > d) \\ &\quad\le P_{\tilde{\mu}}((k+1)u < \tau^{(0,k\mu)} < \infty) \\ &\quad\quad + P_{\tilde{\mu}}\big(\tau^{(0,ku)} = \infty;\, r\big(\hat{\xi}_s^{(0,ku)}\big) \le r\big(\hat{\xi}_{ku}^{(0,ku)}\big) \text{ for some } s \ge (k+1)u\big) \\ &\quad\quad + P_{\tilde{\mu}}\big(\tau^{(0,ku)} = \infty;\, r\big(\hat{\xi}_{t-1}^{(0,ku)}\big) - r\big(\hat{\xi}_{ku}^{(0,ku)}\big) \ge \tfrac{3}{2}t\lambda\big) \\ &\quad\quad + P_{\tilde{\mu}}\big(\tau^{(0,ku)} = \infty;\, r\big(\hat{\xi}_t^{(0,ku)}\big) - r\big(\hat{\xi}_{ku}^{(0,ku)}\big) \ge \tfrac{3}{2}t\lambda\big) \\ &\quad\quad + P_{\tilde{\mu}}\big(\tau^{(0,ku)} = \infty;\, r\big(\hat{\xi}_s^{(0,ku)}\big) > r\big(\hat{\xi}_{ku}^{(0,ku)}\big) \text{ for all } s \ge (k+1)u; \\ &\quad\quad\quad r\big(\hat{\xi}_s^{(0,ku)}\big) - r\big(\hat{\xi}_{ku}^{(0,ku)}\big) < \tfrac{3}{2}t\lambda \text{ for } s = t-1 \text{ and } s = t; \\ &\quad\quad\quad |r_t - r_{t-1}| > d\big). \end{aligned} \tag{4.3}$$

The last term vanishes because both r_t and r_{t-1} are in

$$\big[r\big(\hat{\xi}_{ku}^{(0,ku)}\big), r\big(\hat{\xi}_{ku}^{(0,ku)}\big) + \tfrac{3}{2}t\lambda\big];$$

hence, $|r_t - r_{t-1}| \le \frac{3}{2}t\lambda = \frac{3}{4}d < d$. The first and second terms in the r.h.s. of (4.3) are both bounded by ae^{-bu} [cf. the proof of Lemma 4 for the first one and (3.3b) for the second one]. The third and fourth terms are bounded by the probability of the same events in the contact process with no deaths. We again get a bound like ae^{-bu}.

Since the number of terms in the sum in (4.2) grows like $d^{1/2}$, the lemma is proved. □

So far we know that $\varepsilon[r_{\varepsilon^{-2}t} - E_{\tilde{\mu}}(r_{\varepsilon^{-2}t})]$ converges in law when ε goes to zero to a Brownian motion. The diffusion coefficient however might be zero and the Brownian motion degenerate. In several cases this turns out to be a very delicate point (cf. for instance [7], [8], [19] and [20]).

The proof that

$$\sigma^2 = \lim_{s\to\infty} s^{-1} E_{\tilde{\mu}}\left(|r_s - E_{\tilde{\mu}}(r_s)|^2\right) > 0 \tag{4.4}$$

is in our case based on the following strategy. We will fix suitably many of the increments of the edge and consider the remaining ones as variable. In such a way we "decrease" the diffusion. The increments which have not been fixed are chosen so that they are mutually independent, after conditioning on the others. Hence it is easy to compute the corresponding diffusion coefficient, which turns out to be positive. To implement the preceding strategy we introduce a measurable partition π on the space $\tilde{\Omega}$ (cf. [22, Section 3]). The conditional law of r_t on π will be equal to the law of a sum of i.i.d. random variables (plus a constant which depends on the fixed atom a of π). The number of such i.i.d. variables also depends on the atom a; their law is however independent of a.

In order to define the partition π we realize the contact process as in Section 3, choosing $u = 1$. We then introduce an increasing sequence of positive integer random times t_n: $\Omega \to \mathbb{N}$, $n = 0, 1, \dots,$ as follows. For any ω in Ω,

$$t_1(\omega) = \min\left\{t = 1, 2\dots: \sum_{k=1}^{\infty} 1_{(t \le \dot{U}_k^0(\omega) \le t+1)} = 0 \text{ and } \tau^{(0, t+1)}(\omega) = \infty\right\},$$

$$t_{n+1}(\omega) = \min\left\{t \in \mathbb{N}: t > t_n, \sum_{k=1}^{\infty} 1_{(t \le \dot{U}_k^0(\omega) \le t+1)} = 0 \text{ and } \tau^{(0, t+1)}(\omega) = +\infty\right\}.$$

Notice that for any η in E the increments of $r_t(\eta)$, for t in $[t_n, t_n + 1]$, $n = 1, 2\dots,$ are independent of $\hat{\xi}_{t_n}^{\eta}$. Moreover $r_t(\eta) = r(\hat{\xi}_t^{(0, t_n+1)})$ for all $t \ge t_n = 1$.

The partition π of $\tilde{\Omega}$ is defined so that (η, ω) and (η', ω') are in the same atom a of π if and only if the following three conditions are fulfilled:

1. $\eta = \eta'$.
2. $t_n(\omega) = t_n(\omega')$ for all $n \ge 1$. [We shall therefore write t_n for $t_n(\omega) = t_n(\omega')$ in the following text.]
3. $r_t(\eta, \omega) - r_{t_n+1}(\eta, \omega) = r_t(\eta', \omega') - r_{t_n+1}(\eta', \omega')$ for all t in $[t_n + 1, t_{n+1}]$, for all $n \ge 1$.

Therefore, in each atom a we have

$$r_t = K_t(a) + \sum_{n=1}^{N_t} d_n + D_t,$$

where

$$d_n = r_{t_n+1} - r_{t_n},$$

$$N_t = \sum_{k=1}^{\infty} 1_{\{t_k \le t\}},$$

$$D_t = \sum_{k=1}^{\infty} 1_{\{t-1 \le t_k \le t\}}(r_t - r_{t_k})$$

and $K_t(a)$ is a term which only depends on a (as well as N_t).

Note that after conditioning on π the event $\{\tau^{(0, t_k+1)}(\omega) = \infty\}$ for $k < N_t$ becomes, modulo zero, the same as $\{\tau^{(0, t_k+1)}(\omega) > t_{k+1} - (t_k + 1)\}$. As a consequence, conditioned on π, the increments d_n, $n \geq 1$, are i.i.d.; they have the same law as $r(\xi_1^0)$, given that $\overset{+}{U}{}_1^0 > 1$. They are also independent of K_t, N_t and D_t. From the preceding remarks we get that

$$E_{\tilde{\mu}}\left((r_t - s_t)^2|\pi\right) = E_{\tilde{\mu}}\left\{ \sum_{n=1}^{N_t} \left[d_n - E_{\tilde{\mu}}(d_n|\pi)\right]^2|\pi \right\} + E_{\tilde{\mu}}\left\{\left[R_t - E_{\tilde{\mu}}(R_t|\pi)\right]^2|\pi\right\}, \tag{4.5}$$

where

$$s_t = E_{\tilde{\mu}}(r_t|\pi).$$

Since

$$E_{\tilde{\mu}}\left\{\left[r_t - E_{\tilde{\mu}}(r_t)\right]^2\right\} = E_{\tilde{\mu}}\left\{E_{\tilde{\mu}}\left[\left(r_t - E_{\tilde{\mu}}(r_t)\right)^2|\pi\right]\right\} \geq E_{\tilde{\mu}}\left\{E_{\tilde{\mu}}\left[\left(r_t - s_t\right)^2|\pi\right]\right\}$$

the proof of (4.4) will be concluded once we show that

$$\lim_{t\to\infty} t^{-1}E_{\tilde{\mu}}\left\{E_{\tilde{\mu}}\left[\sum_{n=1}^{N_t}\left(d_n - E_{\tilde{\mu}}(d_n|\pi)\right)^2|\pi\right]\right\} > 0. \tag{4.6}$$

It is, therefore, enough to show that there exists $\beta > 0$, sufficiently small, such that

$$P(N_t \geq [\beta t]) > \tfrac{1}{2}. \tag{4.7}$$

Obviously

$$P(N_t \geq [t\beta]) = P\left(t_{[\beta t]} < t\right).$$

Therefore, using the Chebyshev inequality, we get

$$P\left(t_{[\beta t]} > t\right) \leq t^{-1}E\left(t_{[\beta t]}\right).$$

Since

$$t^{-1}E\left(t_{[\beta t]}\right) = [\beta t]E(t_1)t^{-1} = [\beta t]e(t\rho)^{-1},$$

this concludes the proof of (4.4). □

PROOF OF THEOREM 2. We use again the realization of the contact process described in Section 3. We set

$$r_t(\eta, \omega) = r\left(\xi_t^\eta(\omega)\right)$$

for η in E and ω in Ω. We have shown during the proof of Proposition 5 that given any two configurations η and η' in E, there is a time $T(\eta, \eta')$ which is a.s. finite w.r.t. P and such that the increments of $r_t(\eta, \omega)$ and of $r_t(\eta', \omega)$ are the same for $t \geq T(\eta, \eta')$. On the other hand, the value of $r_t(\eta, \omega) - r_t(\eta', \omega)$ at $t = T(\eta, \eta')$ is also a.s. finite. Hence, the limiting Brownian motions starting from η and η' have the same law which is the same as that obtained in Theorem 6. The proof of Theorem 2 is therefore completed. □

PROOF OF THEOREM 3. Our argument adapts Wick's proof [25] to our context. Let $\mathbb{1}$ be the configuration where all sites are occupied. From the very construction of the contact process described in Section 2, we know that for almost all ω in Ω and for all $t \geq 0$,

$$\xi_t^{\bar\eta}(\omega, x) = \xi_t^{\mathbf{1}}(\omega, x) \tag{4.8}$$

holds for all $x \leq \bar{r}_t(\omega)$, where $\bar{r}_t = r(\xi_t^{\bar\eta})$ (cf. [21, Chapter 6, Theorem 2.2]). We also know that if F is a finite subset of Z and f the function defined by

$$f = \prod_{x \in F} \eta(x), \tag{4.9}$$

then, for any $\delta > 0$,

$$\lim_{T \to \infty} \sup_{t \geq 0} P\left\{ \left| T^{-1} \int_t^{t+T} ds\, f(\xi_s^{\mathbf{1}}) - \int d\mu\, f \right| > \delta \right\} = 0. \tag{4.10}$$

(For details see [21] or [15]).

Theorem 3 is a consequence of (4.8), (4.10) and Theorem 2, as we are going to show.

Given $t > 0$, z an integer, $T_\varepsilon > 0$ and a finite subset F of Z, let

$$C_1^\varepsilon = \left\{ \bar{r}_s - [\varepsilon^{-2}\alpha t + \varepsilon^{-1}z] < a, \quad \text{for all } s \text{ in } [\varepsilon^{-2}t, \varepsilon^{-2}t + T_\varepsilon] \right\},$$

$$C_2^\varepsilon = \left\{ \bar{r}_s - [\varepsilon^{-2}\alpha t + \varepsilon^{-1}z] > b, \quad \text{for all } s \text{ in } [\varepsilon^{-2}t, \varepsilon^{-2}t + T_\varepsilon] \right\},$$

where $a = \min F$ and $b = \max F$. From (4.8) [cf. (4.9) for notation] we get

$$\left| E[A_\varepsilon(t, z)(f) - f(\mathbf{0})P(C_1^\varepsilon) - E\left[T^{-1} \int_{\varepsilon^{-2}t}^{\varepsilon^{-2}t + T_\varepsilon} ds\, f(\xi_s^{\mathbf{1}} - y); C_2^\varepsilon \right] \right| \leq 1 - P(C_1^\varepsilon \cap C_2^\varepsilon), \tag{4.11}$$

where $y = [\alpha\varepsilon^{-2}t + \varepsilon^{-1}z]$ and $\mathbf{0}$ is the zero configuration.

Now by (4.10) and the invariance by spatial shifts of the law of $\xi_s^{\mathbf{1}}$, by letting $T_\varepsilon \to \infty$ when $\varepsilon \to 0$, we get

$$\lim_{\varepsilon \to 0} \left| E\left\{ T_\varepsilon^{-1} \int_{\varepsilon^{-2}t}^{\varepsilon^{-2}t + T_\varepsilon} ds\, f(\xi_s^{\mathbf{1}} - [d\xi^{-2}t + \varepsilon^{-1}z]; C_2^\varepsilon \right\} - \int d\mu\, f P(C_2^\varepsilon) \right| = 0.$$

Finally by Theorem 2, if T_ε is such that $\varepsilon T_\varepsilon \to 0$, when ε goes to zero, then

$$\lim_{\varepsilon \to 0} P(C_1^\varepsilon) = P(B_t < z), \qquad \lim_{\varepsilon \to 0} P(C_2^\varepsilon) = P(B_t > z)$$

and the r.h.s. of (4.10) vanishes when ε goes to zero.

This concludes the proof of Theorem 3. □

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