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ON THE MEANS OF APPROACH TO THE BOUNDARY OF BROWNIAN MOTION[1]

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For a simply connected plane Greenian domain D we give a natural description of the means of exit of a Brownian path from D. We give a related discussion of the connection between classical and probabilistic Fatou theorems.

In this paper we discuss the means of approach of Brownian motion in a simply connected plane Greenian domain D to the boundary. One motivation for this is to compare classical and probabilistic Fatou results.

The classical Fatou theorem on $D = \{z\colon |z| \le 1\}$ is the following. Given $1 > \sigma > 0$, define $\Gamma(\sigma; e^{i\theta})$ to be the convex hull of $\{z\colon |z| < \sigma\}$ and $\{e^{i\theta}\}$. Then given any bounded harmonic function h on D,

$$\lim_{z \to \partial D,\ z \in \Gamma(\sigma;\ e^{i\theta})} h(z)$$

exists for a.e. $\theta \in [0, 2\pi)$. This was proved by Fatou (1906). In Littlewood (1927) this was supplemented by showing that given any tangential means of approach, there is a bounded harmonic function on D for which that tangential approach fails to give a limit at a.e. boundary point. This will be discussed in more detail later.

Doob (1957) provided a probabilistic version of Fatou's theorem. Suppose (W_t, P_z) is Brownian motion killed on exiting D, with $\tau_D = \inf\{t > 0\colon W_t \notin D\}$. Then $\lim_{t \uparrow \tau_D} h(W_t)$ exists P_z a.s. for any bounded harmonic function h. More can be said when W is conditioned on the value of $\lim_{t \uparrow \tau_D} W_t$, i.e., by looking at conditioned Brownian motion. If $p(t, z, w)$ is the transition density for W under P_z and u is the Poisson kernel with pole at $e^{i\theta}$, then $p^u(t, z, w) = u(z)^{-1} p(t, z, w) u(w)$ is the transition density for W conditioned to exit D at the point $e^{i\theta}$. It is easy to see that this is a diffusion on D with generator $\frac{1}{2}\Delta + \nabla u/u \cdot \nabla$. This diffusion solves the equation $dX_t = dW_t + \nabla u/u(X_t)\,dt$. Then we have that for a.e. $\theta \in [0, 2\pi)$, $\lim_{t \uparrow \tau_D} h(X_t)$ exists and is equal to

$$\lim_{\substack{y \to \partial D,\\ y \in \Gamma(\sigma;\ e^{i\theta})}} h(y) \quad P_z \text{ a.s.}$$

This motivates studying the means of approach of Brownian motion in D to ∂D. In what follows D will be a simply connected plane Greenian domain and u will be the Martin kernel associated to a point ξ in the Martin boundary. By X

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we denote the u-process. It was shown in Cranston and McConnell (1983) that $u(X_t) \to \infty$ as $t \uparrow \tau_D$. Also keep in mind that $X_t \to \xi$ in the Martin topology as $t \uparrow \tau_D$. Applying Itô's formula, $u^2(X_t) = u^2(X_0) + 3\int_0^t |\nabla u(X_s)|^2\, ds + 2\int_0^t u(X_s) \nabla u(X_s)\, dW_s$, for $0 \le t < \tau_D$. We now claim that $\int_0^t |\nabla u(X_s)|^2\, ds \to \infty$ as $t \to \tau_D$. Since $u(X_t) \to \infty$ as $t \to \tau_D$ and $\int_0^t |\nabla u(X_s)|^2\, ds$ is increasing, the claim hinges on the behavior of the martingale term $M_t = 2\int_0^t u(X_s) \nabla u(X_s)\, dW_s$. Define $\eta = 4\int_0^{\tau_D} u^2(X_s) |\nabla u(X_s)|^2\, ds$. On $\{\eta < \infty\}$, $M_{\tau_D} = \lim_{t \to \tau_D} M_t$ is finite and thus $\int_0^t |\nabla u(X_s)|^2\, ds \to \infty$ as $t \to \tau_D$ on $\{\eta < \infty\}$. On $\{\eta = \infty\}$, M_t can be time changed to a Brownian motion run up to $\eta = \infty$. Thus $M_t = 0$ infinitely often as $t \to \tau_D$, where by "infinitely often as $t \to \tau_D$" we mean for every $t < \tau_D$ there is an s, $t < s < \tau_D$, such that $M_s = 0$. Consequently, on $\{\eta = \infty\}$, $u^2(X_t) = u^2(X_0) + 3\int_0^t |\nabla u(X_s)|^2\, ds$ infinitely often as $t \to \tau_D$ and since it is increasing, it follows that $3\int_0^t |\nabla u(X_s)|^2\, ds \to \infty$ on $\{\eta = \infty\}$. This establishes the claim which will be useful later. Also, $\eta = \infty$ a.s. now follows since $\int_0^{\tau_D} |\nabla u(X_s)|^2\, ds = \infty$ and $u(X_s) \to \infty$ as $s \to \tau_D$. Now let v be the harmonic conjugate to u, so that $u + iv$ is analytic, or u and v satisfy the Cauchy–Riemann equations, $u_x = v_y$, $u_y = -v_x$. For example, when $D = \{z\colon |z| < 1\}$ and $\xi = e^{i\theta}$ then

$$u(re^{i\psi}) = (1 - r^2)\big(1 - 2r\cos(\theta - \psi) + r^2\big)^{-1}$$

and

$$v(re^{i\psi}) = -2r\sin(\theta - \psi)\big(1 - 2r\cos(\theta - \psi) + r^2\big)^{-1}.$$

In the case when $D = \{z\colon \operatorname{Im} z > 0\}$ and $\xi = (t, 0)$ then

$$u(x, y) = y\big(y^2 + (x - t)^2\big)^{-1} \quad \text{and} \quad v(x, y) = x\big(y^2 + (x - t)^2\big)^{-1}.$$

This example will eventually be discussed in more detail. When D is simply connected, u and v give an orthogonal coordinate system on D. In the first example mentioned above, $D = \{z\colon |z| < 1\}$, the u-level curves are circles internally tangent to ∂D at ξ. The v-level curves are the orthogonal circles with center on the tangent to ∂D at ξ and touching ξ. The positive v-circles are on one side of ξ, the negative on the other. For general simply connected Greenian domains just use conformal mapping to conclude that the functions u and v provide a coordinate system. When the generator for the u-process is written in (u, v)-coordinates something nice happens and this is used to prove the main result.

THEOREM 1. *Suppose D is a simply connected plane Greenian domain and let ξ be any point in its Martin boundary. Denote by u, v and X the Martin kernel with pole at ξ, its harmonic conjugate and the u-process, respectively. Let $g\colon [0, \infty) \to [0, \infty)$ be such that $g(t) \downarrow 0$ as $t \uparrow \infty$ and F be an increasing function. If $\int^\infty g(t) t^{-1}\, dt < \infty$ and $\limsup_{t \uparrow \infty} \sqrt{t \log\log t}\,(F(\sqrt{t}\, g(t)))^{-1} = 0$, then*

$$\lim_{t \uparrow \tau_D} \frac{v(X_t)}{F(u(X_t))} = 0, \quad P_z \text{ a.s.}$$

PROOF. The process $(U_t, V_t) = (u(X_t), v(X_t))$, for $0 \le t < \tau_D$, tells on which (u, v)-level the process X_t is sitting. Writing the generator for X in (u, v)-coordinates yields

$$\frac{1}{2}\Delta + \frac{\nabla u}{u}\nabla = [|\nabla u|^2]\left(\frac{1}{2}\frac{\partial^2}{\partial u^2} + \frac{1}{u}\frac{\partial}{\partial u} + \frac{1}{2}\frac{\partial^2}{\partial v^2}\right)$$
$$= H(u, v)\left(\frac{1}{2}\frac{\partial^2}{\partial u^2} + \frac{1}{u}\frac{\partial}{\partial u} + \frac{1}{2}\frac{\partial^2}{\partial v^2}\right).$$

Here $[|\nabla u|^2]$ has been expressed as a function, $H(u, v)$, of the new coordinates. Thus (u_t, v_t), whose generator is $\frac{1}{2}\partial^2/\partial u^2 + 1/u\, \partial/\partial u + \frac{1}{2}\partial^2/\partial v^2$, is a time change of (U_t, V_t). Furthermore the time change is the inverse of the additional functional $\int_0^t|\nabla u(X_s)|^2\, ds$ and as previously observed this tends to infinity P_z a.s. as $t \uparrow \tau_D$. Thus (U_t, V_t) traces out the same path as $t \uparrow \tau_D$ as does (u_t, v_t) when $t \uparrow \infty$. One sees from the generator of (u_t, v_t) that the coordinate processes are independent, the first is Bess (3), the radial part of a three-dimensional Brownian motion, and the second is one-dimensional Brownian motion.

It is known, Itô and McKean (1974), page 164, that if $g(t) \downarrow 0$ as $t \uparrow \infty$ and $\int^\infty g(t)t^{-1}\, dt < \infty$, then $P_z(u(t) > \sqrt{t}g(t)$ eventually$) = 1$. Thus, if F and g are as assumed

$$\limsup_{t \uparrow \tau_D} \frac{|v(X_t)|}{F(u(X_t))} = \limsup_{t \uparrow \infty} \frac{|v_t|}{F(u_t)}$$
$$= \limsup_{t \uparrow \infty} \frac{|v_t|}{\sqrt{2t \log\log t}} \frac{\sqrt{2t \log\log t}}{F(u_t)}$$
$$\leq \limsup_{t \uparrow \infty} \frac{|v_t|}{\sqrt{2t \log\log t}} \frac{\sqrt{2t \log\log t}}{F(\sqrt{t}g(t))}$$
$$= 0 \quad \text{a.s.}$$ □

Corollary 2. *For every* $\delta > 0$

$$\limsup_{t \uparrow \tau_D} \frac{|v(X_t)|}{u(X_t)(\log u(X_t))^{1+\delta}} = 0,$$
$$\limsup_{t \uparrow \tau_D} \frac{|v(X_t)|}{u(X_t)\log u(X_t)} = \infty,$$

and

$$\liminf_{t \uparrow \tau_D} \frac{|v(X_t)|}{u(X_t)\log u(X_t)} = 0, \quad P_z \text{ a.s.}$$

Proof. The first statement follows from Theorem 1 by selecting $g(t) = (\log t)^{-(1+\delta')}$ with any $0 < \delta' < \delta$ and $F(u) = u(\log u)^{1+\delta}$.

For the second claim we will show $\limsup_{t \uparrow \tau_D}|v(X_t)|/u(X_t)\log u(X_t) \geq M$ a.s. for any $M > 0$. We will abuse notation slightly and let the subscript on P_{z_0},

$z_0 = 1 + 0i$, denote the starting position of the process $u(t) + iv(t)$ and then drop the subscript and write only P, u and v will still be independent Bess (3) and one-dimensional Brownian motion, respectively.

Now $P(u(t) < \sqrt{t}/\log t \text{ i.o. } t \uparrow \infty) = 1$ since $\int^\infty 1/t \log t\, dt = \infty$. Thus let σ_n be a set of times $\sigma_n \uparrow \infty$ such that $u(\sigma_n) < \sqrt{\sigma_n}/\log \sigma_n$. The process v is independent of the times $\{\sigma_n\}$. Thus if $A_n = \{|v(\sigma_n)| \geq \frac{1}{2}M\sqrt{\sigma_n}\}$, then

$$\begin{aligned} P(A_n) &= \int_0^\infty P\left(|v(t)| \geq \tfrac{1}{2}M\sqrt{t}\right) P(\sigma_n \in dt) \\ &= c(M), \end{aligned}$$

where $c(M)$ is a positive constant depending on M. In order to show $P(A_n \text{ i.o.}) = 1$ it is convenient to apply a version of the Borel–Cantelli lemma due to Kochen and Stone (1964). This says if $\Sigma P(A_n) = \infty$ and for some positive c, $P(A_n \cap A_m) \leq cP(A_n)(P(A_m) + P(A_{m-1}))$ for $m > n$, then it follows that $P(A_n \text{ i.o.}) > 0$. Since $\{A_n \text{ i.o.}\}$ is in the tail σ-field for a four-dimensional Brownian motion, which is trivial, we would then conclude that $P(A_n \text{ i.o.}) = 1$. The conditions are easy to verify, $\Sigma P(A_n) = \infty$ is obvious and $P(A_n \cap A_m) \leq 1 = cP(A_n)(P(A_m) + P(A_{m-1}))$ if $c = 1/2c(M)^2$. Thus $P(A_n \text{ i.o.}) \doteq 1$ and

$$\begin{aligned} \limsup_{t \uparrow \tau_D} \frac{|v(X_t)|}{u(X_t)\log u(X_t)} &= \limsup_{t \uparrow \infty} \frac{|v(t)|}{u(t)\log u(t)} \\ &\geq \limsup_{n \uparrow \infty} \frac{|v(\sigma_n)|}{u(\sigma_n)\log u(\sigma_n)} \\ &\geq \lim_{n \uparrow \infty} \frac{\frac{1}{2}M\sqrt{\sigma_n}}{\left(\sqrt{\sigma_n}/\log \sigma_n\right)\log\left(\sqrt{\sigma_n}/\log \sigma_n\right)} \\ &= M, \quad P \text{ a.s., as desired.} \end{aligned}$$

The lim inf result is obvious since $v(X_t) = 0$ i.o. as $t \uparrow \tau_D$ and $u(X_t) \to \infty$. □

The behavior of the time change transforming (U_t, V_t) into (u_t, v_t) has interest of its own. This time change is the inverse of the additive functional $\int_0^t |\nabla u(X_s)|^2\, ds$ which as we have already noted tends to infinity as $t \uparrow \tau_D$. Writing the Itô formula for $v(X_t)$ we see $v(X_t) = v(X_0) + \int_0^t \nabla v(X_s)\, dW_s$, for $0 \leq t < \tau_D$, since $\nabla u \cdot \nabla v = 0$. Thus, $v(X_t)$ is a local martingale and the inverse of the additive functional $\int_0^t |\nabla u(X_s)|^2\, ds$ takes $v(X_t)$ into a one-dimensional Brownian motion. Thus we have the following easy consequence of the law of the iterated logarithm.

Proposition 3. *If $A_t = \int_0^t |\nabla u(X_s)|^2\, ds$, then*

$$\limsup_{t \uparrow \tau_D} \frac{v(X_t)}{\sqrt{2A_t \log\log A_t}} = +1, \quad a.s.$$

Example. Consider the domain $D = \{z\colon \operatorname{Im} z > 0\}$ and take the Poisson kernel with pole at $(0, 0)$, $u(x, y) = y(x^2 + y^2)^{-1}$ together with its harmonic conjugate $v(x, y) = x(x^2 + y^2)^{-1}$. Corollary 2 says that if the Brownian motion exits D at $(0,0)$ it will hit the curve C: $|x| = y\log[y(x^2+y^2)^{-1}]$ i.o. as $t \uparrow \tau_D$ but will ultimately lie "inside" any curve C^δ: $|x| = cy(\log(y(x^2+y^2)^{-1}))^{1+\delta}$ whenever c and δ are positive. An easy computation shows that C is tangential to $\operatorname{Im} z = 0$ at 0. Conformal mapping shows the curve $|v| = u \log u$ is tangential in the disc $D = \{z: |z| < 1\}$ where u and v are the Poisson kernel with pole at $(1,0)$ and its conjugate, respectively. Thus by Corollary 2 the Brownian path in $D = \{z: |z| < 1\}$ which exits at $e^{i\theta}$ will spill out of $\Gamma(\sigma; e^{i\theta})$ arbitrarily close to $e^{i\theta}$ a.s.

We now return to the classical result of Littlewood (1927) in the disc $D = \{z: |z| < 1\}$. Given C: $v = u \log u$, with u, v as above, Littlewood showed there exists a bounded harmonic function h with the following property. Define C_ψ to be C rotated through an angle ψ. Thus for each ψ we have a tangential means of approach to $e^{i\psi}$. Then for a.e. $\psi \in [0, 2\pi)$, $\lim_{|z|\to 1,\ z\in C_\psi} h(z)$ does not exist. On the other hand, h has limits along Brownian paths and a typical Brownian path exiting D at $e^{i\psi}$ will hit C_ψ infinitely often on the way to the boundary. Again by infinitely often we mean for each $t < \tau_D$ the Brownian path hits C_ψ after t. This follows from Corollary 2. The upshot is that the set of C_ψ where h oscillates is too small for the Brownian path to hit infinitely often. In potential theoretic terms we can say that if $H(\psi) = \lim_{|z|\to 1,\ z\in\Gamma(\sigma;\, e^{i\psi})} h(z)$ and

$$C(\psi, \varepsilon) = \left\{z \in C_\psi\colon h(z) > H(\psi) + \varepsilon \text{ or } h(z) < H(\psi) - \varepsilon\right\},$$

then $C(\psi, \varepsilon)$ is minimal thin at $e^{i\psi}$. A good discussion of minimal thinness is given in Doob (1983). Also the paper of Brelot and Doob (1963) discusses the connection between fine limits and nontangential convergence. Burdzy (1986) has obtained results similar in flavor to Corollary 2 for Brownian excursions from smooth surfaces.

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