The Annals of Probability
1987, Vol. 15, No. 3, 1072–1096

DOUBLE STOCHASTIC INTEGRALS, RANDOM QUADRATIC FORMS AND RANDOM SERIES IN ORLICZ SPACES

BY STANISLAW KWAPIEN[1] AND WOJBOR A. WOYCZYNSKI[2]

Case Western Reserve University

Let $X(t)$, $t \geq 0$, be a process with independent, symmetric and stationary increments and let (ξ_i) be i.i.d. symmetric real random variables. We provide a characterization of functions $f(s, t)$, $s, t \geq 0$, such that the double integral $\int\int f(s, t)\, dX(s)\, dX(t)$ exists, a characterization of infinite matrices (α_{ij}) such that the double series $\sum \alpha_{ij}\xi_i\xi_j$ converges a.s. and a characterization of Orlicz space l_ψ valued sequences (a_i) for which the series $\sum a_i \xi_i$ converges a.s. in l_ψ. The above three problems are closely related.

1. Introduction. The aim of this paper is to characterize measurable real functions of two variables $f = f(t, s)$, $t, s \geq 0$, for which the double integral

$$J_X(f) = \int\int f(s, t)\, dX(t)\, dX(s) \tag{1.1}$$

exists in the sense defined in Section 5. Here $X(t)$, $t \geq 0$, is a stochastic process with symmetric, independent and stationary increments, and the characterization is obtained in terms of the Lévy measure of X. This goal is accomplished by studying first series of independent random variables with coefficients in a sequence Orlicz space and then by providing a full description of infinite real matrices $\alpha = (\alpha_{ij})_{i,\, j=1,2,\dots}$ for which the double random series

$$Q_X(\alpha) = \sum_{i,\, j=1}^{\infty} \alpha_{ij}\xi_i\xi_j \tag{1.2}$$

converges a.s.; throughout this paper the convergence of (1.2) is understood as existence of the limit $\lim_n \sum_{1 \leq i,\, j \leq n} \alpha_{ij}\xi_i\xi_j$. Here, the random variables $\xi_1, \xi_2, \dots$ are assumed to be symmetric, independent and identically distributed. The basic tools we use are refinements of Hoffmann-Jørgensen inequalities which we discuss in Section 2 together with other preliminaries on real random series and a decoupling inequality for special martingale transforms. Truncation techniques play a crucial role throughout the paper and the (somewhat unorthodox) notation we use is as follows:

$$[\xi]_\beta^\alpha = \begin{cases} \alpha, & \text{if } \xi \geq \alpha, \\ \xi, & \text{if } \alpha > |\xi| \geq \beta, \\ 0, & \text{if } \beta > |\xi|, \\ -\alpha, & \text{if } -\alpha \geq \xi, \end{cases}$$

where $0 \leq \beta \leq \alpha$. Also, routinely, we will write $[\xi]^\alpha$ instead of $[\xi]_0^\alpha$.

Received July 1985; revised April 1986.

[1]On leave from Warsaw University, Poland.

[2]Supported in part by an ONR contract.

AMS 1980 *subject classifications.* Primary 60H05; secondary 60E07, 60B11, 60B12.

Key words and phrases. Double stochastic integral, random quadratic form, process with independent increments, Orlicz space.

Our results extend earlier results on p-stable, $p > 1$, double stochastic integrals and series obtained by Cambanis, Rosinski and Woyczynski (1985), Rosinski and Woyczynski (1986) [see also McConnell and Taqqu (1984)]. They give a deeper understanding of the structure of double stochastic integrals and provide a more direct approach in comparison with the previous work in this area. The results also offer a better promise in an effort to create a general theory of multiple stochastic integrals.

2. Preliminaries: Hoffmann-Jørgensen type inequalities and real random series. This section contains a summary of inequalities originally studied by Hoffmann-Jørgensen (1974) in the case of random variables taking values in a Banach space. The slightly more general setup here is necessitated by the needs of applications to the study of series of independent real- and vector-valued random variables which follow in Sections 3, 4 and 5.

Let F be a linear metric space and let $\Psi\colon F \to \mathbb{R}^+$ be a continuous function which satisfies the following conditions: There exists a constant $C > 0$ such that for all $x, y \in F$

$$\Psi(x + y) \le C(\Psi(x) + \Psi(y)), \tag{2.1}$$

and, for all $x \in F$, and $\alpha \in \mathbb{R}$, $|\alpha| \le 1$,

$$\Psi(\alpha x) \le \Psi(x). \tag{2.2}$$

PROPOSITION 2.1. *Let $\xi_1, \xi_2, \dots$ be a sequence of independent, symmetric random variables with values in F, such that the partial sums $S_n = \xi_1 + \cdots + \xi_n$, $n = 1, 2, \dots,$ converge a.s. to S as $n \to \infty$. Then*

$$P\Big(\sup_n \Psi(S_n) > t\Big) \le 2P(\Psi(S/2) > t/2C), \qquad t > 0, \tag{2.3}$$

$$P\Big(\sup_n \Psi(\xi_n) > t\Big) \le 2P(\Psi(S/2) > t/2C), \qquad t > 0, \tag{2.4}$$

and, whenever the right-hand side below is positive,

$$E\Psi(S) \le \frac{CE\,\sup_n \Psi(\xi_n) + 4a}{1/3 - 4C^2 P\big(\Psi(S) > a/2C^2\big)}, \qquad a > 0, \tag{2.5}$$

where C is the constant appearing in (2.1).

The proof of the above proposition is almost identical to the proof in the case when F is a Banach space and Ψ is a norm. An even more general situation was considered by Kwapien (1973).

The inequality (2.5) immediately implies the following

COROLLARY 2.1. *If $\xi_1, \xi_2, \dots$ are as in Proposition 2.1 and if $\Psi(\xi_i)$, $i = 1, 2, \dots,$ are uniformly bounded by a constant, then $E\Psi(S) < \infty$.*

Also, integrating both sides of inequality (2.4) one obtains

COROLLARY 2.2. *If* $\xi_1, \xi_2, \dots$ *are as in Proposition* 2.1, *then*

$$E \sup_n \Psi(\xi_n) \le 4CE\Psi(S). \tag{2.6}$$

We will also need the following

PROPOSITION 2.2. *If* $\xi_1, \xi_2, \dots$ *are as in Proposition* 2.1, *then for all* $t \ge 0$,

$$P\Big(\sup_n \Psi(\xi_n) > t\Big) \ge [1 - 2P(\Psi(S/2) > t/2C)] \sum_{n=1}^{\infty} P(\Psi(\xi_n) > t). \tag{2.7}$$

PROOF. Observe that

$$\begin{aligned} P\Big(\sup_n \Psi(\xi_n) > t\Big) &= 1 - \prod_{n=1}^{\infty} (1 - P(\Psi(\xi_n) > t)) \\ &\ge 1 - \exp\left(-\sum_{n=1}^{\infty} P(\Psi(\xi_n) > t)\right) \\ &\ge \exp\left(-\sum_{n=1}^{\infty} P(\Psi(\xi_n) > t)\right) \sum_{n=1}^{\infty} P(\Psi(\xi_n) > t), \end{aligned} \tag{2.8}$$

since $1 - e^{-x} \ge e^{-x}x$ for $x \in \mathbb{R}$. Now the first two lines of (2.8) and (2.4) yield

$$2P(\Psi(S/2) > t/2C) \ge P\Big(\sup_n \Psi(\xi_n) > t\Big) \ge 1 - \exp\left(-\sum_{n=1}^{\infty} P(\Psi(\xi_n) > t)\right),$$

which combined with (2.8) gives (2.7). □

In the remainder of this section, and in the following sections, we assume that $\xi, \xi_1, \xi_2, \dots$ are independent, symmetrically and identically distributed real random variables, and that $\alpha_1, \alpha_2, \dots \in \mathbb{R}$. Define

$$\phi(u) = E\left[(u\xi)^2 \wedge 1\right], \qquad u \in \mathbb{R}, \tag{2.9}$$

and

$$\Phi((\alpha_i)) = \sum_{i=1}^{\infty} \phi(\alpha_i). \tag{2.10}$$

The following proposition gives a familiar description of multiplier sequences (α_i) for which the random series $\Sigma\alpha_i\xi_i$ converges almost surely.

PROPOSITION 2.3. *The series* $\Sigma\alpha_i\xi_i$ *converges almost surely if and only if* $\Phi((\alpha_i)) < \infty$.

The proposition follows immediately from the next theorem which can be thought of as a "uniform" version of Proposition 2.3. It is this "uniform" result that we will need later on.

Theorem 2.1. (a) *If* $0 < b < 1/200$ *and*

$$P\left(\left|\sum_{i=1}^{\infty} \alpha_i \xi_i\right| > b\right) < b,$$

then

$$\Phi((\alpha_i)) < 200b.$$

(b) *For any* $b > 0$,

$$P\left(\left|\sum_{i=1}^{\infty} \alpha_i \xi_i\right| \geq b\right) \leq 2\Phi((\alpha_i)/b).$$

Proof. (a) Applying (2.5) in the case $F = \mathbb{R}$, $\Psi(x) = x^2$ [so that the constant C in (2.1) can be taken equal to 2] and $S = \Sigma[\alpha_i \xi_i]^1$, one obtains for any $a > 0$,

$$\begin{aligned}\Phi((\alpha_i)) &= E\left(\sum_{i=1}^{\infty}[\alpha_i\xi_i]^1\right)^2 \\ &\leq \frac{2E\sup_i\left([\alpha_i\xi_i]^1\right)^2 + 4a}{1/3 - 16P\left(\left(\Sigma_i[\alpha_i\xi_i]^1\right)^2 > a/8\right)}.\end{aligned}$$

Since, by (2.4) applied to $\Psi(x) = |x|$ ($C = 1$), for any $b > 0$,

$$\begin{aligned}E\sup_i\left([\alpha_i\xi_i]^1\right)^2 &\leq b^2 + P\left(\left|\sup_i[\alpha_i\xi_i]^1\right| > b\right) \\ &\leq b^2 + P\left(\sup_i|\alpha_i\xi_i| > b\right) \\ &\leq b^2 + 2P\left(\left|\sum_{i=1}^{\infty}\alpha_i\xi_i\right| > b\right),\end{aligned}$$

and since, by the contraction principle [cf., e.g., de Acosta (1980) or Sztencel (1981)]

$$P\left(\left|\sum_{i=1}^{\infty}[\alpha_i\xi_i]^1\right| > (a/8)^{1/2}\right) \leq 2P\left(\left|\sum_{i=1}^{\infty}\alpha_i\xi_i\right| > (a/8)^{1/2}\right),$$

one gets, selecting $a = 8b^2$,

$$\Phi((\alpha_i)) \leq \frac{34b^2 + 4P(|\Sigma_i\alpha_i\xi_i| > b)}{1/3 - 16P(|\Sigma_i\alpha_i\xi_i| > b)}, \tag{2.11}$$

which immediately yields (a).

(b) The proof here is straightforward. Indeed,

$$P\left(\left|\sum_{i=1}^{\infty} \alpha_i \xi_i\right| \geq 1\right) \leq P\left(\left|\sum_{i=1}^{\infty} [\alpha_i \xi_i]^1\right| > 1\right) + \sum_{i=1}^{\infty} P(|\alpha_i \xi_i| \geq 1)$$

$$\leq E\left(\sum_{i=1}^{\infty} [\alpha_i \xi_i]^1\right)^2 + \sum_{i=1}^{\infty} \phi(\alpha_i)$$

$$= 2\Phi((\alpha_i)). \qquad \square$$

3. Convergence of random series in sequence Orlicz spaces. Let ψ: $\mathbb{R} \to \mathbb{R}^+$ be a continuous function such that $\psi(0) = 0$, $\psi(-x) = \psi(x)$ for $x \in \mathbb{R}$, and such that ψ is increasing for $x \in \mathbb{R}^+$ and

$$\psi(2x) \leq c\psi(x), \tag{3.1}$$

for some $c > 0$ and all $x \in \mathbb{R}$. The last inequality implies that

$$\psi(x + y) \leq c(\psi(x) + \psi(y)),$$

for all $x, y \in \mathbb{R}$. For a sequence $a = (\alpha_i)_{i=1,2,\dots}$ of real numbers let

$$\Psi(a) = \sum_{i=1}^{\infty} \psi(\alpha_i),$$

and define

$$\|a\|_\psi = \inf\{s: \Psi(a/s) \leq \Delta\},$$

where $\Delta < 1/48c^2$ is a number fixed from now on. The sequence Orlicz space $l_\psi \overset{\text{df}}{=} \{a: \psi(a) < \infty\}$ is a linear space. The functional ψ fulfills conditions (2.1) and (2.2). If one introduces a topology in l_ψ by saying that $\lim_n a_n = 0$ if and only if $\lim_n \Psi(a_n) = 0$, then l_ψ equipped with this topology becomes a complete linear metric space. In general, if $\Psi(a_n) \to 0$, then also $\|a_n\|_\Psi \to 0$ but the reverse implication is not always true.

Let $\xi, \xi_1, \xi_2, \dots$ be a sequence of independent, symmetric and identically distributed random variables considered in Section 2. The function $\phi(u) = E[(u\xi)^2 \wedge 1]$ defined in (2.9) fulfills the conditions from the beginning of this section. The constant c in (3.1) for ϕ can be taken to be 2. The set of all sequences (α_i) for which $\sum \alpha_i \xi_i$ converges a.s. is the Orlicz space l_ϕ (cf. Proposition 2.3).

Let $a_i = (\alpha_{ij})_{j=1,2,\dots}$, $i = 1, 2, \dots$, be a sequence of elements of a general sequence Orlicz space l_ψ and let $\alpha_i = \|a_i\|_\psi$, $i = 1, 2, \dots$, and $\beta_j = \|(\alpha_{ij})_{i=1,2,\dots}\|_\phi$, $j = 1, 2, \dots$. We define

$$H(x, u, v) = H_1(x, u, v) + N(1/u)\psi(v), \tag{3.2}$$

where

$$N(t) = P(|\xi| > t), \qquad t > 0,$$

and

$$H_1(x, u, v) = E\psi\left(x[\xi]_{v/|x|}^{1/u}\right).$$

Notice that

$$N(1/u) \le \phi(u). \tag{3.3}$$

THEOREM 3.1. *The series* $\Sigma a_i \xi_i$ *converges almost surely in* l_ψ *if and only if*

$$\sum_{i,j=1}^{\infty} H(\alpha_{ij}, \alpha_i, \beta_j) < \infty.$$

The proof of the above theorem depends on the following lemma which also will be used later on.

LEMMA 3.1. *The following two conditions are equivalent*:

(i) *the series* $\Sigma a_i \xi_i$ *converges a.s. in* l_ψ;
(ii) *the series* $\Sigma a_i [\xi_i]^{1/\alpha_i}$ *converges a.s. in* l_ψ *and*

$$\sum_{i=1}^{\infty} N(1/\alpha_i) < \infty.$$

[*Notice that in view of Corollary* 2.1 *the first condition in* (ii) *is equivalent to the condition* $E\Psi(\Sigma_i a_i [\xi_i]^{1/\alpha_i}) < \infty$.]

Moreover, for each $a > 0$ *there exists a* $b > 0$ (*for each* $b > 0$ *there exists an* $a > 0$) *such that*

$$P\left(\Psi\left(\sum_{i=1}^{\infty} a_i \xi_i\right) > b\right) \le b$$
$$\Rightarrow (\Leftarrow) \sum_{i=1}^{\infty} N(1/\alpha_i) < a \quad \text{and} \quad E\Psi\left(\sum_{i=1}^{\infty} a_i [\xi_i]^{1/\alpha_i}\right) < a. \tag{3.4}$$

PROOF. To prove Lemma 3.1 it is clearly sufficient to show the validity of implications (3.4).

($\Rightarrow$) For $a > 0$ let us put $b = (a/30) \wedge (\Delta/4)$. Then, by (2.7) and (2.4) and the assumption

$$\sum_{i=1}^{\infty} P(\Psi(a_i\xi_i) > 4b) \le \frac{2P(\Psi(\Sigma a_i \xi_i/2) > 2b/c)}{1 - 2P(\Psi(\Sigma a_i \xi_i/2) > 2b/c)}$$
$$\le \frac{2P(\Psi(\Sigma a_i \xi_i) > b)}{1 - 2P(\Psi(\Sigma a_i \xi_i) > b)}$$
$$\le \frac{2b}{1 - 2b}.$$

Since $4b < \Delta < 1/48c^2$ we have

$$\sum_{i=1}^{\infty} N(1/\alpha_i) = \sum_{i=1}^{\infty} P(\Psi(a_i \xi_i) > \Delta) \le \frac{2b}{1 - 2b} < 3b < a. \tag{3.5}$$

By (2.5)

$$E\Psi\left(\sum_{i=1}^{\infty}a_i[\xi_i]^{1/\alpha_i}\right)\le\frac{4E\sup_i\Psi\left(a_i[\xi_i]^{1/\alpha_i}\right)+32b}{1-2P\left(\Psi\left(\Sigma a_i[\xi_i]^{1/\alpha_i}\right)>b\right)}$$

$$\le\frac{4(4b+4bP(\sup_i\psi(a_i\xi_i)>4b))+8b}{1-2P(\Psi(\Sigma a_i\xi_i)>b)-\Sigma P(|\xi_i|>1/\alpha_i)} \tag{3.6}$$

$$\le\frac{24b+\Delta 3b}{1-2b-3b}\le\frac{25b}{1-5b}<30b\le a.$$

($\Leftarrow$) For $b>0$ let us put $a=b^2/(b+1)$. Then, by the assumption

$$P\left(\Psi\left(\sum_{i=1}^{\infty}a_i\xi_i\right)>b\right)\le P\left(\Psi\left(\sum_{i=1}^{\infty}a_i[\xi_i]^{1/\alpha_i}\right)>b\right)+\sum_{i=1}^{\infty}P\left(\xi_i\ne[\xi_i]^{1/\alpha_i}\right)$$

$$\le E\Psi\left(\sum_{i=1}^{\infty}a_i[\xi_i]^{1/\alpha_i}\right)\Big/b+\sum_{i=1}^{\infty}N(1/\alpha_i) \tag{3.7}$$

$$\le a/b+a\le b.$$

□

Proof of Theorem 3.1. *Sufficiency.* Assume that

$$\sum_{i,j}H(\alpha_{ij},\alpha_i,\beta_j)<\infty.$$

Since $H(\alpha_{ij},\alpha_i,\beta_j)\ge N(1/\alpha_i)\psi(\beta_j)$, we also have

$$\sum_{i,j=1}^{\infty}H(\alpha_{i,j},\alpha_i,\beta_j)\ge\sum_{i=1}^{\infty}N(1/\alpha_i)\sum_{j=1}^{\infty}\psi(\beta_j), \tag{3.8}$$

so that

$$\sum\psi(\beta_j)<\infty\quad\text{and}\quad\sum N(1/\alpha_i)<\infty.$$

Also

$$E\Psi\left(\sum_i a_i[\xi_i]^{1/\alpha_i}\right)=\sum_{j=1}^{\infty}E\psi\left(\sum_{i=1}^{\infty}\alpha_{ij}[\xi_i]^{1/\alpha_i}\right).$$

Now, the inequality (2.5) applied in the case $F=R$ gives for each $j=1,2,\dots,$

$$E\psi\left(\sum_{i=1}^{\infty}\alpha_{ij}[\xi_i]^{1/\alpha_i}\right)\le\frac{cE\sup_i\psi\left(\alpha_{ij}[\xi_i]^{1/\alpha_i}\right)+8c^2\psi(\beta_j)}{1/3-4c^2P\left(\psi\left(\Sigma_i\alpha_{ij}[\xi_i]^{1/\alpha_i}\right)>\psi(\beta_j)\right)}. \tag{3.9}$$

On the other hand, the contraction principle and Theorem 2.1(b) give

$$P\left(\psi\left(\sum_{i=1}^{\infty}\alpha_{ij}[\xi_i]^{1/\alpha_i}\right)>\psi(\beta_j)\right)$$

$$=P\left(\left|\sum_{i=1}^{\infty}\alpha_{ij}[\xi_i]^{1/\alpha_i}\right|>\beta_j\right) \tag{3.10}$$

$$\le 2P\left(\left|\sum_{i=1}^{\infty}\alpha_{ij}\xi_i\right|>\beta_j\right)\le 4\Phi\left(\frac{(\alpha_{ij})_{i=1,2,\dots}}{\beta_j}\right)=4\Delta,$$

which, together with (3.9), yields

$$E\psi\left(\sum_{i=1}^{\infty}\alpha_{ij}[\xi_i]^{1/\alpha_i}\right) \le \frac{cE\sup_i\psi\left(\alpha_{ij}[\xi_i]^{1/\alpha_i}\right) + 8c^2\psi(\beta_j)}{3^{-1} - 16c^2\Delta}. \tag{3.11}$$

Since for each $j = 1, 2, \ldots,$

$$\sup_i \psi\left(\alpha_{ij}[\xi_i]^{1/\alpha_i}\right) \le \psi(\beta_j) + \sum_{i=1}^{\infty}\psi\left(\alpha_{ij}[\xi]^{1/\alpha_i}_{\beta_j/|\alpha_{ij}|}\right),$$

(3.11) summed over $j = 1, 2, \ldots$ gives finally

$$\begin{aligned} &E\Psi\left(\sum_{i=1}^{\infty}a_i[\xi_i]^{1/\alpha_i}\right)\\ &\quad\le \frac{3}{1-48c^2\Delta}\left[c\sum_{i,j=1}^{\infty}H_1(\alpha_{ij},\alpha_i,\beta_j) + (c+8c^2)\sum_{j=1}^{\infty}\psi(\beta_j)\right], \end{aligned} \tag{3.12}$$

which gives the desired convergence because $\sum_{i=1}^{\infty}N(1/\alpha_i) < \infty$ (see the statement of Lemma 3.1).

Necessity. Assume that the series $\sum a_i\xi_i$ converges in l_ψ. Then, by Lemma 3.1, we have for each $t > 0$,

$$\sum_{i=1}^{\infty}N(1/t\alpha_i) < \infty \quad\text{and}\quad E\Psi\left(\sum_{i=1}^{\infty}a_i[\xi_i]^{1/t\alpha_i}\right) < \infty.$$

Select t so that $\sum_{i=1}^{\infty}N(1/t\alpha_i) < b/2 < \Delta/400$. Then, by Theorem 2.1,

$$\begin{aligned} &E\Psi\left(\sum_{i=1}^{\infty}a_i[\xi_i]^{1/t\alpha_i}\right)\\ &\quad= \sum_{j=1}^{\infty}E\psi\left(\sum_{i=1}^{\infty}\alpha_{ij}[\xi_i]^{1/t\alpha_i}\right)\\ &\quad\ge \sum_{j=1}^{\infty}\psi(b\beta_j)P\left(\left|\sum_{i=1}^{\infty}\alpha_{ij}[\xi_i]^{1/t\alpha_i}\right| > b\beta_j\right)\\ &\quad\ge \sum_{j=1}^{\infty}\psi(b\beta_j)\left(P\left(\left|\sum_{i=1}^{\infty}\alpha_{ij}\xi_i/\beta_j\right| > b\right) - \sum_{i=1}^{\infty}P\left(\xi_i \ne [\xi_i]^{1/t\alpha_i}\right)\right)\\ &\quad\ge \sum_{j=1}^{\infty}\psi(b\beta_j)(b - b/2), \end{aligned} \tag{3.13}$$

which proves that

$$\sum_{i,j=1}^{\infty}\psi(\beta_j)N(1/\alpha_i) < \infty. \tag{3.14}$$

On the other hand, by (2.4)

$$
\begin{aligned}
4cE\Psi\Big(\sum_i a_i[\xi_i]^{1/\alpha_i}\Big) &= 4c\sum_{j=1}^{\infty} E\psi\Big(\sum_{i=1}^{\infty}\alpha_{ij}[\xi_i]^{1/\alpha_i}\Big) \\
&\geq \sum_{j=1}^{\infty} E\sup_i \psi\big(\alpha_{ij}[\xi]^{1/\alpha_i}\big).
\end{aligned}
\tag{3.15}
$$

Applying Proposition 2.2 in the case $F = R$ and $\Psi(x) = |x|$, we obtain for each $j = 1, 2, \dots,$ and each $t > \beta_j$

$$
\begin{aligned}
&P\Big(\sup_i \big|\alpha_{ij}[\xi_j]^{1/\alpha_i}\big| > t\Big) \\
&\quad\geq \Big(1 - 2P\Big(\Big|\sum_{i=1}^{\infty}\alpha_{ij}[\xi_i]^{1/\alpha_i}\Big| > \beta_j\Big)\Big)\sum_i P\big(\big|\alpha_{ij}[\xi_i]^{1/\alpha_i}\big| > t\big).
\end{aligned}
\tag{3.16}
$$

Now, as in (3.10) we get

$$
P\Big(\Big|\sum \alpha_{ij}[\xi_i]^{1/\alpha_i}\Big| > \beta_j\Big) \leq 4\Delta,
$$

so that, for $t > \beta_j$,

$$
P\Big(\sup_i \big|\alpha_{ij}[\xi_i]^{1/\alpha_i}\big| > t\Big) \geq (1 - 8\Delta)\sum_{i=1}^{\infty} P\Big(\big|\alpha_{ij}[\xi]_{\beta_j}^{1/\alpha_i}\big| > t\Big)
\tag{3.17}
$$

and, consequently, for all $t > 0$ one has

$$
P\Big(\sup_i \big|\alpha_{ij}[\xi_i]^{1/\alpha_i}\big| > t\Big) \geq (1 - 8\Delta)\sum_{i=1}^{\infty} P\Big(\big|\alpha_{ij}[\xi]_{\beta_j/|\alpha_{ij}|}^{1/\alpha_i}\big| > t\Big).
\tag{3.18}
$$

Multiplying both sides of (3.18) by $\psi'(t)$ and integrating over $\mathbb{R}^+$, we obtain

$$
\begin{aligned}
E\sup_i \psi\big(\alpha_{ij}[\xi]^{1/\alpha_i}\big) &\geq (1 - 8\Delta)\sum_i E\psi\big(\alpha_{ij}[\xi]_{\beta_j/|\alpha_{ij}|}^{1/\alpha_i}\big) \\
&= (1 - 8\Delta)\sum_i H_1(\alpha_{ij}, \alpha_i, \beta_j),
\end{aligned}
\tag{3.19}
$$

so that by the above inequality and (3.15) we conclude that

$$
\frac{4c}{1 - 8\Delta} E\Psi\Big(\sum_{i=1}^{\infty} a_i[\xi_i]^{1/\alpha_i}\Big) \geq \sum_{i,j=1}^{\infty} H_1(\alpha_{ij}, \alpha_i, \beta_j),
\tag{3.20}
$$

which, together with (3.14), proves the theorem. □

REMARK 3.1. Formally speaking, the functions H_1 and ψ depend on our initial (and arbitrary) choice of Δ. However, it follows from the main results that the classes of corresponding single and double sequences are independent of Δ as long as it is sufficiently small.

4. Double random series. Let $\xi, \xi_1, \xi_2, \dots$ be a sequence of independent, symmetric and identically distributed random variables (as in Sections 2 and 3) and let $\eta, \eta_1, \eta_2, \dots$ be another, independent of (ξ_n), sequence of independent, symmetric and identically distributed random variables. Define

$$\psi(u) = E\left[(u\eta)^2 \wedge 1\right], \qquad u \in \mathbb{R}, \tag{4.1}$$

and denote by l_ψ the Orlicz sequence space associated with ψ. In this case, the constant C from (3.1) can be chosen to be equal to 2 so that Δ, from now on, can be chosen, say, to be 1/200.

PROPOSITION 4.1. *For an infinite matrix $(\alpha_{ij})_{i,j=1,2,\dots}$ of real numbers, the following conditions are equivalent:*

(i) $\lim_{n\to\infty}\sum_{1\le i,\,j\le n}\alpha_{ij}\xi_i\eta_j$ *exists almost surely;*
(ii) *for each $j = 1,2,\dots,$ the series $\sum_{i=1}^{\infty}\alpha_{ij}\xi_i$ is a.s. convergent and the sequence $(\sum\alpha_{ij}\xi_i)_{j=1,2,\dots}$ almost surely belongs to l_ψ;*
(iii) *the series $\sum_{i=1}^{\infty}a_i\xi_i$, where $a_i = (\alpha_{ij})_{j=1,2,\dots}$ for $i = 1,2,\dots,$ converges in l_ψ almost surely.*

PROOF. Notice that, by Fubini's theorem, if

$$P\left(\left|\sum_{i,j=1}^{\infty}\alpha_{ij}\xi_i\eta_j\right| > \varepsilon\right) < \varepsilon^2,$$

then

$$P_\xi\left(\left\{\omega\colon P_\eta\left(\left|\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty}\alpha_{ij}\xi_i(\omega)\right)\eta_j\right| > \varepsilon\right) < \varepsilon\right\}\right) > 1-\varepsilon,$$

so that by Theorem 2.1(a), for ε small enough, we have

$$P\left(\left|\sum_{i,j}\alpha_{ij}\xi_i\eta_j\right| > \varepsilon\right) < \varepsilon \Rightarrow P\left(\Psi\left(\sum_{i=1}^{\infty}a_i\xi_i\right) \ge 200\varepsilon\right) \le \varepsilon, \tag{4.2}$$

where

$$\Psi((\alpha_i)) = \sum_{i=1}^{\infty}\psi(\alpha_i).$$

On the other hand, by Theorem 2.1(b) we have

$$\begin{aligned}
P\left(\left|\sum_{i,j}\alpha_{ij}\xi_i\eta_j\right| > \varepsilon\right) &\le P_\xi \times P_\eta\left(\left|\sum_j\left(\sum_i\alpha_{ij}\xi_i\right)\eta_j\right| > \varepsilon\right)\\
&\le \left[E2\Psi\left(\sum_{i=1}^{\infty}a_i\xi_i/\varepsilon\right)\right] \wedge 1\\
&\le \left[\varepsilon^{-2}E2\Psi\left(\sum_{i=1}^{\infty}a_i\xi_i\right)\right] \wedge 1,
\end{aligned}$$

as long as $\varepsilon \le 1$. The last inequality follows from the fact that $\psi(tx) \le t^2\psi(x)$

for $t \geq 1$. Consequently, for each $0 \leq \varepsilon, \delta \leq 1$, we get

$$P\left(\left|\sum_{i,j=1}^{\infty} \alpha_{ij}\xi_i\eta_j\right| > \varepsilon\right) \leq \varepsilon^{-2} P\left(\Psi\left(\sum_{i=1}^{\infty} a_i\xi_i\right) > \delta/2\right) + \delta, \tag{4.3}$$

and the proposition follows from inequalities (4.2) and (4.3). □

It follows directly from Proposition 4.1 and Theorem 3.1 that

$$\lim_{n\to\infty} \sum_{1\leq i,\, j\leq n} \alpha_{ij}\xi_i\eta_j$$

exists a.s. if and only if

$$\sum_{i,j=1}^{\infty} H(\alpha_{ij}, \alpha_i, \beta_j) < \infty,$$

where $\alpha_i = \|(\alpha_{ij})_{j=1,2,\dots}\|_\psi$, $\beta_j = \|(\alpha_{ij})_{i=1,2,\dots}\|_\phi$ and

$$H(x, u, v) = H_1(x, u, v) + N(1/u)\psi(v),$$

where $N(t) = P(|\xi| > t)$ and

$$H_1(x, u, v) = E\psi\left(x[\xi]_{v/|x|}^{1/u}\right) = E\left(\left(x\xi_{v/|x|}^{1/u}\eta\right)^2 \wedge 1\right).$$

The function H, however, is not satisfactory since it does not depend symmetrically on ξ and η although $\sum\alpha_{ij}\xi_i\eta_j$ does. To remedy this situation we will modify the function H as follows. Let

$$F(x, u, v) = F_1(x, u, v) + \phi(u)\psi(v), \tag{4.4}$$

where

$$F_1(x, u, v) = E\left[\left(x[\xi]^{1/u}[\eta]^{1/v}\right)^2 \wedge 1\right],$$

and where ξ, η are independent. In other words,

$$F(x, u, v) = E\left(\left(x[\xi]^{1/u}[\eta]^{1/v}\right)^2 \wedge 1\right) + E\left((u\xi)^2 \wedge 1\right)E\left((u\eta)^2 \wedge 1\right).$$

THEOREM 4.1. *The series $\sum\alpha_{ij}\xi_i\eta_j$ converges a.s.* [*as defined in* (1.2)] *if and only if*

$$\sum_{i,j=1}^{\infty} F(\alpha_{i,j}, \alpha_i, \beta_j) < \infty.$$

PROOF. In view of the preceding remarks, it is enough to show that $\sum_{i,j} F(\alpha_{ij}, \alpha_i, \beta_j) < \infty$ if and only if $\sum_{i,j} H(\alpha_{ij}, \alpha_i, \beta_j) < \infty$. To prove this equivalence let us observe that

$$\begin{aligned} H_1(x, u, v) &= E\left(x[\xi]_{v/|x|}^{1/u}\eta\right)^2 \wedge 1 \\ &\leq E\left(x[\xi]_{v/|x|}^{1/u}[\eta]^{1/v}\right)^2 \wedge 1 + E\left(x[\xi]_{v/|x|}^{1/u}[\eta]_{1/v}\right)^2 \wedge 1 \\ &\leq E\left(x[\xi]^{1/u}[\eta]^{1/v}\right)^2 \wedge 1 + E\left(\chi_{(|\xi| > v/|x|)}\chi_{(\eta > 1/v)}\right) \\ &\leq F_1(x, u, v) + \phi(x/|v|)\psi(v), \end{aligned} \tag{4.5}$$

and that, similarly,

$$
\begin{aligned}
F_1(x, u, v) &= E\big(x[\xi]^{1/u}[\eta]^{1/v}\big)^2 \wedge 1 \\
&\le E\big(x[\xi]^{1/u}_{v/|x|}\eta\big)^2 \wedge 1 + E\Big(\frac{x}{v}[\xi]^{v/|x|}v[\eta]^{1/v}\Big)^2 \wedge 1 \\
&\le H_1(x, u, v) + \phi(x/v)\psi(v).
\end{aligned} \tag{4.6}
$$

Also, for any matrix $(\alpha_{ij})_{i,\, j=1,2,\dots}$, we have

$$
\begin{aligned}
\sum_{i,\, j=1}^{\infty} \phi(\alpha_{ij}/\beta_j)\psi(\beta_j) &= \sum_{j=1}^{\infty}\Bigg(\sum_{i=1}^{\infty} \phi(\alpha_{ij}/\beta_j)\Bigg)\psi(\beta_j) \\
&= \Delta \sum_{j=1}^{\infty} \psi(\beta_j).
\end{aligned} \tag{4.7}
$$

Now, assume $\sum_{ij} H(\alpha_{ij}, \alpha_i, \beta_j) < \infty$. Since $H(x, u, v) \ge N(1/u)\psi(v)$, the convergence of the series $\sum_{i,\, j} H(\alpha_{ij}, \alpha_i, \beta_j)$ implies the convergence of the series $\sum_j \psi(\beta_j)$. On the other hand, the condition $\sum_{i,\, j} H(\alpha_{ij}, \alpha_i, \beta_j) < \infty$, being equivalent to the a.s. convergence of the series $\sum_{ij} \alpha_{ij}\xi_i\eta_j$, has to depend symmetrically on ξ and η so that we also have $\sum_i \phi(\alpha_i) < \infty$, which together with (4.6) and (4.7) gives the convergence of $\sum_{i,\, j} F(\alpha_{ij}, \alpha_i, \beta_j)$.

The converse implication follows directly from (4.5), (4.7) and definitions of H and F. □

In what follows we will assume that $(\eta_i) = (\xi_i')$ is an independent copy of (ξ_i). The function F, in this particular case, is

$$
F(x, u, v) = E\Big[\big(x[\xi]^{1/u}[\xi']^{1/v}\big)^2 \wedge 1\Big] + \phi(u)\phi(v). \tag{4.8}
$$

THEOREM 4.2. *For a symmetric matrix $(\alpha_{ij})_{i,\, j=1,2,\dots}$ such that $\alpha_{ii} = 0$, $i = 1, 2, \dots$, the following conditions are equivalent*:

(i) *the series $\sum_{i,\, j} \alpha_{ij}\xi_i\xi_j$ converges a.s.*;
(ii) *the series $\sum_{i,\, j} \alpha_{ij}\xi_i\xi_j'$ converges a.s.*;
(iii) $\sum_{i,\, j} F(\alpha_{ij}, \alpha_i, \alpha_j) < \infty$.

Moreover, for each $\varepsilon > 0$ there exists δ depending only on ε (for each $\delta > 0$ there exists $\varepsilon > 0$ depending only on δ) such that the condition

$$
P\Bigg(\Bigg|\sum_{i,\, j=1}^{\infty} \alpha_{ij}\xi_i\xi_j\Bigg| > \delta\Bigg) < \delta
$$

implies (is implied by) the condition

$$
\sum_{i,\, j=1}^{\infty} F(\alpha_{ij}, \alpha_i, \beta_j) < \varepsilon.
$$

PROOF. The series in (i) [resp. in (ii)] converges a.s. if and only if $\sum_{j=1}^{\infty}(\sum_{i<j} \alpha_{ij}\xi_i)\xi_j$ is a.s. convergent [resp. $\sum_{j=1}^{\infty}(\sum_{i<j} \alpha_{ij}\xi_i)\xi_i'$ is a.s. convergent].

But, by the decoupling principle observed first by Krakowiak and Szulga (1986) [cf. also Kwapien and Woyczynski (1986)], the two series above converge a.s. together, i.e., convergence a.s. of one of them implies the a.s. convergence of the other. This proves the equivalence of (i) and (ii). The equivalence of (ii) and (iii) is a special case of Theorem 4.1.

To prove the final assertion of Theorem 4.2 suppose first that

$$P\left(\left|\sum_{i,j}\alpha_{ij}\xi_i\xi_j'\right| > \delta'\right) < \delta'.$$

By (4.2), we have

$$P\left(\Phi\left(\sum_i a_i\xi_i\right) \geq 200\delta'\right) < \delta',$$

and from (3.5) we obtain that if $\delta' < \Delta/800$, then

$$\sum_{i=1}^{\infty} N(1/\alpha_i) < 3\cdot 200\delta',$$

and if $3\cdot 200\delta' < \Delta/400$, then

$$\sum_i N(1/\alpha_i) < \Delta/400,$$

so that putting $b = 2\Delta/400$ we obtain by (3.13),

$$\sum_{i=1}^{\infty}\frac{b}{2}\phi(b\alpha_i) \leq E\Phi\left(\sum_{i=1}^{\infty} a_i[\xi_i]^{1/\alpha_i}\right) < 6\cdot 10^3\delta'. \tag{4.9}$$

Since $\phi(bx) \geq b^2x$ for $b \leq 1$, we get

$$\sum_{i=1}^{\infty}\phi(\alpha_i) \leq 2b^{-3}\cdot 6\cdot 10^3\delta' \leq 10^{13}\Delta^{-3}\delta'. \tag{4.10}$$

By (4.6) and (4.7) and (3.20)

$$\begin{aligned}\sum_{i,j=1}^{\infty} F_1(\alpha_{ij},\alpha_i,\alpha_j) &\leq \sum_{i,j=1}^{\infty} H_1(\alpha_{ij},\alpha_i,\alpha_j) + \Delta\sum_{i=1}^{\infty}\phi(\alpha_i)\\ &\leq \frac{8}{1-8\Delta}E\Phi\left(\sum_{i=1}^{\infty} a_i[\xi_i]^{1/\alpha_i}\right) + \Delta\sum_{i=1}^{\infty}\phi(\alpha_i)\\ &\leq \frac{8}{1-8\Delta}\cdot 6\cdot 10^3\delta' + \Delta\cdot 10^{13}\Delta^{-3}\delta',\end{aligned}$$

the last inequality following from (4.9) and (4.10).

Hence, by (4.10)

$$\sum_{i=1}^{\infty} F(\alpha_{ij},\alpha_i,\alpha_j) \leq C_1\delta',$$

where the constant C_1 depends only on Δ.

On the other hand, if one assumes that $\Sigma_{i,j}F(\alpha_{ij}, \alpha_i, \alpha_j) < \varepsilon$, then $\Sigma_i\phi(\alpha_i) < \sqrt{\varepsilon}$ which implies that $\Sigma_i N(1/\alpha_i) < \sqrt{\varepsilon}$. By (3.12), (4.5) and (4.7)

$$E\Phi\left(\sum_{i=1}^{\infty} a_i[\xi_i]^{1/\alpha_i}\right) \le \frac{3}{1 - 48 \cdot 4\Delta}\left[\sum_{i,j=1}^{\infty} 2H(\alpha_{ij}, \alpha_i, \alpha_j) + 34\sum_{i=1}^{\infty}\phi(\alpha_i)\right]$$
$$\le C_2\sqrt{\varepsilon},$$

where C_2 depends only on Δ. Now, by Chebyshev's inequality and the above inequalities, we obtain for each $b > 0$,

$$P\left(\Phi\left(\sum_{i=1}^{\infty} a_i\xi_i\right) > b\right) \le C_2\sqrt{\varepsilon}/b + \sqrt{\varepsilon}$$

and, from (4.3) it follows that

$$P\left(\left|\sum_{i,j=1}^{\infty} \alpha_{ij}\xi_i\xi_j'\right| > \delta'\right) \le (\delta')^{-2}\left(C_2\sqrt{\varepsilon}/b + \sqrt{\varepsilon}\right) + 2b.$$

Putting $b = \varepsilon^{1/4}$, we get

$$P\left(\left|\sum_{i,j=1}^{\infty} \alpha_{ij}\xi_i\xi_j'\right| > \delta'\right) \le C_3\varepsilon^{1/4}(\delta')^{-2},$$

where C_3 depends only on Δ.

So, now, Theorem 4.2 follows immediately from the following lemmas.

Lemma 4.1. *Let (α_{ij}) be as in Theorem 4.2. For any $\delta, \delta' > 0$,*

$$P\left(\left|\sum_{i,j=1}^{\infty} \alpha_{ij}\xi_i\xi_j\right| > \delta\right) \le 5\left[(\delta'\delta^{-1})^{2/3} + 3\left(2P\left(\left|\sum_{i,j=1}^{\infty} \alpha_{ij}\xi_i\xi_j'\right| > \delta'\right)\right)^{1/2}\right]$$

and

$$P\left(\left|\sum_{i,j=1}^{\infty} \alpha_{ij}\xi_i\xi_j'\right| > \delta'\right) \le 5\left[(\delta/\delta')^{2/3} + 3\left(P\left(\sup_n\left|\sum_{1\le i,j\le n} \alpha_{ij}\xi_i\xi_j\right| > \delta\right)\right)^{1/2}\right].$$

Proof. This is an immediate consequence of Corollary 2.2 of Kwapien and Woyczynski (1986). □

Lemma 4.2. *Let (α_{ij}) be as in Theorem 4.2. For $\delta > 0$*

$$P\left(\sup_n\left|\sum_{1\le i,j\le n} \alpha_{ij}\xi_i\xi_j\right| > \delta\right) \le 4000P\left(\left|\sum_{i,j=1}^{\infty} \alpha_{ij}\xi_i\xi_j\right| > \delta/2\right).$$

Proof. Let us observe that in view of the symmetry of the ξ_i's it is sufficient to prove the above inequality in the case when (ξ_i) is a sequence of i.i.d.

Rademacher random variables. Let

$$\tau = \inf\left\{k: \left|\sum_{i,\, j\le k} \alpha_{ij}\xi_i\xi_j\right| > \delta\right\}$$

and

$$A = \left\{\left|\sum_{i,\, j} \alpha_{ij}\xi_i\xi_j\right| > \delta/2\right\}.$$

Then

$$P(A) \ge \sum_{k=1}^{\infty} P(A \cap (\tau = k)).$$

On the other hand, if, for a fixed k,

$$A^s = \left\{\left|\sum_{i,\, j=1}^{\infty} \alpha_{ij}\xi_i^s\xi_j^s\right| > \delta/2\right\},$$

where $\xi_i^s = \xi_i$ for $i \le k$ and $\xi_i^s = -\xi_i$ for $i > k$, then

$$P(A^s \cap (\tau = k)) = P(A \cap (\tau = k)),$$

and

$$B_k \stackrel{\text{df}}{=} \left\{\left|\sum_{i,\, j\le k} \alpha_{ij}\xi_i\xi_j + \sum_{i,\, j>k} \alpha_{ij}\xi_i\xi_j\right| > \delta/2\right\} \subset A \cup A^s,$$

so that

$$P(B_k \cap (\tau = k)) \le 2P(A \cap (\tau = k)).$$

Since by Paley and Zygmund's (1932) inequality, for any $X \in L_+^2$ and $0 < \lambda \le 1$,

$$P(X > \lambda EX) \ge (1 - \lambda)^2 E^2X/EX^2,$$

we have $P(|a + Y| > a/2) \ge d/20$ as long as $EY = 0$ and $E^2|Y|/EY^2 > d$, and since, by Bonami's (1970) inequality,

$$\left(E^2\left|\sum_{i,\, j>k} \alpha_{ij}\xi_i\xi_j\right|\right) \Bigg/ E\left(\sum_{i,\, j>k} \alpha_{ij}\xi_i\xi_j\right)^2 > d > 10^{-2},$$

we get

$$\begin{aligned} &P(B_k \cap (\tau = k)) \\ &\quad = P_{(\xi_i,\, i\le k)} \otimes P_{(\xi_i,\, i>k)}\left((\tau = k) \cap \left(\sum_{i,\, j\le k} \alpha_{ij}\xi_i\xi_j + \sum_{i,\, j>k} \alpha_{ij}\xi_i\xi_j\right) > \delta/2\right) \\ &\quad \ge (d/20)P\left((\tau = k) \cap \left(\left|\sum_{i,\, j\le k} \alpha_{ij}\xi_i\xi_j\right| > \delta\right)\right) = dP(\tau = k)/20. \end{aligned}$$

Finally, $P(A) \ge (d/40)\Sigma_k P(\tau = k)$, which yields the assertion of Lemma 4.2. □

REMARK 4.1. If $\xi_1, \xi_2, \dots$ and $\eta_1, \eta_2, \dots$ have finite second moments, then in (4.4)

$$F(x, u, v) \sim x^2 \wedge 1 + (u^2 \wedge 1)(v^2 \wedge 1)$$

and the a.s. convergence of $\sum_{i,j}\alpha_{ij}\xi_i\eta_j$ takes place if and only if $\sum_{i,j}\alpha_{ij}^2 < \infty$. This fact can be established, however, by a much simpler approach than the one used above.

REMARK 4.2. If $\xi_1, \xi_2, \dots,$ and $\eta_1, \eta_2, \dots$ are p-stable random variables $0 < p < 2$, then a relatively straightforward computation (we supply some more details in an analogous case considered in Remark 5.1) gives

$$F(x, u, v) \sim 1 \wedge \left(|x|^p \log^+ \frac{|x|}{uv}\right) + (u^p \wedge 1)(v^p \wedge 1),$$

where $\log^+ x = 0 \vee \log x$, and the a.s. convergence of $\sum_{i,j}\alpha_{ij}\xi_i\eta_j$ takes place if and only if

$$\sum_{i,j=1}^{\infty} |\alpha_{ij}|^p \left(1 + \log^+ \frac{|\alpha_{ij}|}{\left(\sum_{k=1}^{\infty}|\alpha_{kj}|^p\right)^{1/p} \left(\sum_{k=1}^{\infty}|\alpha_{ik}|^p\right)^{1/p}}\right) < \infty.$$

For $1 < p < 2$ and $(\xi_i) = (\eta_i)$, the above result appears in Cambanis, Rosinski and Woyczynski (1985) although, in the context of p-stable series in l_p, it was known earlier to Pisier.

REMARK 4.3. It follows from Theorem 4.2 that if $\bar{\phi}(x) = E(x\xi\eta)^2 \wedge 1$, then the condition $\sum\bar{\phi}(\alpha_{ij}) < \infty$ is sufficient for the a.s. convergence of series $\sum_{i,j}\alpha_{ij}\xi_i\eta_j$ (compare Remark 5.2).

REMARK 4.4. The methods developed in this section can be also used to give necessary and sufficient condition for the a.s. convergence of the series $\sum_{i,j}\alpha_{ij}\xi_i\eta_j$ in the case when $\xi_1, \xi_2, \dots$ and $\eta_1, \eta_2, \dots$ are not necessarily identically distributed. However, the formulas in that case become quite complicated.

5. Double stochastic integral. Let $X(t)$, $0 \le t \le 1$, be a stochastic process with independent, symmetric and stationary increments. In the sequel we assume that X has no Gaussian component (the general case is briefly discussed in Remark 5.3).

In such a case

$$E\exp(iu(X(s) - X(t))) = \exp(s - t)\int_0^{\infty}(\cos uv - 1)L(dv), \qquad 1 \ge s \ge t \ge 0,$$

where L is the Lévy measure of X.

DEFINITION 5.1. Let n be a positive integer and let

$$f(s, t) = \sum_{1 \le i, j \le n} \alpha_{ij}\chi_{((i-1)/n, i/n]}(s)\chi_{((j-1)/n, j/n]}(t) \tag{5.1}$$

be a real step function vanishing on the diagonal and symmetric (i.e., $\alpha_{ii} = 0$, and $\alpha_{ij} = \alpha_{ji}$, $i, j = 1, 2, \ldots, n$). By definition, we set

$$\begin{aligned} J_X(f) &= \int_0^1 \int_0^1 f(s,t)\, dX(s)\, dX(t) \\ &= \sum_{1 \le i,\, j \le n} \alpha_{ij} \left(X\left(\frac{i}{n}\right) - X\left(\frac{i-1}{n}\right) \right) \left(X\left(\frac{j}{n}\right) - X\left(\frac{j-1}{n}\right) \right). \end{aligned} \tag{5.2}$$

DEFINITION 5.2. Let $f = f(s,t)$, $1 \ge s$, $t \ge 0$, be a real, symmetric measurable function, vanishing on the diagonal. f is said to be *doubly integrable* with respect to X [in short, $f \in \mathscr{L}(dX \otimes dX)$] if there exists a sequence (f_n) of step functions of the form (5.1) such that $f_n \to f$ in measure $ds \otimes dt$ as $n \to \infty$, and such that the integrals $J_X(f_n)$, $n = 1, 2, \ldots$, converge in probability. In this case, we define the *double stochastic integral*

$$J_X(f) = \int_0^1 \int_0^1 f(s,t)\, dX(s)\, dX(t) = \lim_{n \to \infty} J_X(f_n),$$

which is independent of the choice of a particular approximating sequence (f_n) (see Proposition 5.1).

Define

$$\phi(x) = \int_0^\infty \left((xu)^2 \wedge 1 \right) L(du), \qquad x \in \mathbb{R}, \tag{5.3}$$

and recall that the class $\mathscr{L}(dX)$ of functions on $[0,1]$ that are integrable with respect to X is identical to the Orlicz space

$$L_\phi = \left\{ f \colon \| f \|_\phi \overset{\mathrm{df}}{=} \inf\left\{ a \colon \int_0^1 \phi(f(s)/a) = \Delta \right\} < \infty \right\},$$

where Δ here is the same as in Section 4 [cf. Urbanik and Woyczynski (1967)]. Furthermore, define

$$F_1(w,u,v) = \int_0^\infty \int_0^\infty \left[\left(w[x]^{1/u}[y]^{1/v} \right)^2 \wedge 1 \right] L(dx) L(dy), \tag{5.4}$$

where $w \in \mathbb{R}$, $u, v \ge 0$, and

$$F(w,u,v) = F_1(w,u,v) + \phi(u)\phi(v).$$

The following theorem is our main result and gives a full analytic description of functions which are doubly integrable with respect to X.

THEOREM 5.1. *Let $f = f(s,t)$, $s, t \ge 0$, be a real, symmetric measurable function. Then $f \in \mathscr{L}(dX \otimes dX)$ if and only if*

$$A_X(f) = \int_0^1 \int_0^1 F\left(f(s,t), \| f(s,\cdot) \|_\phi, \| f(\cdot,t) \|_\phi \right) ds\, dt < \infty. \tag{5.5}$$

The proof of the above theorem will be based on the following sequence of auxiliary results.

LEMMA 5.1. *For each* $\varepsilon > 0$ *there exists a* $\delta > 0$ (*for each* $\delta > 0$ *there exists an* $\varepsilon > 0$) *such that for each step function* f *as in* (5.1) *the inequality*

$$P\left(\left|\int_0^1\int_0^1 f(s,t)\,dX(s)\,dX(t)\right| > \delta\right) < \delta$$

implies (*is implied by*) *the inequality* $A_X(f) < \varepsilon$.

PROOF. For each m which is an integer multiple of n

$$J_X(f) = \sum_{i,j=1}^{n} \sum_{\substack{(i-1)/n<k/m\le i/n\\(j-1)/n<l/m\le j/n}} f\left(\frac{i}{n},\frac{j}{n}\right)\left(X\left(\frac{k}{m}\right) - X\left(\frac{k-1}{m}\right)\right)$$

$$\times\left(X\left(\frac{l}{m}\right) - X\left(\frac{l-1}{m}\right)\right),$$

so that by Theorem 4.2 for each $\varepsilon > 0$ there exists a $\delta > 0$ (for each $\delta > 0$ there exists an $\varepsilon > 0$) such that $P(|J_X(f)| > \delta) < \delta$ implies (is implied by) the condition

$$A_X^m(f) \overset{\text{df}}{=} \sum_{i,j=1}^{n}\left(\frac{m}{n}\right)^2\left[F_1^m\left(f\left(\frac{i}{n},\frac{j}{n}\right),\alpha_i^m,\alpha_j^m\right) + \phi^m(\alpha_i^m)\phi^m(\alpha_j^m)\right] < \varepsilon,$$

where

$$\phi^m(u) = E(uX(1/m))^2 \wedge 1,$$

$$F_1^m(w,u,v) = E\left(w[X(1/m)]^{1/u}[X'(1/m)]^{1/v}\right)^2 \wedge 1,$$

X' is an independent copy of X, and where

$$\alpha_i^m = \left\|\left(f\left(\frac{i}{n},\frac{l}{m}\right)\right)_{l=1,\dots,m}\right\|_{\phi^m}$$

$$= \inf\left\{t\colon \sum_{j=1}^{n}\frac{m}{n}\phi^m\left(f\left(\frac{i}{n},\frac{j}{m}\right)\Big/t\right) \le \Delta\right\}.$$

By the standard theory of infinitely divisible distributions [cf., e.g., Feller (1966)] if ψ is a bounded, continuous function such that $\lim_{s\to 0}\psi(s)/s^2$ exists, then

$$\lim_{n\to\infty} En\psi(X(1/n)) = \int_0^\infty \psi(s)L(ds).$$

Hence, we obtain that for each $u > 0$, $m\phi^m(u) \to \phi(u)$ as $n \to \infty$, and it is easy to see that the convergence is uniform on compact sets of u's. Therefore, for each fixed $t > 0$, we obtain

$$\lim_{m\to\infty}\frac{1}{n}\sum_{j=1}^{n} m\phi^m\left(f\left(\frac{i}{n},\frac{j}{n}\right)\Big/t\right) = \int_0^1 \phi\left(f\left(\frac{i}{n},s\right)\Big/t\right)ds,$$

so that

$$\lim_{m\to\infty}\alpha_i^m = \left\| f\left(\frac{i}{n},\cdot\right)\right\|_\phi.$$

On the other hand, if $\psi(s,t)$ is such that $\psi(s,t)/(s^2\wedge 1)(t^2\wedge 1)$ is continuous and bounded, then

$$\lim_{n\to\infty} En^2\psi(X(1/n),X'(1/n)) = \int_0^\infty\int_0^\infty \psi(s,t)L(ds)L(dt),$$

and, moreover, if

$$\lim_{n\to\infty}\frac{\psi_n(s,t)}{(s^2\wedge 1)(t^2\wedge 1)} = \frac{\psi(s,t)}{(s^2\wedge 1)(t^2\wedge 1)}$$

uniformly for $s,t\geq 0$, then

$$\lim_{n\to\infty} En^2\psi(X(1/n),X'(1/n)) = \int_0^\infty\int_0^\infty \psi(s,t)L(ds)L(dt).$$

Hence,

$$\lim_{m\to\infty} m^2F_1^m\left(f\left(\frac{i}{n},\frac{j}{n}\right),\alpha_i^m,\alpha_j^m\right) = F_1\left(f\left(\frac{i}{n},\frac{j}{n}\right),\left\|f\left(\frac{i}{n},\cdot\right)\right\|_\phi,\left\|f\left(\cdot,\frac{j}{n}\right)\right\|_\phi\right)$$

and

$$\lim_{m\to\infty} A_X^m(f) = A_X(f),$$

which concludes the proof of Lemma 5.1. □

LEMMA 5.2. *If* $\lim_{n,m\to\infty}A_X(f_n-f_m)=0$ *and* $f_1,f_2,\ldots$ *are step functions as in* (5.1) *such that* $f_n\to f$, $n\to\infty$, *in Lebesgue measure on* $[0,1]\times[0,1]$, *then* $\liminf_n A_X(f_n)\geq A_X(f)$.

PROOF. Without loss of generality, one can assume that $f_n\to f$ a.e. on $[0,1]\times[0,1]$. By Fatou's lemma $F(w,u,v)$ is lower semicontinuous in (w,u,v). So using Fatou's lemma one more time we would obtain the assertion of the lemma if we could demonstrate that

(5.6) $\|f_n(s,\cdot)\|_\phi\to\|f(s,\cdot)\|_\phi$ and $\|f_n(\cdot,t)\|_\phi\to\|f(\cdot,t)\|_\phi$ a.e., $n\to\infty$

(the latter following from the former by symmetry of f).

Now, for g, h defined on $[0,1]$ we have the inequality

$$\Phi^{1/2}(g+h)\leq\Phi^{1/2}(g)+\Phi^{1/2}(h),$$

where $\Phi(g)=\int_0^1\phi(g(s))\,ds$, which follows immediately from subadditivity of the function $|x|\wedge 1$. Hence, if $\Phi(h_n-h)\to 0$, then $\|h_n\|_\phi\to\|h\|_\phi$ or, equivalently, if $\lim_{n,m}\Phi(h_n-h_m)=0$ and $h_n\to h$ a.e., then $\|h_n\|_\phi\to\|h\|_\phi$. So, to show (5.6) it suffices to show that

$$\lim_{n,m\to\infty}\Phi(f_n(s,\cdot)-f_m(s,\cdot))=0 \quad s\text{-a.e.}$$

Observe next, that for each $a > 0$ we have

$$
\begin{aligned}
(a^2 \wedge 1)\phi(w)[L((a,\infty)) - \phi(v)] &\\
\le \phi(aw)[L((a,\infty)) - \phi(v)] &\\
\le \int_0^\infty \big[(wxa)^2 \wedge 1\big] L(dx) L((a, 1/v)) &\\
\le \int_0^\infty \int_{a<y<1/v} \big[(wx[y]^{1/v})^2 \wedge 1\big] L(dx) L(dy) & \qquad (5.7)\\
\le F_1(w,u,v) + \int_{x>1/u} \int_{a<y<1/v} L(dx)L(dy) &\\
\le F_1(w,u,v) + \phi(u)\phi(1/a), &
\end{aligned}
$$

for each $u \ge 0$. Finally, note that the assumption $A_X(f_n - f_m) \to 0$ implies that for almost all s, $\| f_n(s,\cdot) - f_m(s,\cdot)\|_\phi \to 0$, $n, m \to \infty$, so that putting in (5.7) $v = \| f_n(s,\cdot)\|_\phi$, $w = f_n(s,t) - f_m(s,t)$ and $u = \| f_n(\cdot,t) - f_m(\cdot,t)\|_\phi$, and using Fubini's theorem we get the desired conclusion. □

PROPOSITION 5.1. *If $h_1, h_2,\dots$ and $g_1, g_2, \dots$ are two sequences of step functions as in* (5.1) *such that* $\lim_n h_n = \lim_n g_n = f$ *in Lebesgue measure on* $[0,1]\times[0,1]$ *and such that $J_X(h_n)$ and $J_X(g_n)$ converge in probability, then* $\lim_n J_X(h_n) = \lim_n J_X(g_n)$.

PROOF. By Lemma 5.1 it suffices to show that for $f_n = h_n - g_n$ we have $A_X(f_n) \to 0$ as $n \to \infty$. Indeed, since $J_X(f_n) \to 0$ in probability, by Lemma 5.1, $\lim_{n,m} A_X(f_n - f_m) = 0$. Now applying Lemma 5.2 to $f_n - f_k$ we get for each fixed k,

$$\liminf_{n\to\infty} A_X(f_n - f_k) \ge A_X(f_k),$$

which implies that $\lim_k A_X(f_k) = 0$. □

LEMMA 5.3. *If $|f| \le |g|$ and $A_X(g) < \infty$, then $A_X(f) < \infty$ and, moreover,*

$$A_X(f) \le A_X(g) + 2\Delta A_X^{1/2}(g).$$

Also, if $f_n \downarrow 0$ and $A_X(f_1) < \infty$, then $A_X(f_n - f_m) \to 0$, $n, m \to \infty$.

PROOF. If $|f| \le |g|$, then $\| f(s,\cdot)\|_\phi \le \|g(s,\cdot)\|_\phi$ for all s and

$$\int_0^1 \phi(\| f(s,\cdot)\|_\phi)\, ds \le \int_0^1 \phi(\|g(s,\cdot)\|)\, ds$$

and, similarly for $f(\cdot,t)$. Also, if $w \le w'$, $u \le u'$, $v \le v'$, then

$$
\begin{aligned}
F_1(w,u,v) &\le F_1(w,u',v') + \iint_{x\ge 1/u'} \left[\left(\frac{w}{u}xy\right)^2 \wedge 1\right] L(dx)L(dy)\\
&\quad + \iint_{y \ge 1/v'} \left[\left(\frac{w}{v}x\right)^2 \wedge 1\right] L(dx)L(dy) \qquad (5.8)\\
&\le F_1(w,u',v') + \phi(w/u)\phi(u') + \phi(w/v)\phi(u')\\
&\le F_1(w',u',v') + \phi(w/u)\phi(u') + \phi(w/v)\phi(u').
\end{aligned}
$$

Substituting in (5.8) $w = |f(s,t)|$, $w' = |g(s,t)|$, $u = \|f(s,\cdot)\|_\phi$, etc., and taking into account the fact that

$$\int_0^1\int_0^1 \phi\left(\frac{f(s,t)}{\|f(s,\cdot)\|_\phi}\right) ds\, dt \le \Delta, \qquad \int_0^1\int_0^1 \phi\left(\frac{g(s,t)}{\|g(\cdot,t)\|_\phi}\right) ds\, dt \le \Delta,$$

we get the first two conclusions of Lemma 5.3. The last conclusion follows from the first inequality in (5.8) used for $w = f_n - f_m$, $w' = f_k$, $k \le m \le m$, and from an application of the Lebesgue dominated convergence theorem. □

PROOF OF THEOREM 5.1. *Necessity.* Assume that $J_X(f)$ exists. Then, by Definition 5.2 and Lemma 5.1 there exists a sequence $f_1, f_2, \dots$ of step functions as in (5.1) such that $f_n \to f$, $n \to \infty$, in Lebesgue measure on $[0,1] \times [0,1]$ and such that $A_X(f_n - f_m) < \varepsilon$ for n, m large enough. In particular, for $n, m \ge n_0$, $A_X(f_n - f_m) \le 1$. Applying Lemma 5.2 to the sequence $(f_n - f_{n_0})$ we get $A_X(f - f_{n_0}) \le 1$ and, finally, Lemma 5.3 gives $A_X(f) < \infty$.

Sufficiency. If $A_X(f) < \infty$, then, in view of Lemma 5.3 we can assume that $f \ge 0$. Let $\tilde{f}_n = [f]^n$. Then $\tilde{f}_n \to f$, $n \to \infty$, in Lebesgue measure. Now, applying Lemma 5.3 to the sequence $f - \tilde{f}_n$ we obtain

$$\lim_{n, m \to \infty} A_X(\tilde{f}_n - \tilde{f}_m) = 0. \tag{5.9}$$

For each n there exists a sequence f_{nk}, $k = 1, 2, \dots$, of step functions as in (5.1) such that $|f_{nk}| \le n$, $k = 1, 2, \dots$, and $f_{nk} \to \tilde{f}_n$ in Lebesgue measure. By the Lebesgue dominated convergence theorem we obtain that for each bounded g

$$\lim_{k \to \infty} A_X(f_{nk} - g) = A_X(\tilde{f}_n - g) \tag{5.10}$$

Therefore, taking into account (5.9) and (5.10) we can construct a sequence of step functions f_n such that $f_n \to f$, $n \to \infty$, in measure and for which $A_X(f_{n+1} - f_n) \to 0$ arbitrarily fast as $n \to \infty$. In view of Lemma 5.1 and in view of the fact the convergence in probability is metrizable, we get $A_X(f_n - f_m) \to 0$, $n, m \to \infty$, so that another application of Lemma 5.1 gives the existence of $J_X(f)$. □

REMARK 5.1. If $X(t)$, $0 \le t \le 1$, is a p-stable process, $0 < p < 2$, then

$$L(dx) = |x|^{-p-1}\, dx,$$

so that, by (5.3)

$$\phi(u) = \int_0^\infty \left((xu)^2 \wedge 1\right) x^{-p-1}\, dx = \frac{2}{p(2-p)} |u|^p,$$

and, by (5.4)

$$\begin{aligned} F_1(w, u, v) &= \int_0^\infty \int_0^\infty \left[\left(w[x]^{1/u}[y]^{1/v}\right)^2 \wedge 1\right] x^{-p-1} y^{-p-1}\, dx\, dy \\ &= |u|^p |v|^p G\left(\frac{w}{uv}\right), \end{aligned}$$

where

$$G(r) = \int_0^\infty \int_0^\infty \left(r[x]^1[y]^1 \wedge 1\right) x^{-p-1} y^{-p-1}\, dx\, dy$$
$$= \frac{2}{p(2-p)} |r|^p \log^+ r + \left(\frac{2}{p(2-p)}\right)^2 |r|^p.$$

Therefore,

$$F(w,u,v) = \frac{2}{p(2-p)} |w|^p \log^+ \frac{|w|}{|u|\,|v|} + \left(\frac{2}{p(2-p)}\right)^2 \left(|w|^p + |u|^p|v|^p\right)$$
$$= \left(\frac{2}{p(2-p)}\right)^2 \left[|w|^p\left(1 + \frac{p(2-p)}{2}\log^+ \frac{|w|}{|u|\,|v|}\right) + |u|^p|v|^p\right].$$

Also, by (5.3)

$$\|f\|_\phi = \left(\frac{2}{p(2-p)\Delta}\int_0^1 |f|^p\, dx\right)^{1/p} = \left(\frac{2}{p(2-p)\Delta}\right)^{1/p} \|f\|_p,$$

so that, by (5.5)

$$A_X(f) = \left(\frac{2}{p(2-p)}\right)^2 \int_0^1\int_0^1 |f(s,t)|^p$$
$$\times \left(1 + \frac{p(2-p)}{2}\log^+ \frac{|f(s,t)|(p(2-p)\Delta)^{2/p}}{\|f(s,\cdot)\|_p \|f(\cdot,t)\|_p}\right) ds\, dt$$
$$+ \left(\frac{2}{p(2-p)\Delta}\right)^{2/p} \left(\int_0^1\int_0^1 |f(s,t)|^p\, ds\, dt\right)^2.$$

Therefore, Theorem 5.1 gives $f \in \mathscr{L}(dX \otimes dX)$ if and only if

$$\int_0^1\int_0^1 |f(s,t)|^p \left(1 + \log^+ \frac{|f(s,t)|}{\left(\int_0^1 |f(u,t)|^p\, du\right)^{1/p}\left(\int_0^1 |f(s,u)|^p\, du\right)^{1/p}}\right) ds\, dt < \infty.$$

For $1 < p < 2$, this result was obtained by Rosinski and Woyczynski (1986) using a different approach.

REMARK 5.2. It follows from Theorem 5.1 that the condition

$$\bar{\Phi}(f) = \int_0^1\int_0^1 \bar{\phi}(f(s,t))\, ds\, dt < \infty, \tag{5.11}$$

where

$$\bar{\phi}(w) = \int_{x \le 1}\int_{y \le 1} \left[(wxy)^2 \wedge 1\right] L(dx) L(dy),$$

is sufficient for the existence of $J_X(f)$.

Indeed, first of all, it is not difficult to see that $J_x(f)$ exists if and only if $J_{\tilde{X}}(f)$ exists, where $\tilde{X}$ is the process corresponding to the Lévy measure $\tilde{L}$ which is the restriction to $[0,1]$ of L. In the latter case,

$$\tilde{F} = \tilde{F}_1(w,u,v) + \phi(u)\phi(v),$$

where

$$(5.12) \qquad \tilde{F}_1(w,u,v) = \int_0^1\int_0^1 \left(w[x]^{1/u}[y]^{1/v}\right)^2 L(dx)L(dy) \le \bar{\phi}(w).$$

Moreover, since

$$\left[(ux)^2 \wedge 1\right]\left[\left(\frac{w}{u}y\right)^2 \wedge 1\right] \le (wxy)^2 \wedge 1,$$

we get

$$\phi(u)\phi\left(\frac{w}{u}\right) \le \bar{\phi}(w),$$

and, similarly,

$$\phi(v)\phi\left(\frac{w}{v}\right) \le \bar{\phi}(w).$$

Hence,

$$\Delta\int_0^1 \phi(\| f(s,\cdot)\|_\phi)\,ds = \int_0^1\int_0^1 \phi\left(\frac{f(s,t)}{\| f(s,\cdot)\|_\phi}\right)\phi(\| f(s,\cdot)\|_\phi)\,ds\,dt$$
$$\le \bar{\Phi}(f),$$

and, in the same fashion we obtain

$$\Delta\int_0^1 \tilde{\phi}(\| f(\cdot,t)\|_{\tilde{\phi}})\,dt \le \bar{\Phi}(f).$$

So, finally,

$$\int_0^1\int_0^1 \phi(\| f(s,\cdot)\|_\phi)\tilde{\phi}(\| f(\cdot,t)\|_\phi)\,ds\,dt \le \Delta^{-2}\bar{\Phi}^2(f),$$

which, taking into account (5.12), gives

$$A_{\tilde{X}}(f) \le \bar{\Phi}(f) + \Delta^{-2}\bar{\Phi}^2(f),$$

so that the sufficiency of the condition (5.11) has been established.

Needless to say, the condition (5.11) is much easier to check than the condition $A_X(f) < \infty$ which appears in Theorem 5.1. For example, if

$$L(dx) \sim |x|^{-p-1}|\log x|^{-c}\,dx,$$

then, by elementary calculations, one obtains

$$\bar{\phi}(w) \sim \begin{cases} |w|^p(\log|w|)^{1-2c}, & \text{if } c < 1,\\ |w|^p(\log|w|)^{-1}\log\log|w|, & \text{if } c = 1,\\ |w|^p|\log w|^{-c}, & \text{if } c > 1.\end{cases}$$

In the particular case of $c = 0$, we recover the known sufficient condition for the existence of a double p-stable integral,

$$\int_0^1 \int_0^1 | f(s,t)|^p \log^+ | f(s,t)| \, ds \, dt < \infty$$

[cf. Rosinski and Woyczynski (1984), Theorem 4.4.2, where also the n-tuple case is considered].

REMARK 5.3. Since $\overline{\Phi}(w) \leq Cw^2$ holds, it follows that the condition $\int\int f^2(s,t)\, ds\, dt < \infty$ is always sufficient for the existence of $\int\int f(s,t)\, dX(s)\, dX(t)$. On the other hand, if X is a Gaussian process this condition is also necessary. Hence, it is not difficult to see that this is a necessary and sufficient condition for the existence of the double integral with respect to X whenever X has a nontrivial Gaussian component.

REMARK 5.4. In Definition 5.2 one could have defined the double integral $\int\int f\, dX\, dX$ as an iterated integral. One can show that our definition is equivalent with the iterated integral definition.

REMARK 5.5. It is obvious that the interval $[0,1]$ in this section (and Theorem 5.1) can be replaced by an arbitrary interval $[0, T]$, $T < \infty$. What is interesting is that our results are also valid for $T = \infty$ but in this case the proofs require essential modifications. In particular, our formulas lead to a nontrivial condition for double integrability of f on $[0,\infty) \times [0,\infty)$ with respect to the Poisson process.

REFERENCES

BONAMI, A. (1970). Etudé des coefficients de Fourier des fonctions de $L_p(G)$. *Ann. Inst. Fourier* (*Grenoble*) **20** 335–402.

CAMBANIS, S., ROSINSKI, J. and WOYCZYNSKI, W. A. (1985). Convergence of quadratic forms in p-stable random variables and θ_p-radonifying operators. *Ann. Probab.* **13** 885–897.

DE ACOSTA, A. (1980). Exponential moments of vector valued random series and triangular arrays. *Ann. Probab.* **8** 381–389.

FELLER, W. (1966). *An Introduction to Probability Theory and Its Applications*. Wiley, New York.

HOFFMANN-JØRGENSEN, J. (1974). Sums of independent Banach space valued random variables. *Studia Math.* **52** 159–186.

KRAKOWIAK, W. and SZULGA, J. (1986). Random multilinear forms. *Ann. Probab.* **14**, 955–973.

KWAPIEN, S. (1973). Sums of independent Banach space valued random variables (after J. Hoffmann-Jørgensen). *Séminaire Maurey–Schwartz 1972–73, Exposé VI.*École Polytechnique, Paris.

KWAPIEN, S. and WOYCZYNSKI, W. A. (1986). Decoupling of martingale transforms and stochastic integrals for processes with independent increments. In *Probability Theory and Harmonic Analysis* (J. A. Chao and W. A. Woyczynski, eds.) 139–148. Dekker, New York.

MCCONNELL, T. and TAQQU, M. (1984). Double integration with respect to symmetric stable processes. Technical Report 618, Cornell Univ.

PALEY, R. E. A. C. and ZYGMUND, A. (1932). A note on analytic functions on the unit circle. *Proc. Cambridge Philos. Soc.* **28** 266–272.

ROSINSKI, J. and WOYCZYNSKI, W. A. (1984). Products of random measures, multilinear random forms and multiple stochastic integrals. *Measure Theory, Oberwelfach* 1983. *Lecture Notes in Math.* **1089** 294–315. Springer, New York.

ROSINSKI, J. and WOYCZYNSKI, W. A. (1986). On Itô stochastic integration with respect to p-stable motion: Inner clock, integrability of sample paths, double and multiple integrals. *Ann. Probab.* **14** 271–286.

SZTENCEL, R. (1981). On boundedness and convergence of some Banach space valued random series. *Probab. Math. Statist.* **2** 83–88.

URBANIK, K. and WOYCZYNSKI, W. A. (1967). A random integral and Orlicz spaces. *Bull. Acad. Polon. Sci. Sér. Sci. Math.* **15** 161–169.

DEPARTMENT OF MATHEMATICS AND STATISTICS
CASE WESTERN RESERVE UNIVERSITY
UNIVERSITY CIRCLE
CLEVELAND, OHIO 44106