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ON THE DISTRIBUTION OF THE SQUARE INTEGRAL OF THE BROWNIAN BRIDGE

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Smirnov obtained the distribution F for his ω^2-test in the form of a certain series. F is identical to the distribution of the the Brownian bridge in the L^2 norm. Smirnov, Kac and Shepp determined the Laplace–Stieltjes transform of F. Anderson and Darling expressed F in terms of Bessel functions. In the present paper we compute the moments of F and their asymptotics, obtain expansions of F and its density f in terms of the parabolic cylinder functions and Laguerre functions, and determine their asymptotics for the small and large values of the argument. A novel derivation of expansions of Smirnov and of Anderson and Darling is obtained.

1. Introduction. Let $B(s)$, $0 < s \leq t$, be the Brownian motion, and $\tilde{B}(s)$, $0 < s \leq t$, be the corresponding Brownian bridge. The interest to the distribution of the Brownian bridge on [0, 1] in the L^2 norm, that is of the functional

$$\int_0^1 \tilde{B}^2(s)\,ds, \tag{1}$$

is motivated by its relationship to the well known Smirnov's (1936) modification of the ω^2-goodness of fit test by Cramér (1928) and von Mises (1931). A discussion on relationships of this kind is given in Kac (1951).

The distribution $F(\lambda)$ which was determined by Smirnov (1936) for his ω^2-test and which coincides with that of (2), has the form

$$F(\lambda) = 1 - \frac{2}{\pi}\sum_{k=1}^{\infty}\int_{(2k-1)\pi}^{2k\pi}\frac{e^{-x^2\lambda/2}}{\sqrt{-x\sin x}}\,dx. \tag{2}$$

REMARK 1. In fact, Smirnov obtained a more general result, which includes a certain weight function $w(t)$. Series (2) is the particular case of this result, when $w(t) \equiv 1$.

Notice that practical applications of (2) formula would require numerical computations of certain definite integrals. This gap was filled by Anderson and

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Darling (1952), who inverted that Laplace–Stieltjes transform in terms of Bessel functions and obtained $F(\lambda)$ in the form

$$F(\lambda) = \frac{1}{\pi\sqrt{\lambda}} \sum_{j=0}^{+\infty} (-1)^j \binom{-1/2}{j} (4j+1)^{1/2} e^{-(4j+1)^2/(16\lambda)} \times K_{1/4}\big((4j+1)^2/(16\lambda)\big), \tag{3}$$

where

$$\binom{\mu}{\nu} = \frac{\Gamma(\mu+1)}{\Gamma(\nu+1)\Gamma(\mu-\nu+1)} \tag{4}$$

are binomial coefficients.

In the cited paper, Anderson and Darling used the Laplace–Stieltjes transform of $F(\lambda)$, which was determined by Smirnov (1936); see also Smirnov (1949) and independently by Kac (1951) and Shepp (1982). Further references regarding this distribution can be found in Kendall and Stuart [(1997), pages 475–476].

The purpose of the present paper is to consider two more methods of inversion: one in terms of the parabolic cylinder functions and another in terms of Laguerre functions. It turns out that the series for $F(\lambda)$ in the parabolic cylinder functions implies the result of Anderson and Darling. We obtain also a novel derivation of the series (2), which appears naturally as a representation which is in a sense a complementary one to the expansion in the parabolic cylinder functions, see Theorem 2. The expansion in parabolic cylinder functions, similarly to the series of Anderson and Darling, is rather satisfactory for computational purposes. Regarding results on the distribution of the Brownian bridge in the L^1 norm and its asymptotics, see Cifarelli (1975), Shepp (1982), Rice (1982), Johnson and Killeen (1983) and Tolmatz (2000).

2. Laplace inversions. Let $f(\lambda) = F'(\lambda)$ be the density of F which is available from (2), with the Laplace transform

$$\bar{f}(p) = \int_0^\infty e^{-p\lambda} f(\lambda)\, d\lambda = \int_0^\infty e^{-p\lambda}\, dF(\lambda). \tag{5}$$

Kac [(1951), Section 6], proved

$$\bar{f}(p) = 2^{3/4} p^{1/4} \sum_{n=0}^{\infty} \frac{D_n^2(0)}{n!} e^{-(n+1/2)\sqrt{2p}} = \left(\frac{\sinh\sqrt{2p}}{\sqrt{2p}}\right)^{-1/2}, \tag{6}$$

where

$$D_n(x) = 2^{-n/2} e^{-x^2/4} H_n(x/\sqrt{2}) \tag{7}$$

are Weber–Hermite functions. [In the middle part of (6) we discarded the erroneous factor $1/2$ which stole into Kac's remarkable computations.]

The following two theorems offer another method to invert (6); the result turn out in terms of the parabolic cylinder functions. An identity relating the parabolic cylinder functions with Bessel functions gives another proof of the result of Anderson and Darling.

THEOREM 1.

$$f(\lambda) = \lambda^{-5/4} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} e^{-(n+1/4)^2/\lambda} \frac{1}{\Gamma(1/2 - n)} D_{3/2}\left(\frac{2n + 1/2}{\sqrt{\lambda}}\right). \tag{8}$$

PROOF. According to Erdèlyi [(1954), equation (9), page 246], for $\Re\alpha > 0$ we have

$$p^{\nu - 1/2} e^{-\sqrt{\alpha p}} \doteqdot 2^{-\nu} \pi^{-1/2} t^{-\nu - 1/2} e^{-\alpha/(8t)} D_{2\nu}\left(\sqrt{\frac{\alpha}{2t}}\right), \tag{9}$$

and the termwise inversion of the series in (6) is readily justifiable by elementary estimates. While doing so, we substituted

$$D_n(0) = \frac{2^{n/2}\sqrt{\pi}}{\Gamma((1 - n)/2)}, \tag{10}$$

changed the index of summation in (6) from n to $2n$, and applied Le Gendre's formula to $\Gamma(2n + 1) = (2n)!$ and the complementation formula to $\Gamma((1 - n)/2)$. $\square$

THEOREM 2.

$$F(\lambda) = 2\lambda^{-1/4} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} e^{-(n+1/4)^2/\lambda} \frac{1}{\Gamma(1/2 - n)} D_{-1/2}\left(\frac{2n + 1/2}{\sqrt{\lambda}}\right). \tag{11}$$

PROOF. We apply (9) to

$$\bar{F}(p) = 2^{-1/4} p^{-3/4} \sum_{n=0}^{\infty} \frac{D_n^2(0)}{n!} e^{-(n+1/2)\sqrt{2p}}, \tag{12}$$

and proceed similarly to the proof of the previous theorem. $\square$

We used also the well known property of the Laplace transform: if $f(x) \doteqdot \bar{f}(p)$, then $\int_0^x f(t)\,dt \doteqdot \frac{1}{p}\bar{f}(p)$.

3. An alternative approach. Kac (1951) obtained (6) by considering an eigenfunction expansion of the Green's function of equation (17) below. We show in this Section that one can obtain the same result by a direct inversion of the Green's function, with taking no resort to eigenfunction expansions.

Let $F(\lambda, t)$ be the distribution of

$$\int_0^t \tilde{B}^2(s)\, ds, \tag{13}$$

$f(\lambda, t)$ be its density, and let $F_1(\lambda, t)$, $f_1(\lambda, t)$ denote the distribution and the corresponding density of the functional

$$\int_0^t B^2(s)\, ds. \tag{14}$$

These functions are related by familiar equations

$$F(\lambda, t) = \sqrt{2\pi t}\, F_1(\lambda, t), \tag{15}$$

$$f(\lambda, t) = \sqrt{2\pi t}\, f_1(\lambda, t). \tag{16}$$

In the following $\bar{f}_1(p, q)$ and $\bar{F}_1(p, q)$ denote the Laplace images of $f_1(\lambda, t)$ and $F_1(\lambda, t)$.

Let $\Re p > 0$, $\Re q > 0$. Then a solution of the equation

$$u'' - 2(q + px^2)u = 0, \tag{17}$$

which vanishes as $x \to +\infty$, can be written in the form

$$w(\xi) = D_{\alpha(p,q)}(\xi), \tag{18}$$

where $D_\alpha(\xi)$ is a parabolic cylinder function, $\alpha(p, q) = -\frac{1}{\sqrt{2}} p^{-1/2} q - \frac{1}{2}$, and $\xi = 2^{3/4} p^{1/4} x$.

From this, the Green's function of equation (17) with the source at $x = 0$ takes the form

$$G(x; p, q) = -2^{-3/4} p^{-1/4} \frac{D_{\alpha(p,q)}(2^{3/4} p^{1/4} |x|)}{D'_{\alpha(p,q)}(0)}, \tag{19}$$

thus

$$\bar{f}_1(p, q) = -2^{-3/4} p^{-1/4} \frac{D_{\alpha(p,q)}(0)}{D'_{\alpha(p,q)}(0)}, \tag{20}$$

and correspondingly

$$\bar{F}_1(p, q) = -2^{-3/4} p^{-5/4} \frac{D_{\alpha(p,q)}(0)}{D'_{\alpha(p,q)}(0)}. \tag{21}$$

REMARK 2. Equation (17) with $p = 1$ was solved in terms of $D_\nu(x)$ by Shepp (1982).

THEOREM 3.

$$\bar{f}_1(p,q) = 2^{-5/4} p^{-1/4} \frac{\Gamma(q/(2\sqrt{2p}) + 1/4)}{\Gamma(q/(2\sqrt{2p}) + 3/4)}. \tag{22}$$

PROOF. From the integral representation [see Gradstein and Ryzhik (1980), page 1064],

$$D_p(z) = \frac{e^{-z^2/4}}{\Gamma(-p)} \int_0^\infty e^{-zx - x^2/2} x^{-p-1}\, dx \qquad \text{where } \Re p < 0, \tag{23}$$

we obtain

$$D_p(0) = 2^{-p/2-1} \frac{\Gamma(-p/2)}{\Gamma(-p)} \tag{24}$$

and

$$D'_p(0) = -2^{-p/2-1/2} \frac{\Gamma((1-p)/2)}{\Gamma(-p)}. \tag{25}$$

Substitution of (24) and (25) in (19) with p replaced by $\alpha(p,q)$ yields (22). □

THEOREM 4.

$$\bar{F}_1(p,q) = 2^{-5/4} p^{-5/4} \frac{\Gamma(q/(2\sqrt{2p}) + 1/4)}{\Gamma(q/(2\sqrt{2p}) + 3/4)}. \tag{26}$$

PROOF. The proof is similar to the proof of the previous theorem. □

THEOREM 5.

$$f(\lambda, t) = \sqrt{t}\lambda^{-5/4} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{e^{-t^2(n+1/4)^2/\lambda}}{\Gamma(1/2 - n)} D_{3/2}\left(\frac{(2n+1/2)t}{\sqrt{\lambda}}\right). \tag{27}$$

PROOF. The following steps are easily justifiable.

We begin with the inversion of (22) in q by the residue theorem, where the poles q_n of the integrand in

$$\begin{aligned} &\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \bar{f}_1(p,q) e^{qt}\, dq \\ &\qquad = -\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} 2^{-5/4} p^{-1/4} \frac{\Gamma(q/(2\sqrt{2p}) + 1/4)}{\Gamma(q/(2\sqrt{2p}) + 3/4)} e^{qt}\, dq \end{aligned} \tag{28}$$

are determined by the condition

$$\frac{q_n}{2\sqrt{2p}} + \frac{1}{4} = -n \qquad \text{where } n = 0, 1, \dots . \tag{29}$$

For any p, such that $\Re p > 0$, we obtain

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \bar{f}_1(p,q)e^{qt}\,dq = 2^{1/4}p^{1/4}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{e^{-(2n+1/2)\sqrt{2pt}}}{\Gamma(1/2-n)}, \tag{30}$$

and invert in p similarly to the inversion in Theorem 1, and apply (15). $\square$

THEOREM 6.

$$F(\lambda,t) = 2\sqrt{t}\lambda^{-1/4}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{e^{-t^2(n+1/4)^2/\lambda}}{\Gamma(1/2-n)}D_{-1/2}\Big(\frac{(2n+1/2)t}{\sqrt{\lambda}}\Big). \tag{31}$$

PROOF. The proof similar to the proof of the previous theorem.

By the substitution $t = 1$ in (27) and (31) we obtain $f(\lambda)$ and $F(\lambda)$ in agreement with (8) and (11).

According to the complementation formula $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$ used with $z = n + 1/2$ we can represent the series (8) and (11) in an equivalent form

$$f(\lambda) = \frac{1}{\pi}\lambda^{-5/4}\sum_{n=0}^{\infty}\frac{\Gamma(n+1/2)}{n!}e^{-(n+1/4)^2/\lambda}D_{3/2}\Big(\frac{2n+1/2}{\sqrt{\lambda}}\Big), \tag{32}$$

$$F(\lambda) = \frac{2}{\pi}\lambda^{-1/4}\sum_{n=0}^{\infty}\frac{\Gamma(n+1/2)}{n!}e^{-(n+1/4)^2/\lambda}D_{-1/2}\Big(\frac{2n+1/2}{\sqrt{\lambda}}\Big). \qquad \square \tag{33}$$

4. Smirnov's result reestablished.

THEOREM 7.

$$f(\lambda) = \frac{1}{\pi}\sum_{k=1}^{\infty}\int_{(2k-1)\pi}^{2k\pi}\frac{x^{3/2}e^{-x^2\lambda/2}}{\sqrt{-\sin x}}\,dx. \tag{34}$$

PROOF. We notice that the singularity of $\bar{f}(p)$ at $p = 0$ is removable, and therefore the function $\bar{f}(p)$ is analytical in the circle $|p| < \pi^2/2$.

By inversion of (6) we get, for any $a > 0$,

$$f(\lambda) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\Big(\frac{\sinh\sqrt{2p}}{\sqrt{2p}}\Big)^{-1/2}e^{\lambda p}\,dp, \tag{35}$$

where the integrand has singularities at $p_k = -\pi^2k^2/2$, where $k = 1, 2, \dots .$

Denote $A = \bigcup_{k=1}^{\infty}[-(2k-1)^2\pi^2/2, -(2k)^2\pi^2/2]$. We observe that $\bar{f}(p)$ admits an analytical single-valued continuation from the right-hand side half-plane into the multiply connected domain $\mathbb{C} \setminus A$.

By an easily justifiable deformation of the contour along the edges of the negative real semiaxis, we obtain

$$f(\lambda) = \frac{1}{\pi}\int_0^{\infty} g(t)e^{-\lambda t}\,dt, \tag{36}$$

where

$$g(t) = \begin{cases} \left(\dfrac{-\sin\sqrt{2t}}{\sqrt{2t}}\right)^{-1/2}, & \text{if } (2k-1)^2\pi^2/2 \le t < (2k)^2\pi^2/2\,, \\ 0, & \text{otherwise,} \end{cases} \tag{37}$$

where $k = 1, 2, \dots$, so that (36) takes the form

$$f(\lambda) = \frac{1}{\pi}\sum_{k=1}^{\infty}\int_{(2k-1)^2\pi^2/2}^{(2k)^2\pi^2/2}\left(\frac{-\sin\sqrt{2t}}{\sqrt{2t}}\right)^{-1/2} e^{-\lambda t}\,dt, \tag{38}$$

and the substitution $t = x^2/2$ transforms it into (34). $\square$

THEOREM 8.

$$F(\lambda) = 1 - \frac{2}{\pi}\sum_{k=1}^{\infty}\int_{(2k-1)\pi}^{2k\pi}\frac{e^{-x^2\lambda/2}}{\sqrt{-x\sin x}}\,dx. \tag{39}$$

PROOF. We begin with

$$F(\lambda) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{1}{p}\left(\frac{\sinh\sqrt{2p}}{\sqrt{2p}}\right)^{-1/2} e^{\lambda p}\,dp, \tag{40}$$

observe that the integrand has a simple pole at $p = 0$, and proceed similarly to the proof of the previous theorem. $\square$

5. The result of Anderson and Darling. We apply to (11) the relationship [see Abramowitz and Stegun (1964), page 692]

$$D_{-1/2}(z) = \sqrt{\frac{z}{2\pi}}K_{1/4}\left(\frac{z^2}{4}\right) \tag{41}$$

and obtain the result of Anderson and Darling (3).

6. Asymptotics for small λ's. According to Gradstein and Ryzhik [(1980), page 1065],

$$D_p(z) = e^{-z^2/4} z^p \left[1 + O\left(\frac{p(p-1)}{2z^2}\right)\right], \tag{42}$$

where $|z| \gg 1$, $|z| \gg p$ and $|\arg z| < \frac{3}{4}\pi$.

THEOREM 9.

$$f(\lambda) = \frac{1}{2\sqrt{2\pi}} \frac{1}{\lambda^2} e^{-1/(8\lambda)} [1 + O(\lambda)] \qquad \textit{as } \lambda \to 0+. \tag{43}$$

PROOF. By (42) with $z = (2n + 1/2)/\sqrt{\lambda}$ we have

$$\begin{aligned} &D_{3/2}\left(\frac{2n+1/2}{\sqrt{\lambda}}\right) \\ &\quad = e^{-\frac{1}{4}(2n+1/2)^2/\lambda} \left(\frac{2n+1/2}{\sqrt{\lambda}}\right)^{3/2} \left[1 + O\left(\frac{\lambda}{(2n+1/2)^2}\right)\right], \end{aligned} \tag{44}$$

as $\lambda \to 0+$, which being applied to (8) after obvious elementary estimates implies (43). □

THEOREM 10.

$$F(\lambda) = 2\sqrt{\frac{2}{\pi}} e^{-1/(8\lambda)} [1 + O(\lambda)] \qquad \textit{as } \lambda \to 0+. \tag{45}$$

PROOF. The proof is similar to the proof of the previous result. □

7. Asymptotics for large λ's.

THEOREM 11.

$$f(\lambda) = \frac{\sqrt{\pi}}{\sqrt{\lambda}} e^{-\pi^2\lambda/2} [1 + O(\lambda^{-1})]. \tag{46}$$

PROOF. We apply Laplace's method for integrals, see for example de Bruijn (1961), to the first term in the expansion (38) and obtain for any $0 < \varepsilon < \pi^2/2$,

$$\begin{aligned} \frac{1}{\pi} \int_{\frac{1}{2}\pi^2}^{\pi^2} \left(\frac{-\sin\sqrt{2t}}{\sqrt{2t}}\right)^{-1/2} e^{-\lambda t}\, dt &= \frac{1}{\pi} \int_{\frac{1}{2}\pi^2}^{\frac{1}{2}\pi^2+\varepsilon} + \frac{1}{\pi} \int_{\frac{1}{2}\pi^2+\varepsilon}^{\pi^2} \\ &= \frac{\sqrt{\pi}}{\sqrt{\lambda}} e^{-\pi^2\lambda/2} [1 + O(\lambda^{-1})] \\ &\quad + O\left(e^{-(\frac{1}{2}\pi^2+\varepsilon)\lambda}\right) \qquad \text{as } \lambda \to \infty. \end{aligned} \tag{47}$$

For the remainder of the series have an elementary estimate

$$\sum_{k=2}^{\infty} = O(e^{-9\pi^2\lambda/4}), \tag{48}$$

and the statement follows. □

In a similar fashion we can prove that

$$F(\lambda) = 1 - \frac{2}{\pi\sqrt{\pi\lambda}} e^{-\pi^2\lambda/2}[1 + O(\lambda^{-1})] \qquad \text{as } \lambda \to \infty. \tag{49}$$

8. The moments and their asymptotics. Smirnov (1936) and independently Kac (1949) and Shepp (1982) determined the Laplace–Stieltjes transform of $F(\lambda)$ in the form

$$\bar{f}(p) = \left(\frac{\sinh\sqrt{2p}}{\sqrt{2p}}\right)^{-1/2}. \tag{50}$$

We will use this expression to determine the moments of $F(\lambda)$ in terms of Bernoulli numbers.

LEMMA 8.1. *The function $\bar{f}(p)$ is analytical at $p = 0$.*

PROOF. We notice that the Taylor expansion of $g(z) = \sinh z/z$ at $z = 0$ contains only even powers of z, and $g(0) \neq 0$. Therefore the functions $1/g(z)$ and $1/\sqrt{g(z)}$ are also analytical at $z = 0$, and possess the same property. Hence the singularity of

$$\bar{f}(p) = \left(\frac{\sinh\sqrt{2p}}{\sqrt{2p}}\right)^{-1/2} \tag{51}$$

at $p = 0$ is removable. □

We observe that $\bar{f}(p)$ is analytical in the circle $|p| < \pi^2/2$.

Lemma 8.1 implies that all the moments m_n of F exist and

$$\bar{f}(p) = \left(\frac{\sinh\sqrt{2p}}{\sqrt{2p}}\right)^{-1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} m_n p^n \qquad \text{for } |p| < \pi^2/2. \tag{52}$$

REMARK 3. The existence of the moments of all orders follows also from the asymptotics of $f(\lambda)$ for large λ's. This asymptotics was determined by Tolmatz (2000); see equation (60).

THEOREM 12. *The moments* m_n *of* F, *for* $n = 0, 1, \ldots$, *are determined by the following recursion formulas*:

$$
\begin{aligned}
&m_0 = 1, \\
&m_n = (-1)^{n+1} 2^n (2^{2n-1} - 1) \frac{n!}{(2n)!} B_{2n} - \frac{1}{2} n! \sum_{k=1}^{n-1} \frac{m_k m_{n-k}}{k!(n-k)!},
\end{aligned} \tag{53}
$$

where B_{2n} *are Bernoulli numbers*, *and* $n = 1, 2, \ldots$.

PROOF. According to Gradstein and Ryzhik [(1980), page 35],

$$
\frac{1}{\sinh z} = \frac{1}{z} - \sum_{k=1}^{\infty} \frac{2(2^{2k-1} - 1) B_{2k}}{(2k)!} z^{2k-1}, \tag{54}
$$

where $|z| < \pi$ and B_{2k} are Bernoulli numbers.

This implies

$$
\begin{aligned}
(\bar{f}(p))^2 = \left(\frac{\sinh \sqrt{2p}}{\sqrt{2p}} \right)^{-1} &= 1 - \sum_{k=1}^{\infty} \frac{2(2^{2k-1} - 1) B_{2k}}{(2k)!} (2p)^k \\
&= 1 - \sum_{k=1}^{\infty} \frac{2^{k+1}(2^{2k-1} - 1) B_{2k}}{(2k)!} p^k.
\end{aligned} \tag{55}
$$

The following steps are easily justifiable.

The termwise squaring of the series in (52) yields

$$
(\bar{f}(p))^2 = \left(\frac{\sinh \sqrt{2p}}{\sqrt{2p}} \right)^{-1} = \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} = \sum_{n=0}^{\infty} a_n p^n, \tag{56}
$$

where

$$
a_n = (-1)^n \sum_{k=0}^{n} \frac{m_k m_{n-k}}{k!(n-k)!}. \tag{57}
$$

By equating the corresponding terms in expansions (55) and (56) we get

$$
m_0 = 1, \qquad \frac{2^{n+1}(2^{2n-1} - 1) B_{2n}}{(2n)!} = (-1)^{n+1} \sum_{k=0}^{n} \frac{m_k m_{n-k}}{k!(n-k)!}, \tag{58}
$$

where $n = 1, 2, \ldots$.

We solve it for m_n, and the statement of the theorem follows. □

Bernoulli numbers are rationals, and the moments m_n are also rationals.

Table 1 shows the first ten values of m_n.

TABLE 1
Moments of $F(\lambda)$

n	m_n	m_n
1	$\frac{1}{6}$	0.1666667
2	$\frac{1}{20}$	0.0500000
3	$\frac{61}{2520}$	0.0242063
4	$\frac{1261}{75600}$	0.0166799
5	$\frac{79}{5280}$	0.0149621
6	$\frac{66643}{4036032}$	0.0165120
7	$\frac{16820653}{778377600}$	0.0216099
8	$\frac{3745813}{114566400}$	0.0326956
9	$\frac{1975649524361}{35198235072000}$	0.0561292
10	$\frac{19259487248923}{178698731904000}$	0.1077763

THEOREM 13. *The asymptotics of the moments m_n is given as follows*:

$$m_n = 2^{n+1/2}\pi^{-2n-1/2}\Gamma(n+1/2)\big(1+O(1/n)\big) \qquad \textit{as } n \to +\infty. \tag{59}$$

PROOF. By making use of (2), it is not difficult to show, see Tolmatz (2000), that

$$f(\lambda) = \sqrt{\pi}/\sqrt{\lambda}\exp(-\pi^2\lambda/2)\big(1+O(\lambda^{-1})\big) \qquad \text{as } \lambda \to +\infty. \tag{60}$$

By (60) we obtain

$$\begin{aligned} m_n &= \int_0^{+\infty} \lambda^n f(\lambda)\,d\lambda \\ &= \sqrt{\pi}/\sqrt{\lambda}\int_0^{+\infty} \lambda^n \exp(-\pi^2\lambda/2)\big(1+O(\lambda^{-1})\big)\,d\lambda \\ &= \sqrt{\pi}/\sqrt{\lambda}(2/\pi^2)^{n+1}\int_0^{+\infty} x^n e^{-x}\big(1+O(x^{-1})\big)\,dx \\ &= 2^{(n+1/2)}\pi^{-2n-1/2}\Gamma(n+1/2)\big(1+O(1/n)\big). \qquad \square \end{aligned}$$

By Stirling's formula, $\Gamma(n+1/2) \sim \sqrt{2\pi/(n+1/2)}e^{-n-1/2}(n+1/2)^{(n+1/2)}$, so we get the corollary.

COROLLARY 1.

$$m_n \sim 2\pi^{-2n}e^{-n-1/2}(2n+1)^n, \tag{61}$$

or equivalently,

$$m_n \sim 2^{n+1}\pi^{-2n}e^{-n}n^n. \tag{62}$$

9. Laguerre expansions. We obtain Laguerre series for $f(\lambda)$ and $F(\lambda)$ by the method proposed by Takáacs (1992) and used in a related work by Perman and Wellner (1996).

The generalized Laguerre polynomials are defined as

$$L_n^{(\alpha)}(x) = \sum_{j=0}^{n}(-1)^n\binom{n+\alpha}{n-j}\frac{x^j}{j!}, \tag{63}$$

where $n = 0, 1, 2, \ldots$ and $\alpha > -1$; these polynomials are orthogonal on the interval $0 \le x < +\infty$ with respect to the weight function

$$g_\alpha(x) = e^{-x}x^\alpha/\Gamma(\alpha+1). \tag{64}$$

We define

$$G_\alpha(x) = \int_0^x g_\alpha(s)\,ds \tag{65}$$

and write

$$F(\lambda) = G_{\alpha-1}(b\lambda) + \alpha\sum_{n=1}^{+\infty}\frac{c_n}{n}g_\alpha(b\lambda)L_{n-1}^{(\alpha)}(b\lambda), \tag{66}$$

$$f(\lambda) = g_{\alpha-1}(b\lambda)b\sum_{n=1}^{+\infty}c_nL_n^{(\alpha-1)}(b\lambda), \tag{67}$$

where $\lambda > 0$, $a > 0$, $b > 0$ and c_n are defined by

$$c_n\binom{n+\alpha-1}{n} = \sum_{k=0}^{n}\frac{(-1)^k}{k!}\binom{n+\alpha-1}{n-k}b^km_k, \tag{68}$$

for $n = 0, 1, 2\ldots$; the moments m_k are given by (53).

Similarly to the choice of α, b in the cited paper by Takács, we choose

$$\alpha = \tfrac{3}{4} \tag{69}$$

and

$$b = \tfrac{3}{2}, \tag{70}$$

which results in $c_1 = c_2 = 0$; Table 2 shows the first twenty values of c_n. We see that the coefficients decay too slowly.

10. Graphs and tables. In the cited paper by Anderson and Darling the distribution $F(\lambda)$ was tabulated in the form adjusted for statistical applications.

Here we tabulate the functions $f(\lambda)$ and $F(\lambda)$ per se, by making use of their series in parabolic cylinder functions and the explicit estimates of the remainders.

LEMMA 10.1. *For $p < 0$ and $z > 0$ it holds that*

$$0 < D_p(z) < 2^{-p/2-1}\frac{\Gamma(-p/2)}{\Gamma(-p)}e^{-z^2/4}. \tag{71}$$

PROOF. By (23) we get

$$\begin{aligned} 0 < D_p(z) &= \frac{e^{-z^2/4}}{\Gamma(-p)}\int_0^\infty e^{-zx-x^2/2}x^{-p-1}\,dx < \frac{e^{-z^2/4}}{\Gamma(-p)}\int_0^\infty e^{-x^2/2}x^{-p-1}\,dx \\ &= 2^{-p/2-1}\frac{\Gamma(-p/2)}{\Gamma(-p)}e^{-z^2/4}. \end{aligned} \tag{72}$$

□

COROLLARY 2. *For $z > 0$ it holds that*

$$0 < D_{-1/2}(z) < 2^{-3/4}\pi^{-1/2}\Gamma(1/4)e^{-z^2/4} < 2e^{-z^2/4} \tag{73}$$

and

$$0 < D_{-3/2}(z) < 2^{3/4}\pi^{-1/2}\Gamma(3/4)e^{-z^2/4} < 2e^{-z^2/4}. \tag{74}$$

Denote

$$\varphi_n(\beta) = \frac{e^{-\beta(n+1/4)}}{1-e^{-\beta}}. \tag{75}$$

LEMMA 10.2. *For $\beta > 0$ and $n = 0, 1, \dots$ we have*

$$\sum_{k=n}^{\infty}\left(k+\frac{1}{4}\right)e^{-\beta(k+1/4)} = -\varphi_n'(\beta) \tag{76}$$

and

$$\sum_{k=n}^{\infty}\left(k+\frac{1}{4}\right)^2 e^{-\beta(k+1/4)} = \varphi_n''(\beta). \tag{77}$$

PROOF. Repeated differentiation on β of the identity

$$\varphi_n(\beta) = \sum_{k=n}^{\infty} e^{-\beta(k+1/4)} = \frac{e^{-\beta(n+1/4)}}{1-e^{-\beta}}. \tag{78}$$

□

COROLLARY 3. *For $\beta > 0$, $n = 0, 1, \ldots$ and arbitrary a, b, c it holds that*

$$\sum_{k=n}^{\infty}[a(k+1/4)^2 + b(k+1/4) + c]e^{-\beta(k+1/4)} = a\varphi_n''(\beta) - b\varphi_n'(\beta) + c\varphi_n(\beta). \tag{79}$$

THEOREM 14. *For the remainders s_n and S_n of the series* (8) *and* (11) *for $f(\lambda)$ and $F(\lambda)$ the following estimates take place*:

$$\begin{aligned} |s_n| &= \lambda^{-5/4}\left|\sum_{k=n}^{\infty}\frac{(-1)^k}{k!}e^{-(k+1/4)^2/\lambda}\frac{1}{\Gamma(1/2-k)}D_{3/2}\left(\frac{2k+1/2}{\sqrt{\lambda}}\right)\right| \\ &= \frac{1}{\pi}\lambda^{-5/4}\sum_{k=n}^{\infty}\frac{\Gamma(k+1/2)}{k!}e^{-(k+1/4)^2/\lambda}\left|D_{3/2}\left(\frac{2k+1/2}{\sqrt{\lambda}}\right)\right| \\ &< 2\lambda^{-5/4}e^{-2(n^2+n)/\lambda}[3\varphi_n''(2/\lambda)/\lambda - \varphi_n'(2/\lambda)/\sqrt{\lambda} + \varphi_n(2/\lambda)], \end{aligned} \tag{80}$$

where $\varphi(\cdot)$ is given by (75), *and*

$$\begin{aligned} 0 < S_n &= 2\lambda^{-1/4}\sum_{k=n}^{\infty}\frac{(-1)^k}{k!}e^{-(k+1/4)^2/\lambda}\frac{1}{\Gamma(1/2-k)}D_{-1/2}\left(\frac{2k+1/2}{\sqrt{\lambda}}\right) \\ &= \frac{2}{\pi}\lambda^{-1/4}\sum_{k=n}^{\infty}\frac{\Gamma(k+1/2)}{k!}e^{-(k+1/4)^2/\lambda}D_{-1/2}\left(\frac{2k+1/2}{\sqrt{\lambda}}\right) \\ &< 2\lambda^{-1/4}\frac{e^{-2n^2/\lambda}}{1-e^{-4n/\lambda}}, \end{aligned} \tag{81}$$

where $n = 1, 2, \ldots$.

PROOF. First we prove (81).
By (73) we have

$$\begin{aligned} e^{-(k+1/4)^2/\lambda}D_{-1/2}\left(\frac{2k+1/2}{\sqrt{\lambda}}\right) &< e^{-(k+1/4)^2/\lambda}2e^{-(k+1/4)^2/\lambda} \\ &= 2e^{-2(k+1/4)^2/\lambda}, \end{aligned} \tag{82}$$

and hence

$$\begin{aligned} 0 < S_n &< \frac{2}{\pi}\lambda^{-1/4}\sum_{k=n}^{\infty}\frac{\Gamma(k+1/2)}{k!}2e^{-2(k+1/4)^2/\lambda} < 2\lambda^{-1/4}\sum_{k=n}^{\infty}e^{-2(k+1/4)^2/\lambda} \\ &= 2\lambda^{-1/4}e^{-2(n+1/4)^2/\lambda}\sum_{k=n}^{\infty}e^{-2((k+1/4)^2-(n+1/4)^2)/\lambda} \\ &= 2\lambda^{-1/4}e^{-2(n+1/4)^2/\lambda}\sum_{k=n}^{\infty}e^{-2(k+n+1/2)(k-n)/\lambda} \end{aligned} \tag{83}$$

$$
\begin{aligned}
&< 2\lambda^{-1/4} e^{-2(n+1/4)^2/\lambda} \sum_{k=n}^{\infty} e^{-4n(k-n)/\lambda} \\
&= 2\lambda^{-1/4} e^{-2(n+1/4)^2/\lambda} \frac{1}{1-e^{-4n/\lambda}} \\
&< 2\lambda^{-1/4} \frac{e^{-2n^2/\lambda}}{1-e^{-4n/\lambda}}.
\end{aligned}
$$

We used the fact that $\Gamma(k+1/2)/k! < 1$ for $k \geq 1$.

In order to prove (80), we make use of the following recurrent relationship [see Gradstein and Ryzhik (1980), page 1066]:

$$D_{p+1}(z) = zD_p(z) - pD_{p-1}(z). \tag{84}$$

This, together with (73) and (74), yields, for $z > 0$,

$$
\begin{aligned}
|D_{3/2}(z)| &= \left| zD_{1/2}(z) - \frac{3}{2}D_{-1/2}(z) \right| = \left| \left(z^2 - \frac{3}{2}\right) D_{-1/2}(z) + \frac{z}{2} D_{-3/2}(z) \right| \\
&\leq (2z^2 + z + 3)e^{-z^2/4},
\end{aligned} \tag{85}
$$

and we proceed similary to the proof of (81), with $z = 2(k+1/4)/\sqrt{\lambda}$, while making use of (79):

$$
\begin{aligned}
|s_n| &\leq \frac{1}{\pi} \lambda^{-5/4} \sum_{k=n}^{\infty} [8(k+1/4)^2/\lambda + 2(k+1/4)/\sqrt{\lambda} + 3] 2e^{-(k+1/4)^2/\lambda} \\
&< \lambda^{-5/4} \sum_{k=n}^{\infty} [3(k+1/4)^2/\lambda + (k+1/4)/\sqrt{\lambda} + 1] 2e^{-(k+1/4)^2/\lambda} \\
&= 2\lambda^{-5/4} e^{-2(n+1/4)^2/\lambda} \sum_{k=n}^{\infty} [3(k+1/4)^2/\lambda + (k+1/4)/\sqrt{\lambda} + 1] \\
&\qquad \times e^{-2(k+n+1/2)(k-n)/\lambda} \\
&= 2\lambda^{-5/4} e^{-2(n+1/4)^2/\lambda} e^{-2(n+1/4)/\lambda} \\
&\qquad \times \sum_{k=n}^{\infty} [3(k+1/4)^2/\lambda + (k+1/4)/\sqrt{\lambda} + 1] e^{-2(k+1/4)/\lambda} \\
&= 2\lambda^{-5/4} e^{-2(n+1/4)^2/\lambda} e^{-2(n+1/4)/\lambda} \\
&\qquad \times [3\varphi_n''(2/\lambda)/\lambda - \varphi_n'(2/\lambda)/\sqrt{\lambda} + \varphi_n(2/\lambda)] \\
&< 2\lambda^{-5/4} e^{-2(n^2+n)/\lambda} [3\varphi_n''(2/\lambda)/\lambda - \varphi_n'(2/\lambda)/\sqrt{\lambda} + \varphi_n(2/\lambda)].
\end{aligned} \tag{86}
$$

The estimates (80) and (81) show that the corresponding series for $f(\lambda)$ and $F(\lambda)$ converge very rapidly. $\square$

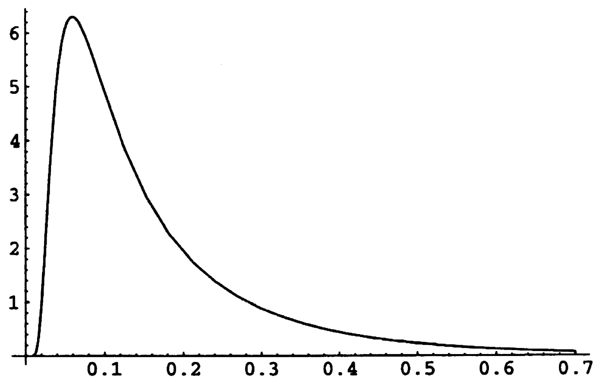

FIG. 1.

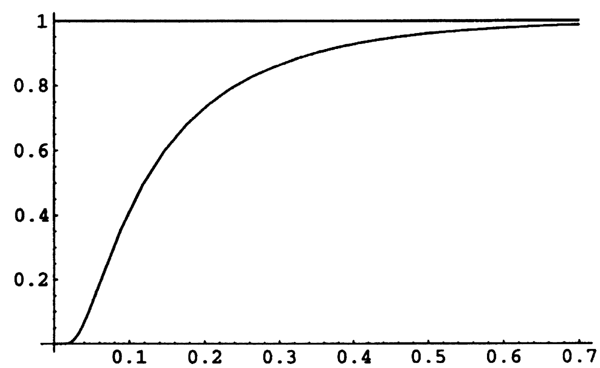

FIG. 2.

Tables 3, 4 and Figures 1, 2 show tables and graphs for $f(\lambda)$ and $F(\lambda)$ which were generated by the series (8) and (11) and the above estimates of their remainders.

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