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DECOUPLING INEQUALITIES FOR POLYNOMIAL CHAOS[1]

BY STANISLAW KWAPIEN

University of Warsaw

Let $X, X_1, \dots, X_d$ be a sequence of independent, symmetric, identically distributed random vectors with independent components. The main subject of this paper is the so-called decoupling inequalities, i.e., inequalities of the form

$$E\phi(cQ(X, X, \dots, X)) \le E\phi(Q(X_1, X_2, \dots, X_d))$$
$$\le E\phi(CQ(X, X, \dots, X)),$$

where Q is a symmetric multilinear form with values in a vector space F with all "diagonal" terms equal to zero and ϕ is a convex function on F.

1. Introduction. Decoupling inequalities, the main subject of this paper, are useful in the study of multilinear random forms, multiple stochastic integrals or more generally, polynomial chaos. They are particularly useful when one considers problems such as convergence or existence of moments of polynomial chaos. They were introduced by McConnell and Taqqu [8] for the purpose of studying multiple stochastic integrals. In a more recent paper [9], they proved that if $X = (\xi_1, \xi_2, \dots, \xi_n)$ is a random vector with independent and symmetric components and if $X^1, X^2, \dots, X^d$ are independent copies of X, then for each d-multilinear, symmetric form Q on R^n with all diagonal terms equal to zero and each convex, symmetric function ϕ which fulfills the Δ_2 growth condition

$$E\phi(cQ(X, X, \dots, X)) \le E\phi(Q(X_1, X_2, \dots, X_d)) \le E\phi(CQ(X, X, \dots, X)),$$

for certain constants c and C depending only on ϕ.

In the present paper we give a simpler proof of a more general result, namely, we consider forms Q defined on R^n with values in a vector space F and ϕ is an arbitrary, convex and symmetric function on F. We also prove a contraction principle for the polynomial chaos. The principle is essential in our proof of decoupling inequalities and is also of independent interest. The decoupling inequalities are later applied to obtain a comparison of moments of polynomial chaos.

During the preparation of this paper many closely related results were obtained by several authors. We would like to mention, besides the already mentioned papers by McConnell and Taqqu, the papers by de Acosta [3], Krakowiak and Szulga [5], [6], Kwapien and Woyczynski [7] and Zinn [10].

2. Contraction principle for polynomial chaos. A sequence of functions $(f_i)_{i=1,2,\dots,n}$ defined on a measure space (T, Ξ, τ) is said to be a multiplicative

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system of order d iff

$$\int_T f_i(t)\tau(dt) = 0, \quad \text{for } i = 1,2,\ldots,n,$$

and

$$\int_T f_1^{\varepsilon_1} f_2^{\varepsilon_2} \cdots f_n^{\varepsilon_n}(t)\tau(dt)$$
$$= \int_T f_1^{\varepsilon_1}(t)\tau(dt) \int_T f_2^{\varepsilon_2}(t)\tau(dt) \cdots \int_T f_n^{\varepsilon_n}(t)\tau(dt),$$

for each sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ with values in the set $\{0,1,2,\ldots,d\}$.

A sequence of independent random variables with mean value equal to 0 is an example of a multiplicative system of any order, a sequence of martingale differences is a multiplicative system of order 1 and a lacunary trigonometric sequence is a multiplicative system of order d if the degree of lacunarity is sufficiently large.

The following proposition is a slight generalization of a theorem from [4].

Proposition 1. *Let $(f_i)_{i=1,\ldots,n}$ be a multiplicative system of order 1 on (T, Ξ, τ) and $(g_i)_{i=1,\ldots,n}$ a multiplicative system of order 2 on (S, Σ, σ). If, for each i,*

$$\|f_i\|_\infty \le \int_S g_i^2(s)\sigma(ds)/\|g_i\|_\infty,$$

then for each convex function ϕ on R^{2^n-1} we have

$$\int_T \phi(F)\, d\tau \le \int_S \phi(G)\, d\sigma,$$

where

$$F = \left(f_{i_1} f_{i_2} \cdots f_{i_k}\right)_{1 \le i_1 < i_2 \cdots < i_k \le n}, \qquad G = \left(g_{i_1} g_{i_2} \cdots g_{i_k}\right)_{1 \le i_1 < i_2 \cdots < i_k \le n}.$$

Proof. Let K be a kernel on $T \times S$ defined by

$$K(t,s) = \prod_{i=1}^{n} \left(1 + a_i f_i(t) g_i(s)\right), \quad \text{for } t \in T,\ s \in S,$$

where $\alpha_i = (\int_S g_i^2(s)\sigma(ds))^{-1}$.

The kernel K fulfills the conditions

$$K(t,s) \ge 0, \quad \text{for } t \in T,\ s \in S,$$

$$\int_T K(t,s)\tau(dt) = 1, \quad \text{for } s \in S,$$

$$\int_S K(t,s)\sigma(ds) = 1, \quad \text{for } t \in T,$$

and

$$\int_S K(t,s) g_{i_1}(s) g_{i_2}(s) \cdots g_{i_k}(s)\sigma(ds) = f_{i_1}(t) f_{i_2}(t) \cdots f_{i_k}(t),$$

for each $1 \le i_1 < i_2 < i_3 \cdots < i_k \le n$ and $t \in T$. Hence, by Jensen's inequality, we obtain

$$\begin{aligned}\int_T \phi(F)\,d\tau &= \int_T \phi\left(\int_S K(t,s)G(s)\sigma(ds)\right)\tau(dt)\\ &\le \int_T\left(\int_S K(t,s)\phi(G(s))\sigma(ds)\right)\tau(dt)\\ &= \int_S \phi(G)\,d\sigma. \qquad \square\end{aligned}$$

As an immediate consequence of Proposition 1, we obtain the following corollaries:

COROLLARY 1. *Let $(f_i)_{i=1,\dots,n}, (g_i)_{i=1,\dots,n}$ be as in Proposition 1 and let Q be a tetrahedral polynomial on R^n with values in a vector space F, i.e.,*

$$Q(x_1, x_2, \dots, x_n) = \sum_{k=0}^{n} \sum_{1\le i_1 < \cdots < i_k \le n} c_{i_1,\dots,i_k} x_{i_1} x_{i_2} \cdots x_{i_k},$$

where the coefficients $c_{i_1,i_2,\dots,i_k}$ are elements of F. Then for each convex function ϕ on F we have

$$\int_T \phi(Q(f_1, f_2, \dots, f_n))\,d\tau \le \int_S \phi(Q(g_1, g_2, \dots, g_n))\,d\sigma.$$

COROLLARY 2. *Let $(r_i)_{i=1,\dots,n}$ be a Rademacher sequence on a probability space (Ω, Σ, P) and let $(\alpha_i)_{i=1,\dots,n}$ be a sequence of real numbers such that $|\alpha_i| \le 1$ for $i = 1, 2, \dots, n$. If ϕ and Q are as in Corollary 1, then*

$$E\phi(Q(\alpha_1 r_1, \alpha_2 r_2, \dots, \alpha_n r_n)) \le E\phi(Q(r_1, r_2, \dots, r_n)).$$

Let $(\xi_i)_{i=1,\dots,n}$ be a symmetric sequence of random variables, i.e., for each sequence $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n = \pm 1$ the sequence $(\varepsilon_i \xi_i)_{i=1,\dots,n}$ is equidistributed with the sequence $(\xi_i)_{i=1,\dots,n}$, and let $(\theta_i)_{i=1,\dots,n}$ be a sequence of random variables such that $|\theta_i| \le 1$ for $i = 1, 2, \dots, n$ and such that the sequence $(\theta_i \xi_i)_{i=1,\dots,n}$ is a symmetric sequence of random variables.

LEMMA 1. *Under the above assumptions on $(\theta_i)_{i=1,\dots,n}$ and $(\xi_i)_{i=1,\dots,n}$ we have, for each Q and ϕ as in Corollary 1,*

$$E\phi(Q(\theta_1\xi_1, \theta_2\xi_2, \dots, \theta_n\xi_n)) \le E\phi(Q(\xi_1, \xi_2, \dots, \xi_n)).$$

PROOF. Let $(r_i)_{i=1,\dots,n}$ be a Rademacher sequence, independent of the sequences $(\theta_i)_{i=1,\dots,n}$ and $(\xi_i)_{i=1,\dots,n}$. The sequence $(\xi_i)_{i=1,\dots,n}$ is equidistributed with the sequence $(r_i\xi_i)_{i=1,\dots,n}$ and the sequence $(\theta_i\xi_i)_{i=1,\dots,n}$ is equidistributed with $(r_i\theta_i\xi_i)_{i=1,\dots,n}$. Hence, if G is the σ-field generated by $(\theta_i, \xi_i)_{i=1,\dots,n}$

by Corollary 2 we get

$$\begin{aligned} E\phi(Q(\theta_1\xi_1, \theta_2\xi_2, \dots, \theta_n\xi_n)) &= E\phi(Q(r_1\theta_1\xi_1, r_2\theta_2\xi_2, \dots, r_n\theta_n\xi_n)) \\ &= E\big(E\{\phi(Q(r_1\theta_1\xi_1, r_2\theta_2\xi_2, \dots, r_n\theta_n\xi_n))|G\}\big) \\ &\le E\big(E\{\phi(Q(r_1\xi_1, r_2\xi_2, \dots, r_n\xi_n))|G\}\big) \\ &= E\phi(Q(\xi_1, \xi_2, \dots, \xi_n)). \qquad \square \end{aligned}$$

Theorem 1 (Contraction principle). *Let $(\eta_i)_{i=1,\dots,n}$ and $(\xi_i)_{i=1,\dots,n}$ be two sequences of independent, symmetric random variables such that for some constants K, L and for $i = 1, 2, \dots, n$,*

$$P(|\eta_i| \ge t) \le KP(L|\xi_i| \ge t), \quad \text{for } t \ge 0.$$

Then for ϕ and Q as in Corollary 1 we have

$$E\phi(Q(\eta_1, \eta_2, \dots, \eta_n)) \le E\phi(Q(KL\xi_1, KL\xi_2, \dots, KL\xi_n)).$$

Proof. Let $(\delta_i)_{i=1,\dots,n}$ be a sequence of independent, symmetric random variables which is independent of the sequence $(\eta_i)_{i=1,\dots,n}$ such that each δ_i is distributed according to the law $P(|\delta_i| = 1) = 1 - P(\delta_i = 0) = K^{-1}$. For $i = 1, 2, \dots, n$ and for $t \ge 0$, we have

$$P(|\delta_i\eta_i| \ge t) \le P(|L\xi_i| \ge t).$$

Hence, we deduce that there exist a sequence $(\bar{\xi}_i)_{i=1,\dots,n}$ and a sequence $(\bar{\theta}_i)_{i=1,\dots,n}$ of random variables (which, perhaps, are defined on another probability space) such that the sequence $(\bar{\xi}_i)_{i=1,\dots,n}$ is equidistributed with the sequence $(\xi_i)_{i=1,\dots,n}$ and the sequence $(\bar{\theta}_i L\bar{\xi}_i)_{i=1,\dots,n}$ is equidistributed with the sequence $(\delta_i\eta_i)_{i=1,\dots,n}$ and $|\bar{\theta}_i| \le 1$ for $i = 1, \dots, n$. By Lemma 1 this implies that

$$\begin{aligned} E\phi(Q(\delta_1\eta_1, \delta_2\eta_2, \dots, \delta_n\eta_n)) &= E\phi\big(Q(\bar{\theta}_1 L\bar{\xi}_1, \bar{\theta}_2 L\bar{\xi}_2, \dots, \bar{\theta}_n L\bar{\xi}_n)\big) \\ &\le E\phi\big(Q(L\bar{\xi}_1, L\bar{\xi}_2, \dots, L\bar{\xi}_n)\big) \\ &= E\phi(Q(L\xi_1, L\xi_2, \dots, L\xi_n)). \end{aligned}$$

By Corollary 1 applied to the sequences $(g_i)_{i=1,\dots,n} = (\delta_i)_{i=1,\dots,n}$ and $(f_i)_{i=1,\dots,n} = (r_i/K)_{i=1,\dots,n}$ we obtain that for each tetrahedral polynomial $\bar{Q}$ and ϕ as in Corollary 1

$$E\phi\big(\bar{Q}(\delta_1, \delta_2, \dots, \delta_n)\big) \ge E\phi\big(\bar{Q}(r_1/K, r_2/K, \dots, r_n/K)\big).$$

Hence, by the preceding inequality we get

$$\begin{aligned} E\phi\big(Q(\eta_1/K, \eta_2/K, \dots, \eta_n/K)\big) &\le E\phi(Q(\delta_1\eta_1, \delta_2\eta_2, \dots, \delta_n\eta_n)) \\ &\le E\phi(Q(L\xi_1, L\xi_2, \dots, L\xi_n)). \end{aligned}$$

An application of this inequality to the polynomial $Q(Kx_1, Kx_2, \dots, Kx_n)$ concludes the proof of the theorem. $\square$

3. Decoupling inequalities. Let $Q(x)$ be a polynomial of degree d on R^n with coefficients in a vector space F, and let $Q(x) = \sum_{k=0}^{d} Q_k(x)$ be its expansion into homogeneous polynomials. Q_k is a homogeneous polynomial of degree k. We associate with Q a d-affine, symmetric form

$$\hat{Q}(x_1, x_2, \dots, x_d) = \operatorname{Ave} \sum_{k=0}^{d} \frac{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_d}{d!} (\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_d)^{d-k} \times Q_k(\varepsilon_1 x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_d x_d)$$
$$= \operatorname{Ave} \frac{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_d}{d!} (\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_d)^{d} \times Q\left(\frac{\varepsilon_1 x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_d x_d}{\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_d}\right),$$

where the average is extended over all sequences $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d = \pm 1$ and where the last expression under the average is equal to

$$\frac{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_d}{d!} Q_d(\varepsilon_1 x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_d x_d) \quad \text{if } \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_d = 0.$$

It is easy to check that $\hat{Q}$ has the following properties:

(i) $\hat{Q}$ is, in each variable separately, an affine map on R^n, i.e., it is a linear map plus a constant vector.

(ii) $\hat{Q}$ is symmetric, i.e., $\hat{Q}(x_1, x_2, \dots, x_d)$ is invariant under permutations of variables $x_1, x_2, \dots, x_d$.

(iii) $\hat{Q}(x, x, \dots, x) = Q(x)$ for each x in R^n.

A form $\hat{Q}$ with the above properties is unique. If Q is a homogeneous polynomial of degree d, then the above formula is the well-known "polarization formula." We will refer to the above formula as the polarization formula.

Let $X = (\xi_1, \xi_2, \dots, \xi_n)$ be a sequence of independent, symmetric random variables and let $X^1, X^2, \dots, X^d$ be independent copies of X.

THEOREM 2 (Decoupling inequalities). *There are constants c, C depending only on d such that for each tetrahedral polynomial Q on R^n of degree d with coefficients in a vector space F and for each convex function ϕ: $F \to R^+$, with $\phi(x) = \phi(-x)$ for $x \in F$,*

$$E\phi(cQ(X)) \le E\phi(\hat{Q}(X^1, X^2, \dots, X^d)) \le E\phi(CQ(X)).$$

We will need the following lemma.

LEMMA 2. *There is a constant K depending only on d such that for all ϕ, Q, X as in Theorem 2*

$$E\phi(Q_k(X)) \le E\phi(KQ(X)), \quad \text{for } k = 0, 1, 2, \dots, d.$$

(Q_k is the kth homogeneous polynomial from the expansion of Q.)

PROOF. For each k there exists an integrable function g_k on $[-1, 1]$ such that $\int_{-1}^{1} t^i g_k(t)\, dt = 0$ for $i \ne k$, $i \le d$, and $\int_{-1}^{1} t^k g_k(t)\, dt = 1$. Let $C_k = \int_{-1}^{1} |g_k(t)|\, dt$.

By Jensen's inequality we obtain

$$E\phi(Q_k(X)) = E\phi\left(\int_{-1}^{1} Q(tX)g_k(t)\,dt\right)$$
$$\leq C_k^{-1}\int_{-1}^{1}\left(E\phi(C_kQ(tX))\right)|g_k(t)|\,dt \leq E\phi(C_kQ(X)),$$

because, by Theorem 1, for each $t \in [-1,1]$

$$E\phi(C_kQ(tX)) \leq E\phi(C_kQ(X)).$$

Putting $K = \max_{0 \leq k \leq d} C_k$ we conclude the proof of Lemma 2. (It can be shown that K can be taken to be equal to 2^d.) □

PROOF OF THEOREM 2. Let $X^i = (\xi_1^i, \xi_2^i, \dots, \xi_n^i)$ for $i = 1,2,\dots,d$. Then for each $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_k \leq d$ we have

$$E\left\{\xi_{i_1}^{j_1}\xi_{i_2}^{j_2}\cdots\xi_{i_k}^{j_k}|X^1 + X^2 + \cdots + X^d\right\}$$
$$= \frac{1}{d}\left(\xi_{i_1}^1 + \xi_{i_1}^2 + \cdots + \xi_{i_1}^d\right)\frac{1}{d}\left(\xi_{i_2}^1 + \xi_{i_2}^2 + \cdots + \xi_{i_2}^d\right)$$
$$\times \cdots \times \frac{1}{d}\left(\xi_{i_k}^1 + \xi_{i_k}^2 + \cdots + \xi_{i_k}^d\right).$$

Hence, by the tetrahedrality of Q we obtain

$$E\{\hat{Q}(X^1, X^2,\dots, X^d)|X^1 + X^2 + \cdots + X^d\} = Q\left(\frac{X^1 + X^2 + \cdots + X^d}{d}\right).$$

On the other hand,

$$E\left\{Q\left(\frac{X^1 + X^2 + \cdots + X^d}{d}\right)\middle|X^1\right\} = Q\left(\frac{X^1}{d}\right).$$

Thus by Jensen's inequality, we get

$$E\phi\left(Q\left(\frac{1}{d}X\right)\right) \leq E\phi(\hat{Q}(X^1, X^2,\dots, X^d)).$$

By Lemma 2 and by the convexity of ϕ we have

$$E\phi\left((d+1)^{-1}d^{-d}K^{-1}Q(X)\right) \leq \sum_{k=0}^{d}\frac{1}{d+1}E\phi\left(K^{-1}Q_k\left(\frac{1}{d}X\right)\right) \leq E\phi\left(Q\left(\frac{1}{d}X\right)\right),$$

which, together with the preceding inequality, proves the left-hand side inequality in Theorem 2 with $c = (d+1)^{-1}d^{-d}K^{-1}$.

By the polarization formula and by the convexity of ϕ we obtain

$$E\phi(\hat{Q}(X^1, X^2,\dots, X^d))$$
$$\leq \operatorname{Ave}\sum_{k=0}^{d}\frac{1}{d+1}E\phi\left(\frac{d+1}{d!}|\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_d|^{d-k}\right.$$
$$\left.\times Q_k\left(\varepsilon_1X^1 + \varepsilon_2X^2 + \cdots + \varepsilon_dE_dX^d\right)\right).$$

For a fixed sequence $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d$, let us write

$$\bar{\xi}_i = \varepsilon_1 \xi_i^1 + \varepsilon_2 \xi_i^2 + \cdots + \varepsilon_d \xi_i^d, \qquad i = 1, 2, \dots, n,$$

and let $\bar{X} = (\bar{\xi}_1, \bar{\xi}_2, \dots, \bar{\xi}_n)$. For each $t \geq 0$ and $i = 1, 2, \dots, n$, $P(|\bar{\xi}_i| \geq t) \leq dP(|\xi_i| \geq t/d)$. Therefore, by Theorem 1 and Lemma 2 we obtain

$$\begin{aligned} &E\phi\left(\frac{d+1}{d!}|\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_d|^{d-k} Q_k(\bar{X})\right) \\ &\quad \leq E\phi\left(\frac{d+1}{d!} d^{d-k} Q_k(d^2 X)\right) \leq E\phi\left(\frac{d+1}{d!} d^{2d} Q_k(X)\right) \\ &\quad \leq E\phi\left(\frac{d+1}{d!} d^{2d} K Q(X)\right). \end{aligned}$$

Hence, we have

$$E\phi(\hat{Q}(X^1, X^2, \dots, X^d)) \leq E\phi\left(K \frac{d+1}{d!} d^{2d} Q(X)\right),$$

which proves Theorem 2. □

REMARK 1. If, in addition to the assumptions of Theorem 2, Q is homogeneous of degree d, then the above proof can be made much more direct and the constants c and C can be improved to be equal to d^{-d} and $d^{2d}/d!$, respectively. Moreover, if $X = (\xi_1, \xi_2, \dots, \xi_n)$ is a p-stable symmetric random vector, then the constants c and C can be even further improved to $d^{d(1/p-1)}$ and $d^{d/p}/d!$, respectively.

As an application of the decoupling inequalities we obtain the following:

COROLLARY 3. *Let us assume that* $X = (\xi_1, \xi_2, \dots, \xi_n)$ *is a sequence of independent symmetric random variables such that for some* $p \geq q \geq 1$ *and for a constant* $C_{p,q}$

$$\left(E\left\|x_0 + \sum_{i=1}^n x_i \xi_i\right\|^p\right)^{1/p} \leq C_{p,q}\left(E\left\|x_0 + \sum_{i=1}^n x_i \xi_i\right\|^q\right)^{1/q},$$

for each Banach space F *and each sequence* $x_0, x_1, \dots, x_n \in F$.

If Q *is a tetrahedral polynomial of degree* d *with coefficients in a Banach space, then*

$$\left(E\|Q(X)\|^p\right)^{1/p} \leq (C_{p,q})^d \frac{C}{c}\left(E\|Q(X)\|^q\right)^{1/q},$$

where c *and* C *are the constants from Theorem* 2.

PROOF. By induction on d we obtain easily

$$\left(E\|\hat{Q}(X^1, X^2, \dots, X^d)\|^p\right)^{1/p} \leq (C_{p,q})^d \left(E\|\hat{Q}(X^1, X^2, \dots, X^d)\|^q\right)^{1/q}.$$

Combining this with Theorem 2 we conclude the proof. □

REMARK 2. A sequence $X = (\xi_1, \xi_2, \dots, \xi_n)$ of independent, symmetric and identically distributed random variables satisfies the assumption of Corollary 3 with a constant $C_{p,q}$ independent of n if and only if there exists a constant C such that $\int_t^\infty s^{p-1}N(s)\,ds \le Ct^pN(t)$ for sufficiently large t, where $N(t) = P(|\xi_i| \ge t)$.

For a special class of random variables (ξ_i) a result similar to the one in Corollary 3 was obtained by Borell [1], by the method of hypercontractive operators. Later it was extended by Krakowiak and Szulga [5] to a larger class of random variables.

REMARK 3. If Q is a polynomial of degree d on R^n and X is a Gaussian vector in R^n, then there exist a sequence of tetrahedral, homogeneous polynomials Q^m on R^n of degree d and a sequence of Gaussian, symmetric vectors X^m in R^n such that

$$E\|EQ(X) + Q^m(X^m) - Q(X)\|^p \to 0, \quad \text{for each } p;$$

cf. Borell [2].

Since for a Gaussian random variable $(\xi_i)_{i=1,2,\dots,n}$, the constant $C_{p,q}$ appearing in Corollary 3 may be taken to be equal to $c\sqrt{p}$ where c is some universal constant, we obtain by Corollary 3 that

$$\left(E\|Q(X)\|^p\right)^{1/p} \le \left(c\sqrt{p}\,e\right)^d\left(E\|Q\|^q\right)^{1/q}, \quad \text{for } 1 \le q \le p,$$

holds for all polynomials of degree d.

Hence, using the power series expansion we obtain

$$E\exp\lambda\|Q(X)\|^{2/d} < C, \quad \text{for } \lambda > 0 \text{ and } C < \infty,$$

which depends only on d and $E\|Q(X)\|$.

We believe that inequalities similar to the decoupling inequalities hold for a much more general class of functions ϕ than the class of convex functions. The following result, which is a generalization of an observation made by Borell, shows that this is true at least in the case of Gaussian random vectors.

PROPOSITION 2. *Let us assume that Q and X, in addition to the assumptions made in Theorem 2, satisfy the following: Q is homogeneous and X is a Gaussian random vector. Then for each convex, symmetric subset K of F*

$$\frac{1}{4\cdot 3^d}P\left(d^{-d/2}Q(X) \notin K\right) \le P\left(\hat{Q}(X^1, X^2, \dots, X^d) \notin K\right) \le 2^dP\left(\frac{d^{d/2}}{d!}Q(X) \notin K\right).$$

PROOF. By the polarization formula we have

$$\hat{Q}(X^1, X^2, \dots, X^d) = \operatorname{Ave}\frac{\varepsilon_1\varepsilon_2\cdots\varepsilon_d}{d!}Q\left(\varepsilon_1X^1 + \varepsilon_2X^2 + \cdots + \varepsilon_dX^d\right).$$

For each $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d = \pm 1$ the random vector $\varepsilon_1X^1 + \varepsilon_2X^2 + \cdots + \varepsilon_dX^d$ is equidistributed with $d^{1/2}X$. Hence, we obtain

$$P\big(\hat{Q}(X^1,\dots,X^d)\notin K\big)\le P\bigg(\frac{1}{d!}Q\big(\varepsilon_1X^1+\varepsilon_2X^2+\cdots+\varepsilon_dX^d\big)\notin K,\ \text{for some } \varepsilon_1,\varepsilon_2,\dots,\varepsilon_d=\pm1\bigg)$$

$$\le 2^dP\bigg(\frac{1}{d!}Q(d^{1/2}X)\notin K\bigg)=2^dP\bigg(\frac{d^{d/2}}{d!}Q(X)\notin K\bigg).$$

This proves the right-hand side inequality. To prove the left-hand side inequality we need the following lemma.

LEMMA 3. *If Y is a random vector in a finite-dimensional vector space F and K is a convex subset of F such that $EY\notin K$, then*

$$P(Y\notin K)\ge\tfrac{1}{4}\inf_{x'\in F',\,\alpha\in R^1}\big(E|x'(Y)-\alpha|\big)^2\Big/E|x'(Y)-\alpha|^2.$$

PROOF. This result follows easily from the theorem on separation of convex sets and the fact that for each real random variable ξ with $E\xi=0$ the inequality $P(\xi\le 0)\ge\frac{1}{4}(E|\xi|)^2/E\xi^2$ holds. □

Let $\overline{X}=(X^1+X^2+\cdots+X^d)/d$. Then the random vector $(X^1-\overline{X},X^2-\overline{X},\dots,X^d-\overline{X})$ is Gaussian and independent of $\overline{X}$. Hence, if we put $Y_a=\hat{Q}(X^1-\overline{X}+a,X^2-\overline{X}+a,\dots,X^d-\overline{X}+a)$, then $P\{\hat{Q}(X^1,X^2,\dots,X^d)\notin K|\overline{X}\}=G(\overline{X})$, where $G(a)=P(Y_a\notin K)$. By Corollary 3 we deduce that for each $x'\in F'$ and $\alpha\in R^1$, $3^{-d}E(x'(Y_a)-\alpha)^2\le(E|x'(Y_a)-\alpha|)^2$, because for a Gaussian sequence X the constant $C_{2,1}$ in Corollary 3 may be taken to be equal to $\sqrt{3}$. Moreover, we have $E\{\hat{Q}(X^1,X^2,\dots,X^d)|\overline{X}\}=Q(\overline{X})$ (cf. the proof of Theorem 2). Hence, by Lemma 2, we obtain that on the set $\{Q(\overline{X})\notin K\}$, $P\{\hat{Q}(X^1,X^2,\dots,X^d)\notin K|\overline{X}\}\ge 3^{-d}/4$ and this yields

$$P\big(\hat{Q}(X^1,X^2,\dots,X^d)\notin K\big)\ge 3^{-d}/4P\big(Q(\overline{X})\notin K\big)$$
$$=3^{-d}/4P\big(d^{-d/2}Q(X)\notin K\big).\qquad\square$$

REMARK 4. Another "nonconvex result" was obtained by de Acosta [3]. He proved that in the case when X is a p-stable random vector, then decoupling inequalities are valid for $\phi(x)=\|x\|^r$ where $r<p$.

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DEPARTMENT OF MATHEMATICS
WARSAW UNIVERSITY
WARSAW 00901
POLAND