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A CENTRAL LIMIT THEOREM FOR STATIONARY ρ-MIXING SEQUENCES WITH INFINITE VARIANCE[1]

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A central limit theorem is proved for some strictly stationary ρ-mixing sequences with infinite second moments. The condition on the tails of the marginal distribution is the same as in the corresponding classic result for i.i.d. sequences. The mixing rate is essentially the slowest possible for this result.

1. Introduction. Khinchin (1935), Lévy (1935) and Feller (1935) showed that for some sequences of i.i.d. random variables with (just barely) infinite second moments, the partial sums are attracted to a normal law. A convenient reference for this result is Ibragimov and Linnik (1971), page 83, last four lines, and page 85, Theorem 2.6.3. Lin (1981) extended this classic result to some strictly stationary m-dependent sequences. Samur (1985), Corollary 3.3, extended it to some strictly stationary sequences which are ϕ-mixing with a polynomial mixing rate, with the random variables taking their values in a separable Hilbert space. The purpose of this note is to extend this classic result to some strictly stationary sequences of real-valued random variables which are ρ-mixing with a certain logarithmic mixing rate.

A number of other papers have been devoted to convergence in distribution for mixing sequences of random variables when second moments are not assumed to be finite. For stationary sequences, limit theorems involving the attraction of the partial sums to nonnormal stable laws can be found, e.g., in Davis (1983), Samur (1984) and the survey by Philipp (1986). For stationary ϕ-mixing sequences, the attraction of "trimmed" partial sums to normal laws was studied by Hahn, Kuelbs and Samur (1987). For ϕ-mixing arrays of random variables, attraction of the row sums to infinitely divisible laws (including special conditions for attraction to normal laws) was studied in, e.g., Bergström (1972), Krieger (1984) and Samur (1984, 1985).

Suppose $(X_k,\ k \in \mathbb{Z})$ is a strictly stationary sequence of real-valued random variables on a probability space $(\Omega, \mathscr{F}, P)$. For $-\infty \le J \le L \le \infty$ let $\mathscr{F}_J^L$ denote the σ-field generated by $(X_k,\ J \le k \le L)$. For any two σ-fields $\mathscr{A}$ and $\mathscr{B}$, define the "maximal correlation"

$$\rho(\mathscr{A}, \mathscr{B}) := \sup|\mathrm{Corr}(f, g)|, \qquad \text{real } f \in \mathscr{L}_2(\mathscr{A}), \qquad \text{real } g \in \mathscr{L}_2(\mathscr{B}).$$

For each $n \in \mathbb{N}$ define the dependence coefficient $\rho(n) := \rho(\mathscr{F}_{-\infty}^0, \mathscr{F}_n^\infty)$. The stationary sequence (X_k) is said to be "ρ-mixing" if $\rho(n) \to 0$ as $n \to \infty$.

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In the statement of our main result we shall use the following notation: The partial sums of our given sequence (X_k) are denoted by $S_n := X_1 + \cdots + X_n$. Indicator functions are denoted by $I(\cdot)$. Given a function g: $[0, \infty) \to [0, \infty)$ such that $g(x) > 0$ for all x sufficiently large, we say that $g(x)$ is "slowly varying" as $x \to \infty$ if $\forall\ t > 0$, $\lim_{x\to\infty} g(tx)/g(x) = 1$. Our main result is as follows.

THEOREM 1. *Suppose* $(X_k,\ k \in \mathbb{Z})$ *is a strictly stationary sequence of nondegenerate real-valued random variables. Suppose that*

$$H(c) := EX_0^2 I(|X_0| \le c) \text{ is slowly varying as } c \to \infty, \tag{1.1}$$

$$EX_0 = 0, \tag{1.2}$$

$$\rho(1) < 1, \tag{1.3}$$

and

$$\sum_{n=1}^{\infty} \rho(2^n) < \infty. \tag{1.4}$$

Then there exists a sequence $(a_n,\ n \in \mathbb{N})$ *of positive numbers with* $a_n \to \infty$ *as* $n \to \infty$, *such that* $S_n/a_n \to N(0,1)$ *in distribution as* $n \to \infty$.

In the case $EX_0^2 < \infty$ this result, under the same mixing rate (1.4), was established by Ibragimov (1975), Theorem 2.2 [with (1.3) replaced by an essentially weaker assumption]. Our attention here is on the case $EX_0^2 = \infty$. The assumption (1.1) is equivalent to

$$\lim_{c\to\infty} c^2 P(|X_0| > c)/EX_0^2 I(|X_0| \le c) = 0,$$

and (in either form) it is precisely the assumption used in the classic CLT for i.i.d. r.v.'s with infinite second moments. It implies that $E|X_0|^\alpha < \infty$ for every α, $0 \le \alpha < 2$, a fact which (for $\alpha = 1$) was implicitly used in the assumption (1.2). See Ibragimov and Linnik (1971), page 83, last four lines, and page 84, Theorems 2.6.3 and 2.6.4. The condition (1.3) is a simple and not too restrictive condition whose purpose is [with the assistance of (1.4)] to insure sufficient "growth" (in probability) of $|S_n|$. The mixing rate (1.4), used by Ibragimov (1975), Theorem 2.2, as noted above, is essentially sharp, even in the case of finite second moments. In Bradley (1987), Theorem 2, a strictly stationary sequence (X_k) is constructed with $EX_0^2 < \infty$ and $\operatorname{Var} S_n \to \infty$ as $n \to \infty$, with $\rho(1)$ arbitrarily small, with $\rho(n) \to 0$ at a rate that is essentially arbitrarily close to (1.4), such that S_n is partially attracted to a nondegenerate nonnormal law. Under stronger "moment" assumptions than just second moments, it is fairly well known what mixing rates for ρ-mixing will insure the CLT; cf. the CLT's in Ibragimov (1975), Theorem 2.1, and Peligrad (1987) and the counterexamples in Bradley (1987).

Theorem 1 will be proved in Section 3, after some preliminary work is done in Section 2.

The following notation will be used: The greatest integer $\le x$ will be denoted by $[x]$. Terms like a_b will be written as $a(b)$ when that is needed for typographical convenience. The notation $a_n \sim b_n$ will mean $\lim_{n\to\infty} a_n/b_n = 1$, and the notation $a_n \ll b_n$ will mean $a_n = O(b_n)$. Random variables are real unless specified otherwise.

2. Preliminaries. In this section we shall give some lemmas that will be used in Section 3 in the proof of Theorem 1. The first is a special case of Peligrad (1982), page 973, Lemma 3.4.

LEMMA 2.1. *Suppose $(r(n), n \in \mathbb{N})$ is a nonincreasing sequence of nonnegative numbers such that $\sum_{n=1}^{\infty} r(2^n) < \infty$. Then there exists a positive constant $D = D(r(1), r(2), \dots)$ such that the following holds:*

For every sequence $Y_1, Y_2, \dots, Y_m$ of square-integrable random variables such that the condition

$$\forall\, j, n \text{ such that } 1 \le j < j + n \le m, \quad \rho\big(\sigma(Y_1, \dots, Y_j), \sigma(Y_{j+n}, \dots, Y_m)\big) \le r(n),$$

holds, one has that

$$\operatorname{Var}(Y_1 + Y_2 + \cdots + Y_m) \le Dm \max_{1 \le k \le m} \operatorname{Var}(Y_k).$$

Here the notation $\sigma(\cdots)$ means the σ-field generated by $(\cdots)$. In her proof of this result, Peligrad gives an explicit value of such a constant D, namely $D := 8000\prod_{n=0}^{\infty}(1 + r([2^{n/3}]))$. For stationary sequences we need a similar bound in the opposite direction, and to avoid trivial counterexamples this will require the extra condition $r(1) < 1$. The existence of such a bound will be established in Lemma 2.3 below; Lemma 2.2 will be important in the proof of Lemma 2.3.

LEMMA 2.2. *Suppose $(r(n), n \in \mathbb{N})$ is a nonincreasing sequence of nonnegative numbers such that $r(1) < 1$ and $\lim_{n \to \infty} r(n) = 0$. Suppose $A > 0$. Then there exists a positive integer $N = N(A, r(1), r(2), \dots)$ such that the following holds:*

For every strictly stationary sequence $(X_k, k \in \mathbb{Z})$ of square-integrable random variables such that the condition

$$\forall\, n \ge 1, \qquad \rho(n) \le r(n)$$

holds, one has that

$$\forall\, n \ge N, \qquad \operatorname{Var} S_n \ge A \operatorname{Var} X_0. \tag{2.1}$$

PROOF. Suppose $(r(n), n \in \mathbb{N})$ and A are as in the hypothesis of Lemma 2.2. Our first task is to define the positive integer $N = N(A, r(1), r(2), \dots)$.

First, let J be a positive integer such that

$$J^2/(J^2 + J) > r(1) \tag{2.2}$$

and

$$r(J) \le 1/2. \tag{2.3}$$

Let $\varepsilon > 0$ be such that

$$\frac{J^2 - 4J^4\varepsilon}{(J^2 + J) + 6J^4\varepsilon} > r(1). \tag{2.4}$$

Define the positive number

$$B := 2(A^{1/2} + 2J)^2. \tag{2.5}$$

Let L be a positive integer such that

$$(L-1)\varepsilon/50 \geq B. \tag{2.6}$$

Let $\delta > 0$ be such that

$$L^2\delta \leq 1. \tag{2.7}$$

Let M be a positive integer such that

$$r(M) \leq \min\{\delta, \varepsilon, 1/50\}. \tag{2.8}$$

Finally, let N be a positive integer such that

$$N \geq \max\{LM, 2B^2\}. \tag{2.9}$$

Now suppose that (X_k) is strictly stationary and satisfies the conditions specified in Lemma 2.2. Our task now is to prove that (2.1) holds for this sequence.

The case $\operatorname{Var} X_0 = 0$ is trivial. Let us assume $\operatorname{Var} X_0 > 0$. Without loss of generality we assume that

$$EX_0 = 0 \quad \text{and} \quad EX_0^2 = 1. \tag{2.10}$$

To prove (2.1) we first need to prove a preliminary statement.

CLAIM 0. *There exists* n, $1 \leq n \leq N$, *such that* $\operatorname{Var} S_n \geq B$.

PROOF. For each $l \in \mathbb{Z}$ define the r.v.

$$Y_l := X_{M(l-1)+1} + X_{M(l-1)+2} + \cdots + X_{Ml}.$$

Then the sequence $(Y_l, l \in \mathbb{Z})$ is strictly stationary. [The notation $\rho(n)$ and S_n will, of course, continue to refer to the original sequence (X_k).] We shall divide our argument into three cases.

CASE I. $\operatorname{Var} Y_0 \leq 1/100$.

We shall first show by induction that $\forall\, j = 0, 1, 2, \ldots,$

$$\operatorname{Var} S(2^j) \geq 2^{j/2}. \tag{2.11}$$

First, (2.11) holds for $j = 0$ by (2.10).

Now suppose $j \geq 0$ is any integer satisfying (2.11). Then

$$\begin{aligned}\|S(2^{j+1})\|_2 &\geq \|S(2^j) + S(2^{j+1} + M) - S(2^j + M)\|_2 \\ &\quad - \|S(2^j + M) - S(2^j)\|_2 - \|S(2^{j+1}) - S(2^{j+1} + M)\|_2 \\ &\geq \|S(2^j) + S(2^{j+1} + M) - S(2^j + M)\|_2 - 2\|Y_0\|_2 \\ &\geq \|S(2^j) + S(2^{j+1} + M) - S(2^j + M)\|_2 - 1/5.\end{aligned}$$

Now by a simple argument and (2.8),

$$\begin{aligned}\operatorname{Var}\big(S(2^j) + S(2^{j+1} + M) - S(2^j + M)\big) &\geq [2 - 2r(M)]\operatorname{Var} S(2^j) \\ &\geq (1.96)\operatorname{Var} S(2^j).\end{aligned}$$

Hence [since $\|S(2^j)\|_2 \geq 2^{j/4} \geq 1$]

$$\|S(2^{j+1})\|_2 \geq (1.4)\|S(2^j)\|_2 - 1/5 \geq (1.2)\|S(2^j)\|_2.$$

Hence $\operatorname{Var} S(2^{j+1}) \geq 2^{1/2} \operatorname{Var} S(2^j) \geq 2^{(j+1)/2}$. This completes the induction step. Hence (2.11) holds for all $j \geq 0$ by induction.

Now $B > 1$ by (2.5). Let m be the positive integer such that $B^2 \leq 2^m < 2B^2$. Then $2^m < N$ by (2.9), and $\operatorname{Var} S(2^m) \geq 2^{m/2} \geq B$ by (2.11). Thus Claim 0 holds (with $n = 2^m$) for Case I.

Case II. $\operatorname{Var} Y_0 > 1/100$ and $\operatorname{Corr}(Y_0, Y_1) \leq -1/2 + \varepsilon$. Define the r.v.'s

$$V := JY_1 + (J-1)Y_2 + \cdots + 1 \cdot Y_J$$

and

$$W := JY_0 + (J-1)Y_{-1} + \cdots + 1 \cdot Y_{-J+1}.$$

Then

$$\begin{aligned}
\operatorname{Var} V &= \sum_{l=1}^{J} l^2 \operatorname{Var} Y_0 + 2\sum_{l=2}^{J} l(l-1)\operatorname{Cov}(Y_0, Y_1) \\
&\quad + 2\sum_{u=2}^{J-1} \sum_{l=u+1}^{J} l(l-u)\operatorname{Cov}(Y_0, Y_u) \\
&\leq (\operatorname{Var} Y_0)\left[\sum_{l=1}^{J} l^2 + 2\sum_{l=2}^{J} l(l-1)(-1/2+\varepsilon) + J^4 r(M)\right] \\
&= (\operatorname{Var} Y_0)\left[\sum_{l=1}^{J} l + 2\sum_{l=2}^{J} l(l-1)\varepsilon + J^4 r(M)\right] \\
&\leq (\operatorname{Var} Y_0)\left[J(J+1)/2 + 3J^4\varepsilon\right],
\end{aligned}$$

where the last step holds by (2.8). Also, clearly $\operatorname{Var} W = \operatorname{Var} V$.

Now $\operatorname{Cov}(V, W) = J^2 \operatorname{Cov}(Y_0, Y_1) + Z$, where Z is the sum of $J^2 - 1$ terms, each of the form $J_1 J_2 \operatorname{Cov}(Y_u, Y_v)$, where $1 \leq J_1, J_2 \leq J$ and $|u - v| \geq 2$. Hence

$$\begin{aligned}
\operatorname{Cov}(V, W) &\leq (\operatorname{Var} Y_0)\left[J^2(-1/2+\varepsilon) + J^4 r(M)\right] \\
&\leq (\operatorname{Var} Y_0)\left[-J^2/2 + 2J^4\varepsilon\right],
\end{aligned}$$

again using (2.8). Putting all this together, we obtain

$$\operatorname{Corr}(V, W) \leq \frac{-J^2 + 4J^4\varepsilon}{J(J+1) + 6J^4\varepsilon} < -r(1),$$

by (2.4). Hence $\rho(1) > r(1)$, but this contradicts the specified restrictions on (X_k) in Lemma 2.2. Consequently, Case II is vacuous.

Case III. $\operatorname{Var} Y_0 > 1/100$ and $\operatorname{Corr}(Y_0, Y_1) > -1/2 + \varepsilon$.

Then by (2.8), (2.7) and (2.6),

$$\begin{aligned}
\operatorname{Var} S_{ML} &= \operatorname{Var} \sum_{j=1}^{L} Y_j \\
&= L \operatorname{Var} Y_0 + 2(L-1)\operatorname{Cov}(Y_0, Y_1) \\
&\quad + \sum_{\substack{1 \le j, k \le L \\ |j-k| \ge 2}} \operatorname{Cov}(Y_j, Y_k) \\
&\ge (\operatorname{Var} Y_0)\left[L + 2(L-1)(-1/2 + \varepsilon) - L^2 r(M)\right] \\
&\ge (\operatorname{Var} Y_0)\left[1 + 2(L-1)\varepsilon - L^2\delta\right] \\
&\ge (1/100)\left[2(L-1)\varepsilon\right] \\
&\ge B.
\end{aligned}$$

Since $ML \le N$ by (2.9), Claim 0 holds for Case III (with $n = ML$). This completes the proof of Claim 0. □

Now let us complete the proof of Lemma 2.2. Using Claim 0, let $m \le N$ be a positive integer such that $\operatorname{Var} S_m \ge B$.

Suppose $n \ge N$. Then

$$\begin{aligned}
\|S_n\|_2 &\ge \|S_m + (S_{n+J} - S_{m+J})\|_2 - \|S_{m+J} - S_m\|_2 - \|S_n - S_{n+J}\|_2 \\
&\ge \|S_m + (S_{n+J} - S_{m+J})\|_2 - 2J,
\end{aligned}$$

by (2.10). Now, by (2.3) and (2.5),

$$\begin{aligned}
\operatorname{Var}(S_m + S_{n+J} - S_{m+J}) &\ge \operatorname{Var} S_m + \operatorname{Var}(S_{n+J} - S_{m+J}) \\
&\quad - 2r(J)(\operatorname{Var} S_m)^{1/2}(\operatorname{Var}(S_{n+J} - S_{m+J}))^{1/2} \\
&\ge (1 - r(J))\left[\operatorname{Var} S_m + \operatorname{Var}(S_{n+J} - S_{m+J})\right] \\
&\ge (1/2)B \\
&= (A^{1/2} + 2J)^2.
\end{aligned}$$

Hence $\|S_n\|_2 \ge (A^{1/2} + 2J) - 2J = A^{1/2}$; that is, $\operatorname{Var} S_n \ge A$. Since $n \ge N$ was arbitrary, (2.1) holds [by (2.10)]. This completes the proof of Lemma 2.2. □

Lemma 2.3. *Suppose $(r(n), n \in \mathbb{N})$ is a nonincreasing sequence of nonnegative numbers such that $r(1) < 1$ and $\sum_{n=1}^{\infty} r(2^n) < \infty$. Then there exists a positive constant $C = C(r(1), r(2), \dots)$ such that the following holds:*

For every strictly stationary sequence $(X_k, k \in \mathbb{Z})$ of square-integrable random variables such that the condition

$$\forall\, n \ge 1, \qquad \rho(n) \le r(n)$$

holds, one has that

$$\forall\, n \ge 1, \qquad \operatorname{Var} S_n \ge Cn \operatorname{Var} X_0. \tag{2.12}$$

PROOF. Suppose $(r(n),\ n \in \mathbb{N})$ is as in the hypothesis of Lemma 2.3. Our first task is to define the positive constant $C = C(r(1), r(2), \dots)$.

First, let ε, $0 < \varepsilon < 1/2$, be such that

$$[2(1-\varepsilon)]^{1/2} - \varepsilon \geq 2^{1/3}. \tag{2.13}$$

Let L be a positive integer such that

$$r([2^{L/6}]) \leq \varepsilon. \tag{2.14}$$

Let $A > 0$ be such that

$$2 \cdot 2^{L/6}/A^{1/2} \leq \varepsilon. \tag{2.15}$$

Define the positive integer N by $N := N(A, r(1), r(2), \dots)$ from Lemma 2.2.
Define the positive integers $K_0, K_1, K_2, \dots$ by $K_l := [2^{(L+l)/6}]$.
Define the positive constant

$$B := \prod_{l=0}^{\infty} [1 - r(K_l)][1 - 2^{-l/6}\varepsilon]^2. \tag{2.16}$$

[Recall that $\varepsilon < 1/2$ and $r(1) < 1$.]

Finally, define the positive constant C by

$$C := \min\{A/N^3, AB/(2N)\}. \tag{2.17}$$

Now suppose that (X_k) is strictly stationary and satisfies the conditions specified in Lemma 2.3. Our task now is to prove that (2.12) holds for this sequence.

The case $\operatorname{Var} X_0 = 0$ is trivial. Let us assume $\operatorname{Var} X_0 > 0$. Without loss of generality we assume that

$$EX_0 = 0 \quad \text{and} \quad EX_0^2 = 1. \tag{2.18}$$

By the definition of N we have

$$\forall\, n \geq N, \qquad ES_n^2 \geq A. \tag{2.19}$$

If $n \geq N$ and $m = 2n$ or $2n+1$, then

$$\|S_m\|_2 \geq \|S_n + S_{2n+K(0)} - S_{n+K(0)}\|_2 - \|S_{n+K(0)} - S_n\|_2 - \|S_m - S_{2n+K(0)}\|_2$$

$$\begin{aligned} &\geq [2(1 - r(K_0))]^{1/2}\|S_n\|_2 - 2K_0 \\ &\geq [2(1-\varepsilon)]^{1/2}\|S_n\|_2 - 2K_0 A^{-1/2}\|S_n\|_2 \\ &\geq \left[(2(1-\varepsilon))^{1/2} - \varepsilon\right]\|S_n\|_2 \\ &\geq 2^{1/3}\|S_n\|_2, \end{aligned} \tag{2.20}$$

by (2.18), the definition of K_0, (2.14), (2.19), (2.15) and (2.13). Now for any $j \geq 0$ and any m, $2^jN \leq m < 2^{j+1}N$, one can define the integers $m_0, m_1, \dots, m_j$ such that $m_j = m$ and (if $j \geq 1$) $[m_i/2] = m_{i-1}$, $\forall\, i = 1, \dots, j$; these integers satisfy $2^iN \leq m_i < 2^{i+1}N$, $\forall\, i = 0, \dots, j$. Therefore by (2.19) and repeated applications of (2.20) one has

$$\forall\, j \geq 0, \forall\, m \in \{2^jN, \dots, 2^{j+1}N - 1\}, \qquad \|S_m\|_2 \geq 2^{j/3}A^{1/2}. \tag{2.21}$$

If $j \geq 0$, $2^j N \leq n < 2^{j+1}N$, and $m = 2n$ or $2n+1$, then by (2.18), the definition of K_j, (2.21) and (2.15), one has that

$$\begin{aligned} \|S_m\|_2 &\geq \|S_n + S_{2n+K(j)} - S_{n+K(j)}\|_2 - \|S_{n+K(j)} - S_n\|_2 - \|S_m - S_{2n+K(j)}\|_2 \\ &\geq \left[2\left(1 - r(K_j)\right)\right]^{1/2}\|S_n\|_2 - 2K_j \\ &\geq \left[2\left(1 - r(K_j)\right)\right]^{1/2}\|S_n\|_2 \\ &\quad -2 \cdot 2^{(L+j)/6}\left(2^{j/3}A^{1/2}\right)^{-1}\|S_n\|_2 \\ &\geq \left[\left[2\left(1 - r(K_j)\right)\right]^{1/2} - 2^{-j/6}\varepsilon\right]\|S_n\|_2 \\ &\geq \left[2\left(1 - r(K_j)\right)\right]^{1/2}\left[1 - 2^{-j/6}\varepsilon\right]\|S_n\|_2, \end{aligned} \tag{2.22}$$

where in the last step we are using the fact $r(K_j) \leq r(K_0) \leq \varepsilon < 1/2$ [see (2.14)]. Consequently,

$$\begin{aligned} &\forall\, j \geq 0, \forall\, m \in \{2^j N, \dots, 2^{j+1}N - 1\}, \\ &\|S_m\|_2 \geq 2^{j/2}A^{1/2}\prod_{i=0}^{j-1}\left[1 - r(K_i)\right]^{1/2}\left[1 - 2^{-i/6}\varepsilon\right] \\ &\qquad \geq 2^{j/2}(AB)^{1/2}. \end{aligned} \tag{2.23}$$

Here the first inequality is derived from (2.19) and (2.22) in the same way that (2.21) was derived from (2.19) and (2.20); and the second inequality comes from (2.16). By (2.23) we have that

$$\begin{aligned} &\forall\, j \geq 0, \forall\, n \in \{2^j N, \dots, 2^{j+1}N - 1\}, \\ &n^{-1}\operatorname{Var} S_n \geq \left(2^{j+1}N\right)^{-1}\left(2^j AB\right) = AB/(2N). \end{aligned} \tag{2.24}$$

Also, if $1 \leq n \leq N-1$, then $Nn \geq N$ and $\operatorname{Var} S_{Nn} \leq N^2 \operatorname{Var} S_n$ and hence $n^{-1}\operatorname{Var} S_n \geq N^{-1}N^{-2}\operatorname{Var} S_{Nn} \geq N^{-3}A$ by (2.19). This and (2.24) and (2.17) together imply (2.12). This completes the proof of Lemma 2.3. □

The next three lemmas are elementary.

Lemma 2.4. *If V is a bounded random variable, then*

$$E\min\{|V - EV|^2, |V - EV|^3\} \leq 4(EV)^2 + 8E\min\{V^2, |V|^3\}.$$

Proof. Suppose x and r are real numbers. If $|x| \leq |r|$, then $\min\{|x-r|^2, |x-r|^3\} \leq |x-r|^2 \leq 4r^2$. If $|x| \geq |r|$, then $\min\{|x-r|^2, |x-r|^3\} \leq \min\{4x^2, 8|x|^3\}$. In either case, $\min\{|x-r|^2, |x-r|^3\} \leq 4r^2 + 8\min\{x^2, |x|^3\}$. Letting $r = EV$ and $x = V(\omega)$ for sample points $\omega \in \Omega$ and taking expectations, we obtain Lemma 2.4. □

The next lemma holds by Withers (1981), page 512, Theorem 1.1, and trivial arithmetic.

LEMMA 2.5. *Suppose $\mathscr{A}$ and $\mathscr{B}$ are σ-fields, V and W are complex-valued random variables such that $V \in \mathscr{L}_2(\mathscr{A})$ and $W \in \mathscr{L}_2(\mathscr{B})$, and y and z are complex numbers. Then* $|EVW - EVEW| \le \rho(\mathscr{A}, \mathscr{B})\|V - y\|_2\|W - z\|_2$.

LEMMA 2.6. *If $y_1, y_2, \dots, y_m, z_1, z_2, \dots, z_m$ are complex numbers in the closed unit disc, then*

$$\left|\prod_{k=1}^{m} y_k - \prod_{k=1}^{m} z_k\right| \le \sum_{k=1}^{m} |y_k - z_k|.$$

The proof is a trivial induction argument.

3. Proof of Theorem 1. Suppose (X_k) is a strictly stationary sequence satisfying the hypothesis of Theorem 1.

It follows from (1.1) that $\lim_{x \to \infty} x^{-2}H(x) = 0$. Let M^* be a positive integer such that

$$\sup_{x>0} x^{-2}H(x) > 1/M^*.$$

For each $n \ge M^*$ define the positive number

$$t_n := \sup\{x > 0 \colon x^{-2}H(x) \ge 1/n.\}. \tag{3.1}$$

As in the classic i.i.d. case [under (1.1)], these are the values at which the X_k's will be truncated in our proof. We need a few standard elementary properties of these t_n's; for the reader's convenience, we shall quickly give standard proofs of these properties here.

Note that by (3.1),

$$t_n \to \infty \text{ monotonically as } n \to \infty. \tag{3.2}$$

At each $c > 0$, $\lim_{x \to c-} H(x) \le \lim_{x \to c+} H(x)$ [both limits exist, and the latter one is $H(c)$], and hence $\lim_{x \to c-} x^{-2}H(x) \le \lim_{x \to c+} x^{-2}H(x)$. This and (3.1) together imply, by a trivial argument,

$$\forall\, n \ge M^*, \qquad t_n^{-2}H(t_n) = 1/n. \tag{3.3}$$

Next we need a couple of lemmas.

LEMMA 3.1.

(a) $$\lim_{n \to \infty} nP(|X_0| > t_n) = 0$$

and

(b) $$\lim_{n \to \infty} n^{1/2}(H(t_n))^{-1/2} E|X_0|I(|X_0| > t_n) = 0.$$

PROOF. Let ε, $0 < \varepsilon < 1$, be arbitrary but fixed. It suffices to prove that

$$\limsup_{n \to \infty} nP(|X_0| > t_n) \le \varepsilon \tag{3.4}$$

and

$$\limsup_{n \to \infty} n^{1/2}(H(t_n))^{-1/2} E|X_0|I(|X_0| > t_n) \le \varepsilon. \tag{3.5}$$

Let $\alpha > 0$ be such that $\alpha \sum_{j=0}^{\infty} 2^{-j}(1+\alpha)^j \le \varepsilon$. Let $c_0 > 0$ be such that $\forall\ c \ge c_0$, $H(2c) \le (1+\alpha)H(c)$. For each $c \ge c_0$ we have by induction that $\forall\ j \ge 1$, $H(2^j c) \le (1+\alpha)^j H(c)$. Hence $\forall\ c \ge c_0$, $\forall\ j \ge 0$,

$$\begin{aligned} E|X_0|^2 I\big(2^j c < |X_0| \le 2^{j+1}c\big) &= H(2^{j+1}c) - H(2^j c) \\ &\le \alpha H(2^j c) \le \alpha(1+\alpha)^j H(c). \end{aligned}$$

For each $n \ge M^*$ such that $t_n \ge c_0$,

$$\begin{aligned} P(|X_0| > t_n) &= \sum_{j=0}^{\infty} P\big(2^j t_n < |X_0| \le 2^{j+1} t_n\big) \\ &\le \sum_{j=0}^{\infty} \big(2^j t_n\big)^{-2} E|X_0|^2 I\big(2^j t_n < |X_0| \le 2^{j+1} t_n\big) \\ &\le \sum_{j=0}^{\infty} \big(2^j t_n\big)^{-2} \alpha(1+\alpha)^j H(t_n) \\ &\le \varepsilon t_n^{-2} H(t_n) \\ &= \varepsilon/n, \end{aligned}$$

by (3.3). Hence (3.4) holds.

For each $n \ge M^*$ such that $t_n \ge c_0$,

$$\begin{aligned} E|X_0| I(|X_0| > t_n) &= \sum_{j=0}^{\infty} E|X_0| I\big(2^j t_n < |X_0| \le 2^{j+1} t_n\big) \\ &\le \sum_{j=0}^{\infty} \big(2^j t_n\big)^{-1} E|X_0|^2 I\big(2^j t_n < |X_0| \le 2^{j+1} t_n\big) \\ &\le \sum_{j=0}^{\infty} \big(2^j t_n\big)^{-1} \alpha(1+\alpha)^j H(t_n) \\ &\le \varepsilon t_n^{-1} H(t_n). \end{aligned}$$

Hence by (3.3) one has (3.5). This completes the proof of Lemma 3.1. □

LEMMA 3.2. *As $n \to \infty$,*

$$E \min\left\{ \frac{X_0^2 I(|X_0| \le t_n)}{nH(t_n)}, \frac{|X_0|^3 I(|X_0| \le t_n)}{n^{3/2} H^{3/2}(t_n)} \right\} = o\left(\frac{1}{n}\right). \tag{3.6}$$

PROOF. Suppose $0 < c < 1$. Then by (3.3), for all $n \ge M^*$,

$$\begin{aligned} [\text{l.h.s. of (3.6)}] &\le n^{-3/2} H^{-3/2}(t_n) E|X_0|^3 I(|X_0| \le ct_n) \\ &\quad + n^{-1} H^{-1}(t_n) E X_0^2 I(ct_n < |X_0| \le t_n) \\ &\le n^{-3/2} H^{-3/2}(t_n) ct_n E X_0^2 I(|X_0| \le ct_n) \\ &\quad + n^{-1} H^{-1}(t_n) [H(t_n) - H(ct_n)] \\ &= (c/n) H^{-1}(t_n) E X_0^2 I(|X_0| \le ct_n) \\ &\quad + n^{-1} H^{-1}(t_n) [H(t_n) - H(ct_n)] \\ &\le (c/n) + n^{-1} H^{-1}(t_n) [H(t_n) - H(ct_n)]. \end{aligned}$$

Hence [l.h.s. of (3.6)] $\le 2c/n$ for all n sufficiently large, by (3.2) and (1.1). Since c can be taken arbitrarily small, Lemma 3.2 follows. □

Recall that $E|X_0| < \infty$ [a simple consequence of (1.1), as mentioned in Section 1]. By (3.2) and (1.2),

$$\lim_{n\to\infty} EX_0 I(|X_0| \le t_n) = 0. \tag{3.7}$$

It follows that

$$\operatorname{Var} X_0 I(|X_0| \le t_n) \sim H(t_n), \quad \text{as } n \to \infty. \tag{3.8}$$

This completes our review of elementary properties of the t_n's. Now we start the main part of the proof of Theorem 1.

(3.9) Define the positive constants $C := C(r(1), r(2), \dots)$ and $D := D(r(1), r(2), \dots)$ from Lemmas 2.3 and 2.1, with $r(n) := \rho(n)$, $\forall\, n \in \mathbb{N}$.

[Recall (1.3) and (1.4).]

Define the following random variables:

$$\forall\, n \ge M^*, \forall\, k \in \mathbb{Z}, \qquad X_k^{(n)} := X_k I(|X_k| \le t_n) - EX_k I(|X_k| \le t_n);$$

and

$$\forall\, n \ge M^*, \forall\, m \in \mathbb{N}, \qquad S_m^{(n)} := X_1^{(n)} + X_2^{(n)} + \cdots + X_m^{(n)}.$$

Note that by the definition of C and D in (3.9),

$$\forall\, n \ge M^*, \forall\, m \in \mathbb{N}, \qquad Cm\|X_0^{(n)}\|_2^2 \le \|S_m^{(n)}\|_2^2 \le Dm\|X_0^{(n)}\|_2^2; \tag{3.10}$$

and

$$\forall\, n \ge M^*, \forall \text{ finite nonempty sets } S \subset \mathbb{N},$$

$$\Big\| \sum_{k\in S} X_k^{(n)} \Big\|_2^2 \le D\,(\operatorname{card} S)\|X_0^{(n)}\|_2^2. \tag{3.11}$$

For each $n \ge M^*$ define the number

$$A_n := \|S_n^{(n)}\|_2. \tag{3.12}$$

By (3.10) and (3.8),

$$A_n \ll n^{1/2} H^{1/2}(t_n) \ll A_n, \quad \text{as } n \to \infty. \tag{3.13}$$

In particular, $A_n \to \infty$ as $n \to \infty$. Hence, to prove Theorem 1 it suffices to prove that $S_n/A_n \to N(0,1)$ in distribution as $n \to \infty$.

By Lemma 3.1(a),

$$\lim_{n\to\infty} P(|X_k| > t_n \text{ for some } k = 1, \dots, n) = 0.$$

Also, by (1.2), (3.13) and Lemma 3.1(b),

$$\lim_{n\to\infty} A_n^{-1} E \sum_{k=1}^{n} X_k I(|X_k| \le t_n) = 0.$$

Hence, to prove Theorem 1 it suffices to prove that $S_n^{(n)}/A_n \to N(0,1)$ in distribution as $n \to \infty$.

To prove Theorem 1 it suffices to prove that

$$\forall\, t \in \mathbb{R}, \qquad \lim_{n\to\infty} E\left(\exp\left[itS_n^{(n)}/A_n\right]\right) = \exp(-t^2/2).$$

This limit holds trivially for $t = 0$, so it needs to be proved only for $t \neq 0$.

Let $T \neq 0$ be an arbitrary but fixed real number. Let $\varepsilon > 0$ be arbitrary but fixed.

(3.14) To prove Theorem 1 it suffices to prove that $\exists\, N^* \ge 1$ such that $\forall\, n \ge N^*$, $|E \exp[iTS_n^{(n)}/A_n] - \exp(-T^2/2)| \le \varepsilon$.

To carry out this proof, our first step is to define a positive integer N^*. This will require the definition of some other parameters.

Let J be a positive integer such that

$$T^2 D \sum_{j=1}^{\infty} \rho([2^{J+j/2}]) \le \varepsilon/6, \tag{3.15}$$

$$1 - \varepsilon/|6T| \le \prod_{j=1}^{\infty} \left(1 - \rho([2^{J+j/2}])\right) \tag{3.16}$$

and

$$\prod_{j=1}^{\infty} \left(1 + \rho([2^{J+j/2}])\right) \le 1 + \varepsilon/|6T|. \tag{3.17}$$

These conditions are somewhat redundant, but that is harmless.)

Define the positive integers $K_1 \le K_2 \le K_3 \le \cdots$ by

$$K_j := [2^{J+j/2}]. \tag{3.18}$$

Let p^* be a positive integer such that

$$\frac{2D^{1/2}}{C^{1/2}(p^*)^{1/2}} \left(\sum_{j=1}^{\infty} 2^{J-j/2} \right)^{1/2} \le \varepsilon/|6T|. \tag{3.19}$$

Let L^* be a positive integer such that

$$\forall\, j \ge 2^{L^*}, \qquad |1 - T^2/(2j)| \le 1 \tag{3.20}$$

and

$$\left|[1 - T^2/(2j)]^j - \exp(-T^2/2)\right| \le \varepsilon/6.$$

Now, finally, let N^* be a positive integer such that

$$N^* \geq M^*, \tag{3.21}$$

$$N^* \geq (2p^*) \cdot 2^{L^*}, \tag{3.22}$$

$$\forall\, n \geq N^*, \qquad \|X_0^{(n)}\|_2 > 0, \tag{3.23}$$

and

$$\begin{aligned} \forall\, n \geq N^*, \qquad & \frac{128(p^*)^4 T^2}{Cn\|X_0^{(n)}\|_2^2} [EX_0 I(|X_0| \leq t_n)]^2 \\ & + \frac{400(p^*)^4 \max\{T^2, |T|^3\}}{\min\{C, C^{3/2}\}} \\ & \times E \min\left\{ \frac{X_0^2 I(|X_0| \leq t_n)}{n\|X_0^{(n)}\|_2^2}, \frac{|X_0|^3 I(|X_0| \leq t_n)}{n^{3/2}\|X_0^{(n)}\|_2^3} \right\} \\ & \leq \varepsilon/(6n). \end{aligned} \tag{3.24}$$

Condition (3.21) insures that the r.v.'s $X_k^{(n)}$ and $S_m^{(n)}$ are defined $\forall\, n \geq N^*$. Condition (3.23) is justified by (3.2) and our assumption (in Theorem 1) that X_0 is nondegenerate. To justify (3.24), first note that by (3.8) one has that (3.6) in Lemma 3.2 remains valid if $H(t_n)$ and $H^{3/2}(t_n)$ there are replaced by $\|X_0^{(n)}\|_2^2$ and $\|X_0^{(n)}\|_2^3$; this and (3.7) and (3.8) together imply that in (3.24) the l.h.s. is $o(1/n)$ as $n \to \infty$. (Remark: It is easy to see that in fact $C \leq 1$ and hence $\min\{C, C^{3/2}\} = C^{3/2}$, but this is not important.)

Let N be an arbitrary but fixed integer such that $N \geq N^*$. Referring to (3.14), we see that the following is true:

(3.25) To prove Theorem 1 it suffices to prove that $|E \exp[iTS_N^{(N)}/A_N] - \exp(-T^2/2)| \leq \varepsilon$.

To prove this inequality we shall use a blocking argument. This will require the definition of more parameters.

Referring to (3.22), let L be the positive integer such that

$$p^* \leq N/2^L < 2p^*. \tag{3.26}$$

Note that [by (3.22) and the fact $N \geq N^*$]

$$L \geq L^*. \tag{3.27}$$

Let p be the positive integer such that

$$p \cdot 2^L \leq N < (p+1) \cdot 2^L. \tag{3.28}$$

Note that by (3.26) and (3.28),

$$p^* \leq p < 2p^*. \tag{3.29}$$

Let us partition $\mathbb{N}$ into disjoint blocks of consecutive integers, leaving no gaps between the blocks. The order of the blocks is $G(1), G(2), G(3), \ldots,$ and the cardinalities will be

$$(3.30)\quad \operatorname{card} G(j) = \begin{cases} p, & \text{if } j \text{ is odd;} \\ K_l, & \text{where } l \text{ is the positive integer such that } j/2^l \text{ is} \\ & \text{an odd integer, if } j \text{ is even.} \end{cases}$$

Henceforth, we shall deal only with the blocks, $G(1), G(2), G(3), \dots, G(2^{L+1} - 1)$. For each $l = 1, \dots, L$, there are exactly 2^{L-l} (even) integers $j \in \{1, 2, 3, \dots, 2^{L+1} - 1\}$ such that $j/2^l$ is an odd integer. Hence

$$(3.31)\quad \operatorname{card}\big(G(2) \cup G(4) \cup G(6) \cup \cdots \cup G(2^{L+1} - 2)\big) = \sum_{l=1}^{L} 2^{L-l} K_l.$$

Hence by (3.28) [and (3.30)],

$$(3.32)\quad \begin{aligned} N &\le 2^L p + (2^L - 1) = 2^L p + \sum_{l=1}^{L} 2^{L-l} \\ &\le 2^L p + \sum_{l=1}^{L} 2^{L-l} K_l \\ &= \operatorname{card}\big(G(1) \cup G(2) \cup G(3) \cup \cdots \cup G(2^{L+1} - 1)\big) \\ &\le N + \sum_{l=1}^{L} 2^{L-l} K_l \end{aligned}$$

[where the last step comes from (3.28)]. We shall come back to (3.31) and (3.32) later on.

For each $j = 1, 2, \dots, 2^{L+1} - 1$ define the random variable

$$U_j := \sum_{k \in G(j)} X_k^{(N)}.$$

In what follows, we shall work mostly with the r.v.'s U_j [also denoted $U(j)$] for *odd* j. The following (perhaps peculiar) notation will be helpful: For *even* positive integers j_1 and j_2 such that $0 \le j_1 < j_2 \le 2^{L+1}$, define the r.v.

$$V(j_1, j_2) := U(j_1 + 1) + U(j_1 + 3) + U(j_1 + 5) + \cdots + U(j_2 - 1).$$

The sequence $U_1, U_3, U_5, \dots, U(2^{L+1} - 1)$ is in general not stationary, but it has a useful property [to be stated in (3.33) below] akin to stationarity. Suppose l is any integer such that $1 \le l \le L$. If $1 \le j \le 2^l - 1$, then for the integer m such that $j/2^m$ is an odd integer, we have that $m < l$ and hence $(2^l + j)/2^m$ is an odd integer, and hence $\operatorname{card} G(2^l + j) = \operatorname{card} G(j)$ by (3.30). Consequently, if we denote $u := \operatorname{card}(G(1) \cup G(2) \cup G(3) \cup \cdots \cup G(2^l))$ and use the notation $u + S := \{u + s\colon s \in S\}$ for sets S of positive integers, we have (by induction on j) that $G(2^l + j) = u + G(j)$, $\forall\, j = 1, 2, \dots, 2^l - 1$. In particular, if we denote

$$S := G(1) \cup G(3) \cup G(5) \cup \cdots \cup G(2^l - 1)$$

and

$$S^* := G(2^l + 1) \cup G(2^l + 3) \cup G(2^l + 5) \cup \cdots \cup G(2^{l+1} - 1),$$

then $S^* = u + S$, $V(2^l, 2^{l+1}) = \sum_{k \in S^*} X_k^{(N)} = \sum_{k \in S} X_{k+u}^{(N)}$, and $V(0, 2^l) = \sum_{k \in S} X_k^{(N)}$. Because of the stationarity of the sequence $(X_k^{(N)}, k \in \mathbb{Z})$, this implies the following property:

(3.33) For each $l = 1, \ldots, L$, the random variables $V(0, 2^l)$ and $V(2^l, 2^{l+1})$ have the same distribution.

If $1 \le l \le L$, then the random variables $V(0, 2^l)$ and $V(2^l, 2^{l+1})$ are "separated" by the block $G(2^l)$, whose cardinality is K_l [by (3.30)]. Hence, by (3.33), the fact that $V(0, 2^{l+1}) = V(0, 2^l) + V(2^l, 2^{l+1})$, and a simple calculation,

$$\text{(3.34)} \quad \forall\, l=1,\ldots,L, \qquad 2(1 - \rho(K_l))\|V(0,2^l)\|_2^2 \le \|V(0,2^{l+1})\|_2^2 \le 2(1 + \rho(K_l))\|V(0,2^l)\|_2^2.$$

Similarly, using also Lemma 2.5 and the trivial inequality $|e^{i\theta} - 1| \le |\theta|$ for real θ, we also have

$$\text{(3.35)} \quad \begin{aligned} &\forall\, l=1,\ldots,L, \forall\, t \in \mathbb{R}, \\ &\left|E\exp[itV(0,2^{l+1})] - \left[E\exp[itV(0,2^l)]\right]^2\right| \\ &\quad \le \rho(K_l)E\left|\exp[itV(0,2^l)] - 1\right|^2 \\ &\quad \le \rho(K_l)E\left|tV(0,2^l)\right|^2 \\ &\quad = \rho(K_l)t^2\|V(0,2^l)\|_2^2 \\ &\quad \le \rho(K_l)t^2 D \cdot 2^{l-1}\|U_1\|_2^2. \end{aligned}$$

[The last step, $\|V(0,2^l)\|_2^2 \le D \cdot 2^{l-1}\|U_1\|_2^2$, is easy to verify from the definition of D in (3.9).] We shall come back to (3.35) later on. First, we need to get bounds on the variances of some random variables.

In what follows, it should be kept in mind that $\|S_m^{(N)}\|_2 > 0$ for all $m \ge 1$ by (3.23), the fact $N \ge N^*$ and (3.10).

By (3.34) and induction [and the fact that $V(0,2) = U_1$],

$$2^L\left[\prod_{l=1}^{L}(1 - \rho(K_l))\right]\|U_1\|_2^2 \le \|V(0,2^{L+1})\|_2^2 \le 2^L\left[\prod_{l=1}^{L}(1 + \rho(K_l))\right]\|U_1\|_2^2,$$

and hence by (3.16), (3.17) and (3.18),

$$\text{(3.36)} \qquad 1 - \varepsilon/|6T| \le \|V(0,2^{L+1})\|_2 / \left(2^{L/2}\|U_1\|_2\right) \le 1 + \varepsilon/|6T|.$$

Now by (3.31) and (3.11),

$$\|U_2 + U_4 + U_6 + \cdots + U(2^{L+1} - 2)\|_2^2 \le D\left(\sum_{l=1}^{L} 2^{L-l}K_l\right)\|X_0^{(N)}\|_2^2.$$

Also, by (3.32), $U_1 + U_2 + U_3 + \cdots + U(2^{L+1} - 1) - S_N^{(N)}$ is the sum of at most $\sum_{l=1}^{L} 2^{L-l} K_l$ distinct $X_k^{(N)}$'s, and hence

$$\|U_1 + U_2 + U_3 + \cdots + U(2^{L+1} - 1) - S_N^{(N)}\|_2^2 \le D\left(\sum_{l=1}^{L} 2^{L-l} K_l\right) \|X_0^{(N)}\|_2^2.$$

Consequently,

$$\begin{aligned}\|V(0, 2^{L+1}) - S_N^{(N)}\|_2 &\le \|U_1 + U_2 + U_3 + \cdots + U(2^{L+1} - 1) - S_N^{(N)}\|_2 \\ &\quad + \|U_2 + U_4 + U_6 + \cdots + U(2^{L+1} - 2)\|_2 \\ &\le 2D^{1/2}\left(\sum_{l=1}^{L} 2^{L-l} K_l\right)^{1/2} \|X_0^{(N)}\|_2.\end{aligned}$$

Now $\|U_1\|_2 \ge C^{1/2} p^{1/2} \|X_0^{(N)}\|_2$ by (3.10); and hence by (3.18), (3.19) and (3.29) we have

$$\begin{aligned}&\|V(0, 2^{L+1}) - S_N^{(N)}\|_2 \\ &\quad \le 2D^{1/2} \cdot 2^{L/2}\left(\sum_{l=1}^{L} 2^{J-l/2}\right)^{1/2} \|U_1\|_2 / (C^{1/2} p^{1/2}) \\ &\quad \le 2^{L/2} \|U_1\|_2 \varepsilon / |6T|.\end{aligned} \tag{3.37}$$

Let us now turn our attention to characteristic functions. Using (3.12), we first present a detailed version of (3.25).

(3.38) To prove Theorem 1 it suffices to prove the following five statements:

(A) $$\left| E \exp\left[iTS_N^{(N)} / \|S_N^{(N)}\|_2\right] - E \exp\left[iTS_N^{(N)} / \left(2^{L/2} \|U_1\|_2\right)\right] \right| \le \varepsilon/3,$$

(B) $$\begin{aligned}&\Big| E \exp\left[iTS_N^{(N)} / \left(2^{L/2} \|U_1\|_2\right)\right] \\ &\quad - E \exp\left[iTV(0, 2^{L+1}) / \left(2^{L/2} \|U_1\|_2\right)\right] \Big| \le \varepsilon/6,\end{aligned}$$

(C) $$\begin{aligned}&\Big| E \exp\left[iTV(0, 2^{L+1}) / \left(2^{L/2} \|U_1\|_2\right)\right] \\ &\quad - \left[E \exp\left[i(T/2^{L/2}) U_1 / \|U_1\|_2\right]\right]^{2^L} \Big| \le \varepsilon/6,\end{aligned}$$

(D) $$\left| \left[E \exp\left[i(T/2^{L/2}) U_1 / \|U_1\|_2\right]\right]^{2^L} - \left[1 - (1/2) T^2 / 2^L\right]^{2^L} \right| \le \varepsilon/6,$$

and

(E) $$\left| \left[1 - (1/2) T^2 / 2^L\right]^{2^L} - \exp(-T^2/2) \right| \le \varepsilon/6.$$

PROOF OF (A). Using the trivial inequality $|e^{i\theta} - e^{i\phi}| \le |\theta - \phi|$ for real θ and ϕ, we have

$$
\begin{aligned}
[\text{l.h.s. of (A)}] &\le E|TS_N^{(N)}/\|S_N^{(N)}\|_2 - TS_N^{(N)}/(2^{L/2}\|U_1\|_2)| \\
&= |T| \cdot \frac{E|S_N^{(N)}|}{\|S_N^{(N)}\|_2} \cdot \left|1 - \frac{\|S_N^{(N)}\|_2}{2^{L/2}\|U_1\|_2}\right| \\
&\le |T| \cdot 1 \cdot \left[\left|1 - \frac{\|V(0,2^{L+1})\|_2}{2^{L/2}\|U_1\|_2}\right| \right. \\
&\qquad \left. + \left|\frac{\|V(0,2^{L+1})\|_2 - \|S_N^{(N)}\|_2}{2^{L/2}\|U_1\|_2}\right|\right] \\
&\le |T| \cdot \left[\frac{\varepsilon}{|6T|} + \frac{\|V(0,2^{L+1}) - S_N^{(N)}\|_2}{2^{L/2}\|U_1\|_2}\right] \\
&\le |T| \cdot [\varepsilon/|6T| + \varepsilon/|6T|] = \varepsilon/3,
\end{aligned}
$$

by (3.36) and (3.37). □

PROOF OF (B). Using (3.37) and arguing as in the proof of (A),

$$
\begin{aligned}
[\text{l.h.s. of (B)}] &\le E\left|T\left[\frac{S_N^{(N)} - V(0,2^{L+1})}{2^{L/2}\|U_1\|_2}\right]\right| \\
&\le |T| \cdot \frac{\|S_N^{(N)} - V(0,2^{L+1})\|_2}{2^{L/2}\|U_1\|_2} \le \varepsilon/6. \qquad \square
\end{aligned}
$$

PROOF OF (C). For each $t \in \mathbb{R}$ and each $l = 1,2,\dots, L$, by (3.35) and Lemma 2.6,

$$
\begin{aligned}
&\left|[E\exp[itV(0,2^{l+1})]]^{2^{L-l}} - [E\exp[itV(0,2^l)]]^{2^{L+1-l}}\right| \\
&\quad \le 2^{L-l}\rho(K_l)t^2D \cdot 2^{l-1}\|U_1\|_2^2 \\
&\quad \le 2^L\rho(K_l)t^2D\|U_1\|_2^2.
\end{aligned}
$$

Hence by induction, for each t,

$$
\begin{aligned}
&\left|E\exp[itV(0,2^{L+1})] - [E\exp[itV(0,2)]]^{2^L}\right| \\
&\quad \le 2^Lt^2D\|U_1\|_2^2 \sum_{l=1}^{L} \rho(K_l).
\end{aligned}
$$

In particular, letting $t = T/(2^{L/2}\|U_1\|_2)$ and keeping in mind that $U_1 = V(0,2)$, we have by (3.15) and (3.18),

$$[\text{l.h.s. of (C)}] \le 2^L\big[T^2/\big(2^L\|U_1\|_2^2\big)\big] D\|U_1\|_2^2 \sum_{l=1}^{L} \rho(K_l)$$

$$= T^2 D \sum_{l=1}^{L} \rho(K_l) \le \varepsilon/6. \qquad \square$$

PROOF OF (D). First, for each $k = 1,\dots, p$, define the event $F_k := \{|X_k^{(N)}| = \max_{1\le j\le p}|X_j^{(N)}|\}$. To shorten the notation, define $s := T/(2^{L/2}\|U_1\|_2)$. Note that by (3.10) and the second inequality in (3.28),

$$s^2 \le T^2/\big(2^L Cp\|X_0^{(N)}\|_2^2\big) \le 2T^2/\big(CN\|X_0^{(N)}\|_2^2\big).$$

By Lemma 2.4 [with $V = sX_0 I(|X_0| \le t_N)$], (3.29), (3.24) and (3.28), we have that

$$\begin{aligned}
&E\min\{|sU_1|^2, |sU_1|^3\}\\
&\le \sum_{k=1}^{p} EI(F_k)\min\{|sU_1|^2, |sU_1|^3\}\\
&\le \sum_{k=1}^{p} EI(F_k)\min\{|spX_k^{(N)}|^2, |spX_k^{(N)}|^3\}\\
&\le p^3 \sum_{k=1}^{p} EI(F_k)\min\{|sX_k^{(N)}|^2, |sX_k^{(N)}|^3\}\\
&\le p^3 \sum_{k=1}^{p} E\min\{|sX_k^{(N)}|^2, |sX_k^{(N)}|^3\}\\
&= p^4 E\min\{|sX_0^{(N)}|^2, |sX_0^{(N)}|^3\}\\
&\le p^4\cdot\Big[4s^2[EX_0I(|X_0|\le t_N)]^2\\
&\qquad\quad +8E\min\{s^2X_0^2I(|X_0|\le t_N), |s|^3|X_0|^3I(|X_0|\le t_N)\}\Big]\\
&\le (2p^*)^4\cdot\left[\frac{8T^2}{CN\|X_0^{(N)}\|_2^2}[EX_0I(|X_0|\le t_N)]^2\right.\\
&\qquad\quad \left.+8E\min\left\{\frac{2T^2X_0^2I(|X_0|\le t_n)}{CN\|X_0^{(N)}\|_2^2}, \frac{2^{3/2}|T|^3|X_0|^3I(|X_0|\le t_N)}{C^{3/2}N^{3/2}\|X_0^{(N)}\|_2^3}\right\}\right]\\
&\le \frac{128(p^*)^4T^2}{CN\|X_0^{(N)}\|_2^2}[EX_0I(|X_0|\le t_N)]^2\\
&\quad + \frac{400(p^*)^4\max\{T^2,|T|^3\}}{\min\{C, C^{3/2}\}}E\min\left\{\frac{X_0^2I(|X_0|\le t_N)}{N\|X_0^{(N)}\|_2^2}, \frac{|X_0|^3I(|X_0|\le t_N)}{N^{3/2}\|X_0^{(N)}\|_2^3}\right\}\\
&\le \varepsilon/(6N) \le \varepsilon/(6\cdot 2^L).
\end{aligned}$$

By this equation and Billingsley (1979), page 297, equation (26.5), we have that

$$\begin{aligned}
&\left|E\exp\left[i(T/2^{L/2})U_1/\|U_1\|_2\right] - \left[1-(1/2)T^2/2^L\right]\right| \\
&\quad = \left|\left[E\exp(isU_1)\right] - \left[1-(1/2)s^2\|U_1\|_2^2\right]\right| \\
&\quad \le E\min\left\{|sU_1|^2, |sU_1|^3\right\} \\
&\quad \le \varepsilon/(6\cdot 2^L).
\end{aligned}$$

Hence by (3.20) [together with the fact $2^L \ge 2^{L^*}$ by (3.27)] and Lemma 2.6 statement (D) holds. □

PROOF OF (E). Simply use (3.20) and (3.27). □

By (3.38) the proof of Theorem 1 is now complete. □

Note added in proof. The author has learned of three more references giving central limit theorems for weakly dependent stationary sequences in the absence of finite second moments:

M. I. Gordin (1973). Central limit theorems for stationary processes without the assumption of finite variance. In *International Conference on Probability Theory and Mathematical Statistics, June 25–30, 1973, Vilnius.* Abstracts of communications **1**, 173–174.

L. Heinrich (1982). Infinitely divisible distributions as limit laws for sums of random variables connected in a Markov chain. *Math. Nachr.* **107** 103–121.

Q. Shao (1986). An invariance principle for stationary ρ-mixing sequences with infinite variance. Report, Department of Mathematics, Hangzhou University, Hangzhou, People's Republic of China.

The paper of Shao extends Theorem 1 to a weak invariance principle (under the same hypothesis); it also shows that under extra "regularity" conditions on the tail of the marginal distribution (which do not imply, and are not implied by, finite second moments), the mixing rate in (1.4) can be relaxed somewhat.

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