POLYNOMIAL APPROXIMATION BY THE METHOD OF LEAST SQUARES

By H. T. DAVIS

1. *Introduction.* In an earlier article in the Annals of Mathematics the author in collaboration with V. V. Latshaw published formulas and tables for the fitting of polynomials to data by the method of least squares.[1] In that paper two ranges of the independent variable were considered, one from $x = 1$ to $x = p$, and the other from $x = -p$ to $x = p$. For the first range formulas were given for fitting polynomials of first, second and third degrees to data and these formulas were reduced to tables. For the second range formulas were given for polynomials from the first to the seventh degrees, but these formulas were not then reduced to tables.

It is the purpose of the present paper to supply the tables for the second range and hence to furnish a means of reducing to a minimum the numerical labor involved in fitting to data polynomials from the first to the seventh degree inclusive. Incidentally some novel mathematical aspects of the problem of polynomial approximation have been brought to light, particularly as it applies to the existence of a set of polynomials which are orthogonal for a summation over discrete intervals.

The tables have been computed in the statistical laboratory of Indiana University and have been checked by duplicate calculation. The computation has been made possible by grants of funds by the Waterman Institute of the University. The author is particularly indebted to Dr. V. V. Latshaw, Miss Irene Price, Byron Shelley, George Davis, and Miss Anna Lescisin for the work which they have done in connection with the various computations of this paper.

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[1] Volume 31 (1930), pp. 52-78.

2. *Formulas.* Let us first consider the data to be given as a set of equally spaced items:

y	y_1	y_2	y_3	$\cdots\cdots$	y_m
x	x_1	x_2	x_3	$\cdots\cdots$	x_m

in which we assume that the difference $x_{i+1} - x_i$ is constant.

If m is an odd number, $m = 2p+1$, we select zero as the center of the x-range and without loss of generality we replace the table just given by the following:

y	y_{-p}	$y_{-p+1}, \cdots$	y_{-1}	y_0	$y_1 \cdots$	y_{p-1}	y_p
x	$-p$	$-p+1, \cdots$	-1	0	$1 \cdots$	$p-1$	p

Let us designate by M_r the moments,

$$M_r = \sum_{s=-p}^{p} s^r y_s, \quad r = 0, 1, 2, \cdots, m. \tag{1}$$

If m is an even number, $m = 2p$, we must make a slight change in the notation and consider the distribution,

y	$y_{-p} \cdots\cdots$	y_{-2}	y_{-1}	y_1	y_2	$\cdots\cdots y_p$
x	$-(2p-1)/2 \cdots$	$-3/2$	$-1/2$	$1/2$	$3/2$	$\cdots (2p-1)/2$

The r-*th* moments, M'_r, will be correspondingly equal to,

$$M'_r = \left(\tfrac{1}{2}\right)^r \sum_{s=1}^{p} (2s-1)^r \left\{ y_s + (-1)^r y_{-s} \right\}.$$

The method of least squares is then employed as described in the previous paper to determine the coefficients of the polynomial,

$$y = a_0 + a_1 x + a_2 x^2 + \cdots\cdots + a_m x^m, \quad m = 1, 2, \cdots, 7. \tag{2}$$

It will be unnecessary to repeat the explicit formulas obtained since they have been given in the previous paper, but it will be useful in explanation of the notation of the tables to give the following determination of the coefficients as linear functions of the moments:[2]

1. The straight line, $y = a_0 + a_1 x$.

$$(3)\quad a_0 = \frac{M_0}{(2p+1)} = AM_0\ ,\quad a_1 = \frac{3M_1}{p(p+1)(2p+1)} = A'M_1 .$$

2. The parabola, $y = a_0 + a_1 x + a_2 x^2$

$$(4)\quad \begin{aligned} a_0 &= AM_0 + BM_2\ , \\ a_2 &= BM_0 + CM_2\ , \end{aligned}$$

a_1 determined from (3).

3. The cubic, $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$.

$$(5)\quad \begin{aligned} a_1 &= A'M_1 + B'M_3\ , \\ a_3 &= B'M_1 + C'M_3\ , \end{aligned}$$

a_0 and a_2 determined from (4).

4. The quartic, $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$.

$$(6)\quad \begin{aligned} a_0 &= AM_0 + BM_2 + CM_4\ , \\ a_2 &= BM_0 + DM_2 + EM_4\ , \\ a_4 &= CM_0 + EM_2 + FM_4\ , \end{aligned}$$

a_1 and a_3 determined from (5).

5. The quintic, $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$

$$(7)\quad \begin{aligned} a_1 &= A'M_1 + B'M_3 + C'M_5\ , \\ a_3 &= B'M_1 + D'M_3 + E'M_5\ , \\ a_5 &= C'M_1 + E'M_3 + F'M_5\ , \end{aligned}$$

a_0, a_2 and a_4 determined from (6).

6. The sextic, $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6$

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[2] The notation follows that of the previous paper. It should be noted that the coefficients A, B, C, etc. for the straight line, the parabola, the quartic, and the sextic, and the coefficients A', B', C', etc. for the cubic, the quintic, and the septic are all given by different formulas, but it is hoped that the omission of subscripts denoting the degrees of the polynomials will lead to no confusion.

$$(8)\quad \begin{aligned} a_0 &= AM_0 + BM_2 + CM_4 + DM_6, \\ a_2 &= BM_0 + EM_2 + FM_4 + GM_6, \\ a_4 &= CM_0 + FM_2 + HM_4 + IM_6, \\ a_6 &= DM_0 + GM_2 + IM_4 + JM_6, \end{aligned}$$

a_1, a_3 and a_5 determined from (7).

7. *The septic,* $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ $a_5x^5 + a_6x^6 + a_7x^7$.

$$(9)\quad \begin{aligned} a_1 &= A'M_1 + B'M_3 + C'M_5 + D'M_7, \\ a_3 &= B'M_1 + E'M_3 + F'M_5 + G'M_7, \\ a_5 &= C'M_1 + F'M_3 + H'M_5 + I'M_7, \\ a_7 &= D'M_1 + G'M_3 + I'M_5 + J'M_7, \end{aligned}$$

a_0, a_2, a_4 and a_6 determined from (8).

3. *Orthogonal Polynomials.* In a paper the significance of which has perhaps never been fully appreciated, J. P. Gram investigated the problem of polynomial approximation over discrete intervals by means of orthogonal polynomials.[3] This method has since been more fully investigated by Edward Condon[4] and his work was made the basis of a method for obtaining least squares polynomials by R. T. Birge and J. D. Shea.[5] The work of the latter, however, while effecting a simplification, does not reduce the problem to its simplest form.

In a recent paper issued by the Hungarian National Committee on Economic Statistics, Karl Jordan has employed orthogonal functions in connection with binomial moments and has very

[3]Über die Entwickelung reeller Functionen in Reihen mittelst der Methode der kleinsten Quadrate. Journal für Math., vol. 94 (1883), pp. 41-73.

[4]The Rapid Fitting of a Certain Class of Empirical Formulae by the Method of Least Squares. Univ. of California Publications in Mathematics, vol. 2 (1927), pp. 55-66.

[5]A Rapid Method for Calculating the Least Square Solution of a Polynomial of any Degree. *Ibid.*, pp. 67-118.

greatly simplified the numerical work of curve fitting.[6] The polynomials which he employs, however, appear in the form,

$$y = a_0 + a_1 x + a_2 x(x-1) + a_3 x(x-1)(x-2) + \cdots\cdots,$$

although in the final result they are numerically equivalent to the polynomials of the present paper.

Let us begin with a set of polynomials,

$$\phi_0(x),\ \phi_1(x),\ \phi_2(x),\ \cdots,\ \phi_m(x),$$

of degrees $0, 1, 2, \cdots, m$ respectively such that,

$$\sum_{x=-p}^{p} \phi_s(x)\, \phi_t(x) = 0, \quad \text{for } s \neq t.$$

Assuming the existence of such a set of polynomials we can approximate by means of them a function $f(x)$ which is defined over the set of integers from $-p$ to p.

Writing the approximation equation,

$$(10) \quad f(x) = A_0\, \phi_0(x) + A_1\, \phi_1(x) + \cdots + A_m\, \phi_m(x),$$

we multiply by $\phi_r(x)$ and sum from $-p$ to p. We then obtain,

$$(11) \quad \sum_{-p}^{p} f(x)\, \phi_r(x) = A_r\, S_r \quad \text{, where } S_r = \sum_{-p}^{p} \phi_r^2(x).$$

If we represent the polynomial $\phi_r(x)$ by the series,

$$\phi_r(x) = \phi_0 + \phi_1 x + \phi_2 x^2 + \cdots + \phi_r x^r,$$

[6] See Berechnung der Trendlinie auf Grund der Theorie der kleinsten Quadrate, Budapest (1930) and Praktische Anwendung der Trendberechnungs-Methode von Jordan, by A. Šipos, Budapest (1930).

it is clear from the definition of the moments (1) that we get from (10) the evaluation,

$$(12) \qquad A_r = \sum_{s=1}^{r} \phi_s M_s / S_r .$$

That these coefficients are identifiable with those explicitly given in equations (3) to (9) is a consequence of the following consideration:

Let us approximate $f(x)$ by minimizing the following sum:

$$J = \sum_{x=-p}^{p} \left[f(x) - A_0 \phi_0(x) - A_1 \phi_1(x) - \cdots - A_m \phi_m(x) \right]^2 ,$$

which is equivalent in its result to the somewhat different method employed in the actual determination of the formulas (3) to (9).

Taking the derivative of J with respect to A_r and equating the result to zero we get,

$$2 \sum_{-p}^{p} \left[f(x) \phi_r(x) - \sum_{s=0}^{m} A_s \phi_r(x) \phi_s(x) \right] = 0 ;$$

whence, recalling the orthogonality of the polynomials,

$$(13) \qquad A_r = \sum_{-p}^{p} f(x) \phi_r(x) / S_r .$$

We thus see that the ratios ϕ_s / S_r can be written down explicitly by comparing them with the corresponding coefficients of M_s in equations (3) to (9).

In particular, if $r = m$, we find the coefficients of the polynomial $\phi_m(x)$ by equating the right member of (13) with the corresponding last row in the formulas (3) to (9). For example, if $r=5$, we have,

$$\frac{(\phi M_1+\phi_2 M_2+\phi_3 M_3+\phi_4 M_4+\phi_5 M_5)}{S_5}=C'M_1+E'M_3+F'M_5 .$$

Hence we get,

$$\phi_1=C'S,\qquad \phi_3=E'S,\qquad \phi_5=F'S,\qquad \phi_2=\phi_4=0.$$

By means of this identification we obtain as the first seven polynomials the following:[7]

$$\phi_0(x)=1/(2p+1)=A,$$

$$\phi_1(x)=3x/p(p+1)(2p+1)=A'x,$$

$$\phi_2(x)=\left\{\frac{3^2\cdot 5^2}{p(p+1)(4p^2-1)(2p+3)}\right\}\left\{x^2-\frac{p(p+1)}{3}\right\}=Cx^2+B,$$

$$\phi_3(x)=\left\{\frac{5^2\cdot 7}{p(p^2-1)(4p^2-1)(2p+3)(p+2)}\right\}\left\{x^3-\frac{(3p^2+3p-1)x}{5}\right\}=C'x^3+B'x,$$

$$\phi_4(x)=\left\{\frac{15^2\cdot 7^2}{4p(p^2-1)(4p^2\ 1)(4p^2-9)(2p+5)(p+2)}\right\}\cdot$$

$$\left\{x^4-\frac{(6p^2+6p-5)x^2}{7}+\frac{3p(p^2-1)(p+2)}{35}\right\}=Fx^4+Ex^2+C,$$

$$\phi_5(x)=\left\{\frac{3^4\cdot 7^2\cdot 11}{4p(p^2-1)(4p^2-1)(4p^2-9)(p^2-4)(2p+5)(p+3)}\right\}\cdot$$

$$\left\{x^5-\frac{5(2p^2+2p-3)x^3}{9}+\frac{(15p^4+30p^3-35p^2-50p+12)x}{63}\right\}$$

$$=F'x^5+E'x^3+C'x,$$

$$\phi_6(x)=\left\{\frac{3^2\cdot 7^2\cdot 11^2\cdot 13}{4p(p^2-1)(p^2-4)(4p^2-9)(4p^2-25)(p+3)(2p+7}\right\}\cdot$$

$$\left\{x^6-\frac{5(3p^2+3p-7)x^4}{11}+\frac{(5p^4+10p^3-20p^2-25p+14)x^2}{11}\right.$$

$$\left.-\frac{5p(p^2-1)(p^2-4)(p+3)}{3\cdot 7\cdot 11}\right\}=Jx^6+Ix^4+Gx^2+D,$$

$$\phi_7(x)=\left\{\frac{3^3\cdot 5\cdot 11^2\cdot 13^2}{4p(p^2-1)(p^2-4)(p^2-9)(4p^2-1)(4p^2-9)(4p^2-25)(p+4)(2p+7)}\right\}\cdot$$

$$\left\{x^7\,\frac{-7(3p^2+3p-10)x^5}{13}+\frac{7(15p^4+30p^3-90p^2-105p+101)x^3}{11\cdot 13}\right.$$

$$\left.-\frac{35p^6+105p^6-280p^4-735p^3+497p^2+882p-180)x}{3\cdot 11\cdot 13}\right\},$$

$$=J'x^7+I'x^5+G'x^3+D'x.$$

In order to effect the computation of the sum $S_m=\sum_{-p}^{p}\phi_m^2(x)$

[7]See note at end of this section.

we replace the value of A_m as given by (13) in (10) and compare the coefficient of x^m with the corresponding coefficient in the proper formula of the set from (3) to (9). Thus for $m=5$, since $\phi_5(x) = F'x^5 + E'x^3 + C'x$, we have from the quintic approximation,

$$f(x) = x^5(F'M_5 + E'M_3 + C'M_1) \quad + \text{ terms of lower degree,}$$

$$= F'x^5\left(\frac{\phi_5 M_5}{S_5} + \frac{\phi_3 M_3}{S_5} + \frac{\phi_1 M_1}{S_5}\right) \quad + \text{ terms of lower degree,}$$

$$= \left(\frac{F'^2 M_5}{S_5} + \frac{F'E'M_3}{S_5} + \frac{F'C'M_1}{S_5}\right)x^5 \quad + \text{ terms of lower degree,}$$

Equating the coefficients of M_5 it is clear that $S_5 = F'$.

4. *The Recursion Formula.* It will be obvious from the preceding discussion that the polynomials which we have investigated are essentially the analogue of the well-known Legendre polynomials, where the integration between the limits -1 and $+1$ used in the definition of the latter's orthogonality is here replaced by the discrete sum over the integers from $-p$ to $+p$. We might, therefore, expect to find a recursion formula connecting any successive three of the new polynomials similar to the recursion formula which exists for the Legendre case. It turns out that this expectation is justified and we find the following relationship holding between $\phi_{n+1}(x)$, $\phi_n(x)$, and $\phi_{n-1}(x)$:

$$(14) \quad \begin{aligned} &(n+1)^2(2p-n)(2p+n+2)\phi_{n+1}(x) \\ &- 4(2n+1)(2n+3)x\,\phi_n(x) \\ &+ 4(2n+1)(2n+3)\,\phi_{n-1}(x) = 0. \end{aligned}$$

From this equation we easily deduce that the coefficient, ϕ_{n+1}, of x^{n+1} in $\phi_{n+1}(x)$ is related to the coefficient, ϕ_n, of x^n in

$\phi_n(x)$ as follows:

$$\phi_{n+1} = 4(2n+1)(2n+3)\phi_n / (n+1)^2(2p-n)(2p+n+2).$$

From this we obtain by iteration and a proper change in notation, the following value for ϕ_n:

$$\phi_n = \{4^n 1^2 \cdot 3^2 \cdot 5^2 \cdots (2n-1)^2(2n+1)\} / \{(n!)^2 2p(4p^2-1) \cdot (4p^2-4)(4p^2-9) \cdots [4p^2-(n-1)^2](2p+n)(2p+n+1)\}. \tag{15}$$

The value thus obtained is at once seen to be equal to

$$S_n = \sum_{-p}^{p} \phi_n^2(x).$$

As an example consider the case where $n=2$. We then have,

$$\phi_2^2(x) = C^2x^4 + 2CBx^2 + B^2 = a^2\{x^4 - \tfrac{2}{3}p(p+1)x^2 + p^2(p+1)^2/9\},$$

where we abbreviate, $a^2 = 3^4 \cdot 5^2 / p^2(p+1)^2(4p^2-1)^2(2p+3)^2$

We then obtain,

$$\sum_{-p}^{p} \phi_2^2(x) = a^2\{\sum_{-p}^{p} x^4 - 2p(p+1)\sum_{-p}^{p} \frac{x^2}{3} + p^2(p+1)^2(2p+1)/9\},$$

$$= a^2\{\frac{p}{15}(p+1)(2p+1)(3p^2+3p-1) - \frac{2p^2}{9}(p+1)^2(2p+1) + p^2(p+1)^2(2p+1)/9\},$$

$$= a^2\{p(p+1)(4p^2-1)(2p+3)/45\} = 3^2 \cdot 5/p(p+1)(4p^2-1)(2p+3)$$

which is seen to agree with the value of S_2 as calculated directly from (15).[8]

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[8]The definition of the $\phi_n(x)$ which we have given above was chosen for the obvious connection which the functions in that form have with the problem of curve fitting and with the computed values in the tables. If, however, the coefficient of x^n were reduced to unity,

$$\phi_0(x) = 1, \ \phi_1(x) = x, \ \phi_2(x) = x^2 - p(p+1)/3,$$

etc., then the recursion formula (14) would have been,

$$4(4n^2-1)\phi_{n+1}(x) - 4(4n^2-1)x\phi_n(x) + n^2(2p-n+1)(2p+n+1)\phi_{n-1}(x) = 0$$

If, moreover, $\phi_n(x)$ as just defined were multiplied by the coefficient $1 \cdot 3 \cdot 5 \ldots (2n-1)/n!$, which is the multiplier of the corresponding Legendre polynomials, then the recursion formula becomes,

$$4(n+1)\phi_{n+1}(x) - 4(2n+1)\phi_n(x) + n(2p-n+1)(2p+n+1)\phi_{n-1}(x) = 0.$$

5. *The Polynomials of Gram.* It will be at once evident that the results obtained above permit us to define a new set of polynomials which are orthogonal over the discrete range from

$x=1$ to $x=p'$

In the former paper in the Annals it was proved that the formulas for the coefficients of the least square polynomial,

$$y=A_0+A_1x+\cdots\cdots+A_m x^m,$$

fitted to data given over the discrete range $x=1$ to $x=p'$, can be obtained from the coefficients, $a_0, a_1, a_2, \cdots, a_m$, of equation (2), by means of the following substitution:

$$p=(p'-1)/2,$$

$$M_r=m_r-r\left(\frac{p'+1}{2}\right)m_{r-1}+\frac{r(r-1)}{2!}\left(\frac{p'+1}{2}\right)^2 m_{r-2}-\cdots\cdots+(-1)^r\left(\frac{p'+1}{2}\right)^r m_0,$$

where M_r are the moments defined by (1) and m_r are the moments,

$$m_r=\sum_{s=1}^{p'} s^r y_s.$$

Conversely we can pass from the range $x=-p$ to $x=p$ to the range $x=1$ to $x=p'$, by means of the substitution:

$$p'=2p+1,$$

$$m_r=M_r+r\left(\frac{p'+1}{2}\right)M_{r-1}+\frac{r(r-1)}{2!}\left(\frac{p'+1}{2}\right)^2 M_{r-2}+\cdots\cdots\left(\frac{p'+1}{2}\right)^r M_0.$$

Replacing M_r and m_r by x^r, it is clear that new polynomials $\psi_m(x)$ are obtained which belong to the range $x=1$ to $x=p'$. The polynomials may be explicitly evaluated from

$$\phi_m(x)=\phi_0+\phi_1 x+\cdots+\phi_m x^m$$

as follows:

$$
\begin{aligned}
(16)\qquad \psi_m(x) = {} & \phi_0' + \phi_1'(x-b) + \phi_2'(x^2 - 2bx + b^2) + \cdots \\
& + \phi_m'\left[x^m - mbx^{m-1} + m(m-1)b^2x^{m-2}/2 + \cdots\right],
\end{aligned}
$$

where $b = (p'-1)/2$ and ϕ_r' denotes the value of ϕ_r after the substitution $p = (p'-1)/2$.

These polynomials can be proved by the method of section 3 to be orthogonal over the discrete range $x=1$ to $x=p'$ and they are identifiable with the last lines of the formulas (3), (4), and (5) of the Annals paper previously cited where the m_r are replaced by x^r

Polynomials orthogonal over the discrete range $x=0$ to $x=n-1$ were first obtained by Gram in the paper to which we previously referred and hence (16) may properly be called *the Gram polynomial of m th degree.*

The following explicit formula, in the notation of the present paper where the range is from $x=1$ to $x=p$, was derived by Gram:

$$
\begin{aligned}
\psi_m(x) = \left\{\frac{1}{(p-m-1)!}\right\}\Bigg\{ & \frac{(p-1)!}{m!} - \frac{(m+1)(p-2)!(x-1)}{(m-1)!\cdot 1!^2} \\
& + \frac{(m+1)(m+2)(p-3)!\cdot(x-1)(x-2)}{(m-2)!\,2!^2} \\
& - \frac{(m+1)(m+2)(m+3)(p-4)!(x-1)(x-2)(x-3)}{(m-3)!\,3!^2} + \cdots \Bigg\}.
\end{aligned}
$$

Since the coefficient of x^m in $\psi_m(x)$ equals $\phi_m' = 4^m 1^2\cdot 3^2 5^2 \cdots (2m-1)^2(2m+1)/(m!)^2\cdot p(p^2-1)(p^2-4)\cdots(p^2-m^2)$, and since the coefficient of x^m in Gram's definition is $(-1)^m(m+1)(m+2)\cdots\frac{(2m)}{(m!)^2}$, it is clear that the following equation holds between $\Psi_m(x)$ and $\psi_m(x)$:

$$\bar{\psi}_m(x) = \{(-1)^m p(p^2-1)(p^2-4)\cdots(p^2-m^2)(2m)!/4^m \cdot 1^2 \cdot 3^2 \cdot 5^2 \cdots$$

$$(2m-1)^2(2m+1)m!\}\ \psi_m(x),$$

$$= \{(-1)^m p(p^2-1)(p^2-4)\cdots(p^2-m^2)(m!)/(2m)!(2m+1)\}\psi_m(x).$$

By methods previously used it can be shown that,

$$\sum_{x=1}^{p} \psi_m^2(x) = \phi'(p).$$

The first four Gram polynomials are given below explicitly:[9]

$$\psi_0(x) = 1/p,$$

$$\psi_1(x) = \left[12/p(p^2-1)\right]\left[x-(p+1)/2\right],$$

$$\psi_2(x) = \left[180/p(p^2-1)(p^2-4)\right]\left[x^2-(p+1)x+(p+1)(p+2)/6\right],$$

$$\psi_3(x) = \left[2800/p(p^2-1)(p^2-4)(p^2-9)\right]\left[x^3-3(p+1)\frac{x^2}{2}+\right.$$

$$\left.\frac{1}{10}(6p^2+15p+11)x \quad -(p+1)(p+2)(p+3)/20\right].$$

6. *Tables and Numerical Application.* In tables 1 to 7 the numerical values of the coefficients of equations (3) to (9) have been tabulated for values of p by half integers. For the case of the straight line the range of p is from 0.5 to 100.0; for the parabola the range is from 1.0 to 100.0; for the cubic the range is from 1.5 to 50.0; for the other polynomials the range does not exceed $p = 25.0$. The tables have been computed to ten significant figures and have been checked by duplicate calculation.

[9]These polynomials are essentially the same as those employed by Jordan (loc. cit.) except that the summation in his work has taken over the numbers 0, 1, 2, ..., $n-1$. His polynomials are also expressed in terms of the Newton polynomials: $x(x-1)\cdots(x-n)$.

In illustration of the application of these tables to the numerical problem of polynomial approximation, we shall fit polynomials to the data employed by Karl Pearson in the same connection, his method being, however, the method of moments.[10] The data are from T. N. Thiele[11] and consist of a system of frequencies obtained from a game of patience (solitaire):

Value of character	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Frequency	0	3	7	35	101	89	94	70	46	30	15	4	5	1	0
Class marks (x)	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7

Computing the moments, using the values of x for this purpose, we obtain the following:

$$M_0 = 500; \quad M_1 = -570; \quad M_2 = 2728; \quad M_3 = -5508;$$

$$M_4 = 34108; \quad M_5 = -76380; \quad M_6 = 626188; \quad M_7 = -1419708.$$

These values are then substituted in equations (3) to (9) and the coefficients of the desired polynomials thus obtained. In illustration we shall give only the computations for the parabola.

From the value corresponding to $p=7$ in column A' of table 1 we obtain,

$$a_1 = M_1 . (2)357\,1428\,571 = -2.035\,714;$$

Similarly from the values corresponding to $p=7$ in columns A, B, and C of table 2 we compute,

$$a_0 = M_0 . 151\,1312\,217 - M_2 \cdot (2)452\,4886\,878 = 63.221\,719,$$

$$a_2 = -M_0 . (2)\,452\,4886\,878 + M_2 \cdot (3)242\,4046\,542 - 1.601\,1635$$

[10]On the Systematic Fitting of Curves to Observations and Measurements. Biometrika, vol. 1 (1902), pp. 265-303; vol. 2 (1903), pp. 1-27, in particular, p. 18.

[11]Forelaesninger over Almindelig Iagttagelseslaere, Copenhagen, (1889), p. 12.

Proceeding in this manner the other coefficients are easily computed and we obtain the following seven polynomials of approximation:

$$y = 33.333333 - 2.035714\,x,$$

$$y = 63.221719 - 2.035714x - 1.601163x^2,$$

$$y = 63.221719 - 9.924624x - 1.601163x^2 + .236195x^3,$$

$$y = 75.058367 - 9.924624x - 3.760517x^2 + .236195x^3 + .045666x^4,$$

$$y = 75.058367 - 20.405890x - 3.760517x^2 + 1.139793x^3 + .045666x^4 - .014922x^5,$$

$$y = 73.950386 - 20.405890x - 3.320586x^2 + 1.139793x^3 + .020890x^4 - .014922x^5 + .000338551x^6,$$

$$y = 73.950386 - 25.763034x - 3.320586x^2 + 2.070322x^3 + .020890x^4 - .054118x^5 + .000338551x^6 + .0004607106x^7.$$

The following table contains the values computed from these polynomials over the range from $x = -7$ to $x = 7$:

x	y	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
−7	0	47.583	−.985	−26.778	−11.105	3.123	3.916	.591
−6	3	45.548	17.794	14.109	7.393	−8.863	−10.448	−3.483
−5	7	43.512	33.371	43.291	29.685	15.773	15.469	12.432
−4	35	41.476	45.746	62.185	51.163	50.538	51.513	45.975
−3	101	39.440	54.918	72.208	68.309	78.982	80.073	79.537
−2	89	37.405	60.888	74.777	78.707	92.918	93.194	97.660
−1	94	35.369	63.656	71.309	81.032	90.625	89.932	94.397
0	70	33.333	63.222	63.222	75.058	75.058	73.950	73.950
1	46	31.298	59.585	51.932	61.655	52.062	51.370	46.905
2	30	29.262	52.746	38.857	42.787	28.576	28.853	24.388
3	15	27.226	42.704	25.415	21.516	10.843	11.935	12.471
4	4	25.190	29.460	13.021	1.999	2.624	3.599	9.136
5	5	23.155	13.014	3.094	−10.512	3.400	3.095	6.133
6	1	21.119	−6.634	−2.950	−9.667	6.589	5.005	−1.959
7	0	19.083	−29.485	−33.693	11.980	−2.249	−1.457	.869

The approximation attained by these polynomials is exhibited in the following charts, the odd order cases being given in figure 1 and the even order cases in figure 2.

In order to illustrate the case where the number of items is even we shall delete the last value from the series which we have just used. The table must then be arranged as follows:

Frequency	0	3	7	35	101	89	94	70
Class mark	-13/2	-11/2	-9/2	-7/2	-5/2	-3/2	-1/2	1/2
Frequency	46	30	15	4	5	1		
Class mark	3/2	5/2	7/2	9/2	11/2	13/2		

The method will be sufficiently illustrated by means of the first, second, and fifth degree polynomials. To compute these we first obtain the moments:

$$M_0 = 500; \quad M_1 = -320; \quad M_2 = 2283; \quad M_3 = -1781;$$
$$M_4 = 26930.25; \quad M_5 = -1632.5.$$

In order to evaluate the coefficients of the parabola we use the value corresponding to $p = 6.5$ in column A' of table 1 and the values in columns A, B, and C of table 2 as follows:

$$a_1 = M_1 \cdot (2)\,439\,5604\,396 = -1.406\,593,$$

$$a_0 = M_0 \cdot 162\,1093\,750 - M_2 \cdot (2)\,558\,0357\,143 = 68.314\,734,$$

$$a_2 = -M_0 \cdot (2)\,558\,0357\,143 + M_2\,(3)\,3434065\,934 = -2.006\,181.$$

The other coefficients are similarly obtained and we thus derive the following approximating polynomials:

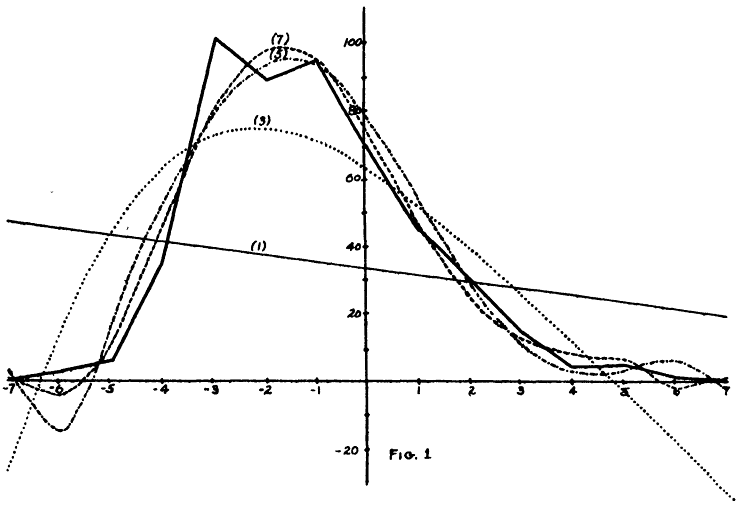
$$y = 35.714286 - 1.406593x,$$
$$y = 68.314734 - 1.406593x - 2.006181x^2;$$
$$y = 83.37250 - 16.63150x - 5.172034x^2 + 1.149221x^3 + .077082x^4$$
$$- .017801x^5.$$

The approximating values obtained from the parabola and the quintic are recorded in the following table:

x	y	$n=2$	$n=5$	x	y	$n=2$	$n=5$
-6.5	0	- 7.304	1.492	0.5	70	67.110	73.912
-5.5	3	15.364	- 12.686	1.5	46	61.691	50.922
-4.5	7	34.019	13.214	2.5	30	52.260	28.698
-3.5	35	48.662	49.869	3.5	15	38.816	13.295
-2.5	101	59.293	79.419	4.5	4	21.360	7.280
-1.5	89	65.911	93.329	5.5	5	-.108	7.592
-0.5	94	68.516	90.257	6.5	1	-25.589	3.407

In the article from which these data are taken, Karl Pearson compares the efficacy of polynomial curves with that of skew-frequency curves and shows the superiority of the latter in the present case. It is worth noting here, however, that the *least square* polynomials of the present paper give a measurably better fit than the *moment* polynomials employed by Pearson. The sum of the squares of the deviations from the data of the values obtained by means of the sixth degree parabola of Pearson is found to be 1402.31; the same sum for the sextic of the present paper is 1091.22. For the septic the sum of the square of the deviations is 926.32, which compares not too unfavorably with the sum 760.91 obtained from the skew-frequency curve.

H. T. Davis


(7)
(5)
(3)
(1)
100
80
60
40
20
-20
-7
-6
-5
-4
-3
-2
-1
1
2
3
4
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6
7

FIG. 1

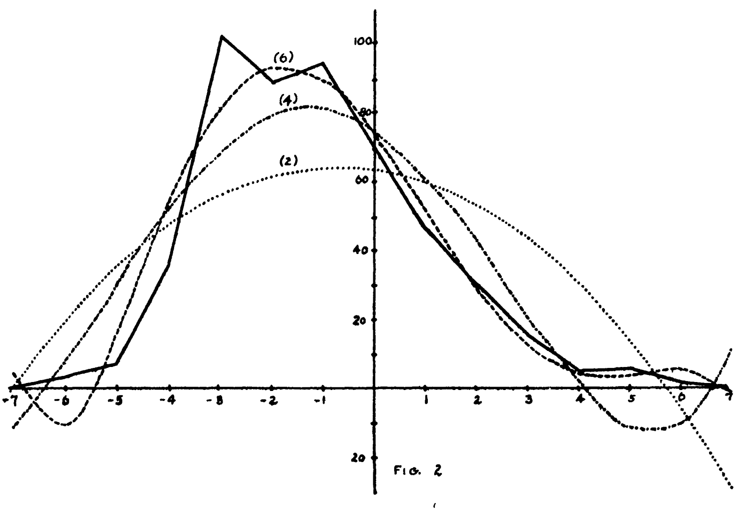
(6)
(4)
(2)
100
80
60
40
20
20
-7
-6
-5
-4
-3
-2
-1
1
2
3
4
5
6
7

FIG. 2

TABLE I

(The numbers in parentheses denote the number of ciphers between the decimal point and the first significant figure.)

p	A	A'
0.5	.500 0000 000	2.000 0000 000
1.0	.333 3333 333	.500 0000 000
1.5	.250 0000 000	.200 0000 000
2.0	.200 0000 000	.100 0000 000
2.5	.166 6666 667	.(1)571 4285 714
3.0	.142 8571 429	.(1)357 1428 571
3.5	.125 0000 000	.(1)238 0952 381
4.0	.111 1111 111	.(1)166 6666 667
4.5	.100 0000 000	.(1)121 2121 212
5.0	.(1)909 0909 091	.(2)909 0909 091
5.5	.(1)833 3333 333	.(2)699 3006 993
6.0	.(1)769 2307 692	.(2)549 4505 495
6.5	.(1)714 2857 143	.(2)439 5604 396
7.0	.(1)666 6666 667	.(2)357 1428 571
7.5	.(1)625 0000 000	.(2)294 1176 471
8.0	.(1)588 2352 941	.(2)245 0980 392
8.5	.(1)555 5555 556	.(2)206 3983 488
9.0	.(1)526 3157 895	.(2)175 4385 965
9.5	.(1)500 0000 000	.(2)150 3759 398
10.0	.(1)476 1904 762	.(2)129 8701 299
10.5	.(1)454 5454 545	.(2)112 9305 477
11.0	.(1)434 7826 087	.(3)988 1422 925
11.5	.(1)416 6666 667	.(3)869 5652 174
12.0	.(1)400 0000 000	.(3)769 2307 692
12.5	.(1)384 6153 846	.(3)683 7606 838
13.0	.(1)370 3703 704	.(3)610 5006 105
13.5	.(1)357 1428 571	.(3)547 3453 749
14.0	.(1)344 8275 862	.(3)492 6108 374
14.5	.(1)333 3333 333	.(3)444 9388 209
15.0	.(1)322 5806 452	.(3)403 2258 064
15.5	.(1)312 5000 000	.(3)366 5689 150
16.0	.(1)303 0303 030	.(3)334 2245 989
16.5	.(1)294 1176 471	.(3)305 5767 762
17.0	.(1)285 7142 857	.(3)280 1120 448
17.5	.(1)277 7777 778	.(3)257 4002 574
18.0	.(1)270 2702 703	.(3)237 0791 844
18.5	.(1)263 1578 947	.(3)218 8423 241
19.0	.(1)256 4102 564	.(3)202 4291 498
19.5	.(1)250 0000 000	.(3)187 6172 608
20.0	.(1)243 9024 390	.(3)174 2160 279
20.5	.(1)238 0952 381	.(3)162 0614 213
21.0	.(1)232 5581 395	.(3)151 0117 789
21.5	.(1)227 2727 273	.(3)140 9443 270
22.0	.(1)222 2222 222	.(3)131 7523 057
22.5	.(1)217 3913 043	.(3)123 3425 840
23.0	.(1)212 7659 574	.(3)115 6336 725
23.5	.(1)208 3333 333	.(3)108 5540 599
24.0	.(1)204 0816 327	.(3)102 0408 163
24.5	.(1)200 0000 000	.(4)960 3841 537
25.0	.(1)196 0784 314	.(4)904 9773 756

TABLE I—(Continued)

p	A	A'
25.5	.(1)192 3076 923	.(4)853 7522 411
26.0	.(1)188 6792 453	.(4)806 3215 610
26.5	.(1)185 1851 852	.(4)762 3403 850
27.0	.(1)181 8181 818	.(4)721 5007 215
27.5	.(1)178 5714 286	.(4)683 5269 993
28.0	.(1)175 4385 965	.(4)648 1721 545
28.5	.(1)172 4137 931	.(4)615 2142 484
29.0	.(1)169 4915 254	.(4)584 4535 359
29.5	.(1)166 6666 667	.(4)555 7099 194
30.0	.(1)163 9344 262	.(4)528 8207 298
30.5	.(1)161 2903 226	.(4)503 6387 903
31.0	.(1)158 7301 587	.(4)480 0307 220
31.5	.(1)156 2500 000	.(4)457 8754 579
32.0	.(1)153 8461 538	.(4)437 0629 371
32.5	.(1)151 5151 515	.(4)417 4929 548
33.0	.(1)149 2537 313	.(4)399 0741 480
33 5	.(1)147 0588 235	.(4)381 7230 981
34.0	.(1)144 9275 362	.(4)365 3635 367
34.5	.(1)142 8571 429	.(4)349 9256 408
35.0	.(1)140 8450 704	.(4)335 3454 058
35.5	.(1)138 8888 889	.(4)321 5640 877
36.0	.(1)136 9863 014	.(4)308 5277 058
36.5	.(1)135 1351 351	.(4)296 1865 976
37.0	.(1)133 3333 333	.(4)284 4950 213
37.5	.(1)131 5789 474	.(4)273 4107 997
38.0	.(1)129 8701 299	.(4)262 8949 997
38.5	.(1)128 2051 282	.(4)252 9116 453
39.0	.(1)126 5822 785	.(4)243 4274 586
39.5	.(1)125 0000 000	.(4)234 4116 268
40.0	.(1)123 4567 901	.(4)225 8355 917
40.5	.(1)121 9512 195	.(4)217 6728 595
41.0	.(1)120 4819 277	.(4)209 8988 288
41.5	.(1)119 0476 190	.(4)202 4906 348
42.0	.(1)117 6470 588	.(4)195 4270 080
42.5	.(1)116 2790 698	.(4)188 6881 457
43.0	.(1)114 9425 287	.(4)182 2555 952
43 5	.(1)113 6363 636	.(4)176 1121 482
44.0	.(1)112 3595 506	.(4)170 2417 433
44.5	.(1)111 1111 111	.(4)164 6293 781
45.0	.(1)109 8901 099	.(4)159 2610 288
45.5	.(1)108 6956 522	.(4)154 1235 763
46.0	.(1)107 5268 817	.(4)149 2047 387
46.5	.(1)106 3829 787	.(4)144 4930 102
47.0	.(1)105 2631 579	.(4)139 9776 036
47.5	.(1)104 1666 667	.(4)135 6483 993
48.0	.(1)103 0927 835	.(4)131 4958 973
48.5	.(1)102 0408 163	.(4)127 5111 732
49.0	.(1)101 0101 010	.(4)123 6858 380
49.5	.(1)100 0000 000	.(4)120 0120 012
50.0	.(2)990 0990 099	.(4)116 4822 365

TABLE I—(Continued)

ρ	A	A'
50.5	.(2)980 3921 569	.(4)113 0895 500
51.0	.(2)970 8737 864	.(4)109 8273 514
51.5	.(2)961 5384 615	.(4)106 6894 271
52.0	.(2)952 3809 524	.(4)103 6699 150
52.5	.(2)943 3962 264	.(4)100 7632 819
53.0	.(2)934 5794 393	.(5)979 6430 181
53.5	.(2)925 9259 259	.(5)952 6803 662
54.0	.(2)917 4311 927	.(5)926 6981 744
54.5	.(2)909 0909 091	.(5)901 6522 778
55.0	.(2)900 9009 009	.(5)877 5008 775
55.5	.(2)892 8571 429	.(5)854 2043 940
56.0	.(2)884 9557 522	.(5)831 7253 310
56.5	.(2)877 1929 825	.(5)810 0281 485
57.0	.(2)869 5652 174	.(5)789 0791 446
57.5	.(2)862 0689 655	.(5)768 8463 461
58.0	.(2)854 7008 547	.(5)749 2994 051
58.5	.(2)847 4576 271	.(5)730 4095 041
59.0	.(2)840 3361 345	.(5)712 1492 665
59.5	.(2)833 3333 333	.(5)694 4926 731
60.0	.(2)826 4462 810	.(5)677 4149 844
60.5	.(2)819 6721 311	.(5)660 8926 677
61.0	.(2)813 0081 301	.(5)644 9033 290
61.5	.(2)806 4516 129	.(5)629 4256 491
62.0	.(2)800 0000 000	.(5)614 4393 241
62.5	.(2)793 6507 937	.(5)599 9250 094
63.0	.(2)787 4015 748	.(5)585 8642 670
63.5	.(2)781 2500 000	.(5)572 2395 166
64.0	.(2)775 1937 984	.(5)559 0339 893
64.5	.(2)769 2307 692	.(5)546 2316 842
65.0	.(2)763 3587 786	.(5)533 8173 277
65.5	.(2)757 5757 576	.(5)521 7763 354
66.0	.(2)751 8796 992	.(5)510 0947 756
66.5	.(2)746 2686 567	.(5)498 7593 361
67.0	.(2)740 7407 407	.(5)487 7572 920
67.5	.(2)735 2941 176	.(5)477 0764 754
68.0	.(2)729 9270 073	.(5)466 7052 476
68.5	.(2)724 6376 812	.(5)456 6324 725
69.0	.(2)719 4244 604	.(5)446 8474 910
69.5	.(2)714 2857 143	.(5)437 3400 975
70.0	.(2)709 2198 582	.(5)428 1005 180
70.5	.(2)704 2253 521	.(5)419 1193 883
71.0	.(2)699 3006 993	.(5)410 3877 343
71.5	.(2)694 4444 444	.(5)401 8969 536
72.0	.(2)689 6551 724	.(5)393 6387 970
72.5	.(2)684 9315 068	.(5)385 6053 522
73.0	.(2)680 2721 088	.(5)377 7890 275
73.5	.(2)675 6756 757	.(5)370 1825 370
74.0	.(2)671 1409 396	.(5)362 7788 863
74.5	.(2)666 6666 667	.(5)355 5713 587
75.0	.(2)662 2516 556	.(5)348 5535 030

TABLE I—(Continued)

ρ	A	A'
75.5	.(2)657 8947 368	.(5)341 7191 206
76.0	.(2)653 5947 712	.(5)335 0622 546
76.5	.(2)649 3506 494	.(5)328 5771 787
77.0	.(2)645 1612 903	.(5)322 2583 868
77.5	.(2)641 0256 410	.(5)316 1005 832
78.0	.(2)636 9426 752	.(5)310 0986 734
78.5	.(2)632 9113 924	.(5)304 2477 550
79.0	.(2)628 9308 176	.(5)298 5431 096
79.5	.(2)625 0000 000	.(5)292 9801 945
80.0	.(2)621 1180 124	.(5)287 5546 354
80.5	.(2)617 2839 506	.(5)282 2622 188
81.0	.(2)613 4969 325	.(5)277 0988 855
81.5	.(2)609 7560 976	.(5)272 0607 240
82.0	.(2)606 0606 061	.(5)267 1439 639
82.5	.(2)602 4096 386	.(5)262 3449 705
83.0	.(2)598 8023 952	.(5)257 6602 389
83.5	.(2)595 2380 952	.(5)253 0863 885
84.0	.(2)591 7159 763	.(5)248 6201 581
84.5	.(2)588 2352 941	.(5)244 2584 010
85.0	.(2)584 7953 216	.(5)239 9980 800
85.5	.(2)581 3953 488	.(5)235 8362 636
86.0	.(2)578 0346 821	.(5)231 7701 211
86.5	.(2)574 7126 437	.(5)227 7969 190
87.0	.(2)571 4285 714	.(5)223 9140 170
87.5	.(2)568 1818 182	.(5)220 1188 642
88.0	.(2)564 9717 514	.(5)216 4089 957
88.5	.(2)561 7977 528	.(5)212 7820 293
89.0	.(2)558 6592 179	.(5)209 2356 621
89.5	.(2)555 5555 556	.(5)205 7676 677
90.0	.(2)552 4861 878	.(5)202 3758 930
90.5	.(2)549 4505 495	.(5)199 0582 554
91.0	.(2)546 4480 874	.(5)195 8127 404
91.5	.(2)543 4782 609	.(5)192 6373 986
92.0	.(2)540 5405 405	.(5)189 5303 438
92.5	.(2)537 6344 086	.(5)186 4897 501
93.0	.(2)534 7593 583	.(5)183 5138 498
93.5	.(2)531 9148 936	.(5)180 6009 315
94.0	.(2)529 1005 291	.(5)177 7493 378
94.5	.(2)526 3157 895	.(5)174 9574 635
95.0	.(2)523 5602 094	.(5)172 2237 531
95.5	.(2)520 8333 333	.(5)169 5466 999
96.0	.(2)518 1347 150	.(5)166 9248 438
96.5	.(2)515 4639 175	.(5)164 3567 692
97.0	.(2)512 8205 128	.(5)161 8411 044
97.5	.(2)510 2040 816	.(5)159 3765 191
98.0	.(2)507 6142 132	.(5)156 9617 233
98.5	.(2)505 0505 051	.(5)154 5954 662
99.0	.(2)502 5125 628	.(5)152 2765 342
99.5	.(2)500 0000 000	.(5)150 0037 501
100.0	.(2)497 5124 378	.(5)147 7759 716

TABLE II

(The number in parentheses denote the number of ciphers between the decimal point and the first significant figure.)

ρ	A	B	C
1.0	1.000 0000 000	− 1.000 0000 000	1.500 0000 000
1.5	.640 6250 000	− .312 5000 000	.250 0000 000
2.0	.485 7142 857	− .142 8571 429	.(1)714 2857 143
2.5	.394 5312 500	−.(1)781 2500 000	.(1)267 8571 429
3.0	.333 3333 333	−.(1)476 1904 762	.(1)119 0476 190
3.5	.289 0625 000	−.(1)312 5000 000	.(2)595 2380 952
4.0	.255 4112 554	−.(1)216 4502 165	.(2)324 6753 247
4.5	.228 9062 500	−.(1)156 2500 000	.(2)189 3939 394
5.0	.207 4592 075	−.(1)116 5501 166	.(2)116 5501 166
5.5	.189 7321 429	−.(2)892 8571 429	.(3)749 2507 493
6.0	.174 8251 748	−.(2)699 3006 993	.(3)499 5004 995
6.5	.162 1093 750	−.(2)558 0357 143	.(3)343 4065 934
7.0	.151 1312 217	−.(2)452 4886 878	.(3)242 4046 542
7.5	.141 5550 595	−.(2)372 0238 095	.(3)175 0700 280
8.0	.133 1269 350	−.(2)309 5975 232	.(3)128 9989 680
8.5	.125 6510 417	−.(2)260 4166 667	.(4)967 4922 601
9.0	.118 9739 054	−.(2)221 1410 880	.(4)737 1369 601
9.5	.112 9734 848	−.(2)189 3939 394	.(4)569 6058 328
10.0	.107 5514 874	−.(2)163 4521 085	.(4)445 7784 778
10.5	.102 6278 409	−.(2)142 0454 545	.(4)352 9079 616
11.0	.(1)981 3664 596	−.(2)124 2236 025	.(4)282 3263 693
11.5	.(1)940 2316 434	−.(2)109 2657 343	.(4)228 0328 367
12.0	.(1)902 4154 589	−.(3)966 1835 749	.(4)185 8045 336
12.5	.(1)867 5309 066	−.(3)858 5164 835	.(4)152 6251 526
13.0	.(1)835 2490 421	−.(3)766 2835 249	.(4)126 3104 711
13.5	.(1)805 2884 615	−.(3)686 8131 868	.(4)105 2587 260
14.0	.(1)777 4069 954	−.(3)617 9705 846	.(5)882 8151 209
14.5	.(1)751 3950 893	−.(3)558 0357 143	.(5)744 8752 582
15.0	.(1)727 0704 824	−.(3)505 6122 965	.(5)632 0153 706
15.5	.(1)704 2738 971	−.(3)459 5588 235	.(5)539 0719 338
16.0	.(1)682 8655 216	−.(3)418 9359 028	.(5)462 0616 575
16.5	.(1)662 7221 201	−.(3)382 9656 863	.(5)397 8864 273
17.0	.(1)643 7346 437	−.(3)351 0003 510	.(5)344 1179 912
17.5	.(1)625 8062 436	−.(3)322 4974 200	.(5)298 8393 081
18.0	.(1)608 8506 089	−.(3)297 0002 970	.(5)260 5265 763
18.5	.(1)592 7905 702	−.(3)274 1228 070	.(5)227 9607 543
19.0	.(1)577 5569 190	−.(3)253 5368 389	.(5)200 1606 623
19.5	.(1)563 0874 060	−.(3)234 9624 060	.(5)176 3320 120
20.0	.(1)549 3258 868	−.(3)218 1596 056	.(5)155 8282 897
20.5	.(1)536 2215 909	−.(3)202 9220 779	.(5)138 1205 295
21.0	.(1)523 7284 931	−.(3)189 0716 582	.(5)122 7738 040
21.5	.(1)511 8047 713	−.(3)176 4539 808	.(5)109 4288 253
22.0	.(1)500 4123 371	−.(3)164 9348 507	.(6)977 8746 091
22.5	.(1)489 5164 279	−.(3)154 3972 332	.(6)876 0126 706
23.0	.(1)479 0852 511	−.(3)144 7387 466	.(6)785 2017 048
23.5	.(1)469 0896 739	−.(3)135 8695 652	.(6)707 9612 604
24.0	.(1)459 5029 501	−.(3)127 7106 587	.(6)638 5532 937
24.5	.(1)450 3004 808	−.(3)120 1923 077	.(6)577 1539 385
25.0	.(1)441 4596 027	−.(3)113 2528 483	.(6)522 7054 537

TABLE II—(Continued)

p	A	B	C
25.5	.(1)432 9594 017	−.(3)106 8376 068	.(6)474 3068 006
26.0	.(1)424 7805 469	−.(3)100 8979 921	.(6)431 1880 006
26.5	.(1)416 9051 435	−.(4)953 9072 039	.(6)392 6890 719
27.0	.(1)409 3166 020	−.(4)902 7715 085	.(6)358 2426 621
27.5	.(1)401 9995 211	−.(4)855 2271 483	.(6)327 3596 740
28.0	.(1)394 9395 832	−.(4)810 9642 365	.(6)299 6173 287
28.5	.(1)388 1234 606	−.(4)769 7044 335	.(6)274 6492 180
29.0	.(1)381 5387 315	−.(4)731 1972 624	.(6)252 1369 870
29.5	.(1)375 1738 042	−.(4)695 2169 077	.(6)231 8033 590
30.0	.(1)369 0178 489	−.(4)661 5594 279	.(6)213 4062 671
30.5	.(1)363 0607 359	−.(4)630 0403 226	.(6)196 7339 024
31.0	.(1)357 2929 802	−.(4)600 4924 038	.(6)181 6005 253
31.5	.(1)351 7056 910	−.(4)572 7639 296	.(6)167 8429 098
32.0	.(1)346 2905 254	−.(4)546 7169 646	.(6)155 3173 195
32.5	.(1)341 0396 474	−.(4)522 2259 358	.(6)143 8969 284
33.0	.(1)335 9456 896	−.(4)499 1763 590	.(6)133 4696 147
33.5	.(1)331 0017 189	−.(4)477 4637 128	.(6)123 9360 708
34.0	.(1)326 2012 046	−.(4)456 9924 414	.(6)115 2081 785
34.5	.(1)321 5379 902	−.(4)437 6750 700	.(6)107 2076 105
35.0	.(1)317 0062 663	−.(4)419 4314 188	.(7)998 6462 352
35.5	.(1)312 6005 470	−.(4)402 1879 022	.(7)931 1701 382
36.0	.(1)308 3156 473	−.(4)385 8769 053	.(7)869 0921 290
36.5	.(1)304 1466 631	−.(4)370 4362 257	.(7)811 9150 152
37.0	.(1)300 0889 521	−.(4)355 8085 750	.(7)759 1932 610
37.5	.(1)296 1381 169	−.(4)341 9411 314	.(7)710 5270 263
38.0	.(1)292 2899 885	−.(4)328 7851 389	.(7)665 5569 614
38.5	.(1)288 5406 123	−.(4)316 2955 466	.(7)623 9596 513
39.0	.(1)284 8862 343	−.(4)304 4306 842	.(7)585 4436 234
39.5	.(1)281 3232 880	−.(4)293 1519 700	.(7)550 4855 968
40.0	.(1)277 8483 837	−.(4)282 4236 468	.(7)516 6286 221
40.5	.(1)274 4582 970	−.(4)272 2125 436	.(7)485 8769 184
41.0	.(1)271 1499 593	−.(4)262 4878 599	.(7)457 2959 232
41.5	.(1)267 9204 481	−.(4)253 2209 708	.(7)430 7052 407
42.0	.(1)264 7669 787	−.(4)244 3852 489	.(7)405 9555 630
42.5	.(1)261 6868 960	−.(4)235 9559 046	.(7)382 8899 060
43.0	.(1)258 6776 671	−.(4)227 9098 389	.(7)361 3792 371
43.5	.(1)255 7368 746	−.(4)220 2255 109	.(7)341 3026 128
44.0	.(1)252 8622 095	−.(4)212 8828 165	.(7)322 5542 998
44.5	.(1)250 0514 657	−.(4)205 8629 776	.(7)305 0198 458
45.0	.(1)247 3025 344	−.(4)199 1484 413	.(7)288 6209 294
45.5	.(1)244 6133 981	−.(4)192 7227 875	.(7)273 2687 523
46.0	.(1)241 9821 265	−.(4)186 5706 450	.(7)258 8861 864
46.5	.(1)239 4068 715	−.(4)180 6776 133	.(7)245 4025 308
47.0	.(1)236 8858 628	−.(4)175 0301 927	.(7)232 7529 158
47.5	.(1)234 4174 039	−.(4)169 6157 186	.(7)220 8777 671
48.0	.(1)231 9998 685	−.(4)164 4223 022	.(7)209 7223 243
48.5	.(1)229 6316 964	−.(4)159 4387 755	.(7)199 2362 081
49.0	.(1)227 3113 909	−.(4)154 6546 407	.(7)189 3730 295
49.5	.(1)225 0375 150	−.(4)150 0600 240	.(7)180 0900 478
50.0	.(1)222 8086 886	−.(4)145 6456 325	.(7)171 3478 030

TABLE II—(Continued)

p	A	B	C
50.5	.(1)220 6235 860	−.(4)141 4027 149	.(7)163 1099 278
51.0	.(1)218 4809 327	−.(4)137 3230 250	.(7)155 3427 884
51.5	.(1)216 3795 035	−.(4)133 3987 877	.(7)148 0152 984
52.0	.(1)214 3181 200	−.(4)129 6226 684	.(7)141 0986 957
52.5	.(1)212 2956 479	−.(4)125 9877 439	.(7)134 5663 486
53.0	.(1)210 3109 957	−.(4)122 4874 757	.(7)128 3935 804
53.5	.(1)208 3631 123	−.(4)119 1156 852	.(7)122 5575 085
54.0	.(1)206 4509 850	−.(4)115 8665 310	.(7)117 0369 000
54.5	.(1)204 5736 382	−.(4)112 7344 877	.(7)111 8120 384
55.0	.(1)202 7301 313	−.(4)109 7143 258	.(7)106 8646 031
55.5	.(1)200 9195 574	−.(4)106 8010 936	.(7)102 1775 591
56.0	.(1)199 1410 418	−.(4)103 9901 001	.(8)977 3505 652
56.5	.(1)197 3937 403	−.(4)101 2768 991	.(8)935 2233 857
57.0	.(1)195 6768 382	−.(5)986 5727 449	.(8)895 2565 744
57.5	.(1)193 9895 490	−.(5)961 2722 631	.(8)857 3219 737
58.0	.(1)192 3311 130	−.(5)936 8295 813	.(8)821 3000 421
58.5	.(1)190 7007 963	−.(5)913 2086 499	.(8)787 0792 070
59.0	.(1)189 0978 896	−.(5)890 3752 219	.(8)754 5552 728
59.5	.(1)187 5217 074	−.(5)868 2967 491	.(8)723 6308 764
60.0	.(1)185 9715 868	−.(5)846 9422 843	.(8)694 2149 871
60.5	.(1)184 4468 866	−.(5)826 2823 903	.(8)666 2224 473
61.0	.(1)182 9469 865	−.(5)806 2890 546	.(8)639 5735 494
61.5	.(1)181 4712 863	−.(5)786 9356 098	.(8)614 1936 467
62.0	.(1)180 0192 049	−.(5)768 1966 583	.(8)590 0127 944
62.5	.(1)178 5901 798	−.(5)750 0480 031	.(8)566 9654 196
63.0	.(1)177 1836 660	−.(5)732 4665 812	.(8)544 9900 158
63.5	.(1)175 7991 358	−.(5)715 4304 029	.(8)524 0288 613
64.0	.(1)174 4360 776	−.(5)698 9184 935	.(8)504 0277 597
64.5	.(1)173 0939 958	−.(5)682 9108 392	.(8)484 9357 992
65.0	.(1)171 7724 099	−.(5)667 3883 359	.(8)466 7051 300
65.5	.(1)170 4708 538	−.(5)652 3327 419	.(8)449 2907 595
66.0	.(1)169 1888 755	−.(5)637 7266 321	.(8)432 6503 610
66.5	.(1)167 9260 366	−.(5)623 5533 562	.(8)416 7440 977
67.0	.(1)166 6819 116	−.(5)609 7969 986	.(8)401 5344 591
67.5	.(1)165 4560 875	−.(5)596 4423 407	.(8)386 9861 091
68.0	.(1)164 2481 635	−.(5)583 4748 260	.(8)373 0657 455
68.5	.(1)163 0577 503	−.(5)570 8805 261	.(8)359 7419 689
69.0	.(1)161 8844 697	−.(5)558 6461 100	.(8)346 9851 615
69.5	.(1)160 7279 547	−.(5)546 7588 138	.(8)334 7673 741
70.0	.(1)159 5878 482	−.(5)535 2064 131	.(8)323 0622 212
70.5	.(1)158 4638 037	−.(5)523 9771 965	.(8)311 8447 829
71.0	.(1)157 3554 838	−.(5)513 0599 408	.(8)301 0915 145
71.5	.(1)156 2625 611	−.(5)502 4438 871	.(8)290 7801 613
72.0	.(1)155 1847 168	−.(5)492 1187 187	.(8)280 8896 796
72.5	.(1)154 1216 409	−.(5)482 0745 403	.(8)271 4001 634
73.0	.(1)153 0730 320	−.(5)472 3018 576	.(8)262 2927 754
73.5	.(1)152 0385 968	−.(5)462 7915 587	.(8)253 5496 829
74.0	.(1)151 0180 498	−.(5)453 5348 963	.(8)245 1539 980
74.5	.(1)150 0111 131	−.(5)444 5234 708	.(8)237 0897 218
75.0	.(1)149 0175 162	−.(5)435 7492 141	.(8)229 3416 916

TABLE II—(Continued)

ρ	A	B	C
75.5	.(1)148 0369 959	−.(5)427 2043 746	.(8)221 8955 328
76.0	.(1)147 0692 956	−.(5)418 8815 026	.(8)214 7376 124
76.5	.(1)146 1141 654	−.(5)410 7734 371	.(8)207 8549 966
77.0	.(1)145 1713 622	−.(5)402 8732 923	.(8)201 2354 107
77.5	.(1)144 2406 486	−.(5)395 1744 458	.(8)194 8672 015
78.0	.(1)143 3217 937	−.(5)387 6705 266	.(8)188 7393 021
78.5	.(1)142 4145 722	−.(5)380 3554 041	.(8)182 8411 989
79.0	.(1)141 5187 645	−.(5)373 2231 778	.(8)177 1629 008
79.5	.(1)140 6341 567	−.(5)366 2681 669	.(8)171 6949 101
80.0	.(1)139 7605 399	−.(5)359 4849 013	.(8)166 4281 950
80.5	.(1)138 8977 106	−.(5)352 8681 120	.(8)161 3541 647
81.0	.(1)138 0454 701	−.(5)346 4127 230	.(8)156 4646 446
81.5	.(1)137 2036 248	−.(5)340 1138 429	.(8)151 7518 541
82.0	.(1)136 3719 855	−.(5)333 9667 569	.(8)147 2083 854
82.5	.(1)135 5503 678	−.(5)327 9669 199	.(8)142 8271 834
83.0	.(1)134 7385 917	−.(5)322 1099 490	.(8)138 6015 271
83.5	.(1)133 9364 812	−.(5)316 3916 169	.(8)134 5250 116
84.0	.(1)133 1438 649	−.(5)310 8078 455	.(8)130 5915 317
84.5	.(1)132 3605 750	−.(5)305 3547 000	.(8)126 7952 663
85.0	.(1)131 5864 481	−.(5)300 0283 827	.(8)123 1306 632
85.5	.(1)130 8213 241	−.(5)294 8252 276	.(8)119 5924 258
86.0	.(1)130 0650 470	−.(5)289 7416 953	.(8)116 1754 993
86.5	.(1)129 3174 642	−.(5)284 7743 676	.(8)112 8750 590
87.0	.(1)128 5784 266	−.(5)279 9199 429	.(8)109 6864 980
87.5	.(1)127 8477 885	−.(5)275 1752 316	.(8)106 6054 166
88.0	.(1)127 1254 075	−.(5)270 5371 515	.(8)103 6276 117
88.5	.(1)126 4111 445	−.(5)266 0027 239	.(8)100 7490 669
89.0	.(1)125 7048 632	−.(5)261 5690 691	.(9)979 6594 350
89.5	.(1)125 0064 308	−.(5)257 2334 033	.(9)952 7457 143
90.0	.(1)124 3157 171	−.(5)252 9930 341	.(9)926 7144 106
90.5	.(1)123 6325 948	−.(5)248 8453 575	.(9)901 5319 538
91.0	.(1)122 9569 394	−.(5)244 7878 546	.(9)877 1662 253
91.5	.(1)122 2886 291	−.(5)240 8180 879	.(9)853 5864 881
92.0	.(1)121 6275 450	−.(5)236 9336 988	.(9)830 7633 199
92.5	.(1)120 9735 702	−.(5)233 1324 043	.(9)808 6685 508
93.0	.(1)120 3265 909	−.(5)229 4119 941	.(9)787 2752 029
93.5	.(1)119 6864 953	−.(5)225 7703 284	.(9)766 5574 344
94.0	.(1)119 0531 742	−.(5)222 2053 346	.(9)746 4904 858
94.5	.(1)118 4265 205	−.(5)218 7150 056	.(9)727 0506 294
95.0	.(1)117 8064 296	−.(5)215 2973 968	.(9)708 2151 209
95.0	.(1)117 1927 988	−.(5)211 9506 240	.(9)689 9621 539
96.0	.(1)116 5855 277	−.(5)208 6728 615	.(9)672 2708 166
96.5	.(1)115 9845 180	−.(5)205 4623 396	.(9)655 1210 509
97.0	.(1)115 3896 733	−.(5)202 3173 428	.(9)638 4936 130
97.5	.(1)114 8008 993	−.(5)199 2362 081	.(9)622 3700 369
98.0	.(1)114 2181 034	−.(5)196 2173 225	.(9)606 7325 988
98.5	.(1)113 6411 951	−.(5)193 2591 218	.(9)591 5642 838
99.0	.(1)113 0700 856	−.(5)190 3600 889	.(9)576 8487 544
99.5	.(1)112 5046 880	−.(5)187 5187 519	.(9)562 5703 199
100.0	.(1)111 9449 168	−.(5)184 7336 824	.(9)548 7139 081

TABLE III

(The numbers in parentheses denote the number of ciphers between the decimal point and the first significant figure.)

ρ	A'	B'	C'
1.5	2.534 7222 222	− 1.138 8888 889	.555 5555 556
2.0	.902 7777 778	−.236 1111 111	.(1)694 4444 444
2.5	.450 6999 559	−.(1)779 3209 877	.(1)154 3209 877
3.0	.262 5661 376	−.(1)324 0740 741	.(2)462 9629 630
3.5	.167 8541 366	−.(1)155 7239 057	.(2)168 3501 684
4.0	.114 3378 227	−.(2)827 7216 611	.(3)701 4590 348
4.5	.(1)816 0531 598	−.(2)474 2942 243	.(3)323 7503 238
5.0	.(1)603 7943 538	−.(2)288 1377 881	.(3)161 8751 619
5.5	.(1)459 7794 181	−.(2)183 4585 167	.(4)863 3341 967
6.0	.(1)358 4609 835	−.(2)121 4063 714	.(4)485 6254 856
6.5	.(1)285 0269 624	−.(3)829 8482 563	.(4)285 6620 504
7.0	.(1)230 4589 927	−.(3)583 0679 850	.(4)174 5712 530
7.5	.(1)189 0399 941	−.(3)419 5222 848	.(4)110 2555 282
8.0	.(1)157 0204 105	−.(3)308 1642 014	.(5)716 6609 334
8.5	.(1)131 8685 978	−.(3)230 5259 336	.(5)477 7739 556
9.0	.(1)111 8316 811	−.(3)175 2561 737	.(5)325 7549 697
9.5	.(2)956 6897 397	−.(3)135 1741 492	.(5)226 6121 528
10.0	.(2)824 8507 009	−.(3)105 6201 476	.(5)160 5169 416
10.5	.(2)716 2246 443	−.(4)835 0091 302	.(5)115 5721 980
11.0	.(2)625 9079 085	−.(4)667 2071 889	.(6)844 5660 620
11.5	.(2)550 1917 791	−.(4)538 3326 639	.(6)625 6044 903
12.0	.(2)486 2354 500	−.(4)438 2359 455	.(6)469 2033 678
12.5	.(2)431 8375 890	−.(4)359 6848 299	.(6)355 9473 824
13.0	.(2)385 2742 263	−.(4)297 4533 626	.(6)272 8929 932
13.5	.(2)345 1820 327	−.(4)247 7164 138	.(6)211 2719 947
14.0	.(2)310 4731 565	−.(4)207 6407 573	.(6)165 0562 459
14.5	.(2)280 2723 203	−.(4)175 1046 701	.(6)130 0443 149
15.0	.(2)253 8698 342	−.(4)148 5029 580	.(6)103 2704 854
15.5	.(2)230 6861 266	−.(4)126 6096 151	.(7)826 1638 832
16.0	.(2)210 2447 093	−.(4)108 4799 077	.(7)665 5209 059
16.5	.(2)192 1513 833	−.(5)933 7977 791	.(7)539 6115 453
17.0	.(2)176 0781 076	−.(5)807 3440 736	.(7)440 2094 185
17.5	.(2)161 7503 875	−.(5)700 9036 937	.(7)361 1974 716
18.0	.(2)148 9373 393	−.(5)610 8752 239	.(7)297 9879 141
18.5	.(2)137 4438 092	−.(5)534 3795 459	.(7)247 1119 288
19.0	.(2)127 1040 798	−.(5)469 1008 114	.(7)205 9266 073
19.5	.(2)117 7768 137	−.(5)413 1653 981	.(7)172 4036 712
20.0	.(2)109 3409 664	−.(5)365 0491 008	.(7)144 9758 144
20.5	.(2)101 6924 641	−.(5)323 5054 757	.(7)122 4240 211
21.0	.(3)947 4149 003	−.(5)287 5101 521	.(7)103 7942 787
21.5	.(3)884 1025 513	−.(5)256 2172 813	.(8)883 3555 638
22.0	.(3)826 3115 900	−.(5)228 9252 750	.(8)754 5328 774
22.5	.(3)773 4526 546	−.(5)205 0496 990	.(8)646 7424 663
23.0	.(3)725 0103 342	−.(5)184 1017 105	.(8)556 1985 210
23.5	.(3)680 5325 601	−.(5)165 6708 183	.(8)479 8575 476
24.0	.(3)639 6217 018	−.(5)149 4110 299	.(8)415 2613 392
24.5	.(3)601 9270 658	−.(5)135 0296 678	.(8)360 4155 020
25.0	.(3)567 1385 543	−.(5)122 2783 008	.(8)313 6949 739

TABLE III—(Continued)

ρ	A'	B'	C'
25.5	.(3)534 9812 832	-.(5)110 9453 570	.(8)273 7701 591
26.0	.(3)505 2110 028	-.(5)100 8500 824	.(8)239 5488 892
26.5	.(3)477 6101 850	-.(6)918 3758 072	.(8)210 1306 046
27.0	.(3)451 9846 731	-.(6)837 7472 451	.(8)184 7700 144
27.5	.(3)428 1608 021	-.(6)765 4677 208	.(8)162 8481 482
28.0	.(3)405 9829 189	-.(6)700 5455 924	.(8)143 8491 976
28.5	.(3)385 3112 389	-.(6)642 1215 945	.(8)127 3419 126
29.0	.(3)366 0199 892	-.(6)589 4492 824	.(8)112 9645 999
29.5	.(3)347 9957 969	-.(6)541 8786 342	.(8)100 4129 777
30.0	.(3)331 1362 854	-.(6)498 8422 596	.(9)894 3030 827
30.5	.(3)315 3488 491	-.(6)459 8437 659	.(9)797 9935 200
31.0	.(3)300 5495 828	-.(6)424 4479 170	.(9)713 3578 436
31.5	.(3)286 6623 418	-.(6)392 2722 841	.(9)638 8279 197
32.0	.(3)273 6179 176	-.(6)362 9801 450	.(9)573 0662 221
32.5	.(3)261 3533 111	-.(6)336 2744 286	.(9)514 9290 691
33.0	.(3)249 8110 923	-.(6)311 8925 371	.(9)463 4361 622
33.5	.(3)238 9388 342	-.(6)289.6019 105	.(9)417 7452 730
34.0	.(3)228 6886 114	-.(6)269 1962 143	.(9)377 1311 492
34.5	.(3)219 0165 553	-.(6)250 4920 591	.(9)340 9678 883
35.0	.(3)209 8824 591	-.(6)233 3261 690	.(9)308 7141 691
35.5	.(3)201 2494 260	-.(6)217 5529 331	.(9)279 9008 467
36.0	.(3)193 0835 554	-.(6)203 0422 839	.(9)254 1205 055
36.5	.(3)185 3536 630	-.(6)189 6778 555	.(9)231 0186 414
37.0	.(3)178 0310 300	-.(6)177 3553 804	.(9)210 2861 992
37.5	.(3)171 0891 785	-.(6)165 9829 799	.(9)191 6532 449
38.0	.(3)164 5036 705	-.(6)155 4715 079	.(9)174 8835 859
38.5	.(3)158 2519 263	-.(6)145 7503 555	.(9)159 7701 896
39.0	.(3)152 3130 623	-.(6)136 7496 434	.(9)146 1312 710
39.5	.(3)146 6677 431	-.(6)128 4078 366	.(9)133 8069 469
40.0	.(3)141 2980 508	-.(6)120 6693 349	.(9)122 6563 680
40.5	.(3)136 1873 638	-.(6)113 4838 362	.(9)112 5552 554
41.0	.(3)131 3202 493	-.(6)106 8057 759	.(9)103 3937 811
41.5	.(3)126 6823 653	-.(6)100 5938 300	.(10)950 7474 123
42.0	.(3)122 2603 712	-.(7)948 1047 667	.(10)875 1197 773
42.5	.(3)118 0418 479	-.(7)894 2160 708	.(10)806 2901 319
43.0	.(3)114 0152 237	-.(7)843 9617 986	.(10)743 5786 772
43.5	.(3)110 1697 079	-.(7)797 0591 436	.(10)686 3803 174
44.0	.(3)106 4952 300	-.(7)753 2501 737	.(10)634 1557 280
44.5	.(3)102 9823 841	-.(7)712 2993 971	.(10)586 4235 765
45.0	.(4)996 2237 840	-.(7)673 9915 889	.(10)542 7537 357
45.5	.(4)964 0698 884	-.(7)638 1298 500	.(10)502 7613 551
46.0	.(4)933 2851 680	-.(7)604 5338 699	.(10)466 1016 730
46.5	.(4)903 7975 033	-.(7)573 0383 707	.(10)432 4654 698
47.0	.(4)875 5392 866	-.(7)543 4917 120	.(10)401 5750 791
47.5	.(4)848 4470 959	-.(7)515 7546 374	.(10)373 1808 816
48.0	.(4)822 4613 955	-.(7)489 6991 482	.(10)347 0582 199
48.5	.(4)797 5262 609	-.(7)465 2074 902	.(10)323 0046 799
49.0	.(4)773 5891 262	-.(7)442 1712 397	.(10)300 8376 920
49.5	.(4)750 6005 505	-.(7)420 4904 806	.(10)280 3924 120
50.0	.(4)728 5140 038	-.(7)400 0730 601	.(10)261 5198 458

TABLE IV

(The numbers in parentheses denote the number of ciphers between the decimal point and the first significant figure.)

ρ	A'	B	C
2.0	1.000 0000 000	− 1.250 0000 000	.250 0000 000
2.5	.705 9936 523	− .495 6054 688	.(1)615 2343 750
3.0	.567 0995 671	− .265 1515 152	.(1)227 2727 273
3.5	.479 4006 348	− .162 3535 156	.(1)102 5390 625
4.0	.417 2494 172	− .107 8088 578	.(2)524 4755 245
4.5	.370 3002 930	−.(1)756 8359 375	.(2)292 9687 500
5.0	.333 3333 333	−.(1)553 6130 536	.(2)174 8251 748
5.5	.303 3523 560	−.(1)418 0908 203	.(2)109 8632 813
6.0	.278 4862 197	−.(1)323 9407 651	.(3)719 8683 669
6.5	.257 4942 453	−.(1)256 3476 563	.(3)488 2812 500
7.0	.239 5159 021	−.(1)206 4885 579	.(3)340 9902 791
7.5	.223 9329 020	−.(1)168 8639 323	.(3)244 1406 250
8.0	.210 2881 638	−.(1)139 9142 653	.(3)178 6139 557
8.5	.198 2357 141	−.(1)117 2614 820	.(3)133 1676 136
9.0	.187 5084 130	−.(2)992 7311 886	.(3)100 9557 141
9.5	.177 8964 418	−.(2)848 0187 618	.(4)776 8110 795
10.0	.169 2325 443	−.(2)730 2463 319	.(4)605 7342 846
10.5	.161 3816 481	−.(2)633 3998 033	.(4)478 0375 874
11.0	.154 2334 096	−.(2)553 0129 672	.(4)381 3882 532
11.5	.147 6967 551	−.(2)485 7203 344	.(4)307 3098 776
12.0	.141 6958 188	−.(2)428 9521 906	.(4)249 8750 625
12.5	.136 1668 726	−.(2)380 7227 928	.(4)204 8732 517
13.0	.131 0559 774	−.(2)339 4807 972	.(4)169 2702 036
13.5	.126 3171 605	−.(2)304 0020 282	.(4)140 8503 606
14.0	.121 9109 898	−.(2)273 3115 358	.(4)117 9762 025
14.5	.117 8034 442	−.(2)246 6262 196	.(5)994 2378 394
15.0	.113 9650 116	−.(2)223 3120 976	.(5)842 6871 608
15.5	.110 3699 628	−.(2)202 8521 369	.(5)718 0606 618
16.0	.106 9957 611	−.(2)184 8217 922	.(5)614 9338 741
16.5	.103 8225 802	−.(2)168 8702 311	.(5)529 0973 297
17.0	.100 8329 073	−.(2)154 7057 999	.(5)457 2585 218
17.5	.(1)980 1121 368	−.(2)142 0846 788	.(5)396 8229 973
18.0	.(1)953 4368 071	−.(2)130 8019 601	.(5)345 7320 530
18.5	.(1)928 1796 986	−.(2)120 6845 814	.(5)302 3413 313
19.0	.(1)904 2302 916	−.(2)111 5856 901	.(5)265 3292 500
19.5	.(1)881 4892 929	−.(2)103 3801 211	.(5)233 6273 923
20.0	.(1)859 8672 410	−.(3)959 6074 542	.(5)206 3671 945
20.5	.(1)839 2833 163	−.(3)892 3550 616	.(5)182 8388 288
21.0	.(1)819 6643 189	−.(3)831 2499 865	.(5)162 4592 807
21.5	.(1)800 9437 893	−.(3)775 6048 512	.(5)144 7474 061
22.0	.(1)783 0612 483	−.(3)724 8225 801	.(5)129 3043 255
22.5	.(1)765 9615 377	−.(3)678 3828 434	.(5)115 7979 249
23.0	.(1)749 5942 464	−.(3)635 8307 796	.(5)103 9505 362
23.5	.(1)733 9132 094	−.(3)596 7675 752	.(6)935 2909 319
24.0	.(1)718 8760 691	−.(3)560 8425 626	.(6)843 3722 746
24.5	.(1)704 4438 900	−.(3)527 7465 684	.(6)762 0889 075
25.0	.(1)690 5808 198	−.(3)497 2062 910	.(6)690 0318 611

TABLE IV—(Continued)

ρ	D	E	F
2.0	2.454 8611 111	– .538 1944 444	.121 5277 778
2.5	.586 3715 278	–.(1)824 6527 778	.(1)121 5277 778
3.0	.214 3308 081	–.(1)211 4898 990	.(2)220 9595 960
3.5	.(1)962 5552 400	–.(2)706 2815 657	.(3)552 3989 899
4.0	.(1)491 2101 788	–.(2)279 2346 542	.(3)169 9689 200
4.5	.(1)274 0445 318	–.(2)124 4415 307	.(4)607 0318 570
5.0	.(1)163 4129 759	–.(3)607 0318 570	.(4)242 8127 428
5.5	.(1)102 6452 931	–.(3)317 9329 351	.(4)106 2305 750
6.0	.(2)672 3770 510	–.(3)176 3963 161	.(5)499 9085 881
6.5	.(2)455 9791 210	–.(3)102 6597 994	.(5)249 9542 941
7.0	.(2)318 3891 488	–.(4)622 0667 018	.(5)131 5548 916
7.5	.(2)227 9369 385	–.(4)390 2012 053	.(6)723 5519 039
8.0	.(2)166 7477 045	–.(4)252 2095 208	.(6)413 4582 308
8.5	.(2)124 3142 044	–.(4)167 3566 157	.(6)244 3162 273
9.0	.(3)942 4020 458	–.(4)113 6601 579	.(6)148 7142 253
9.5	.(3)725 1166 208	–.(5)788 0526 136	.(7)929 4639 082
10.0	.(3)565 4114 845	–.(5)556 6161 004	.(7)594 8569 012
10.5	.(3)446 2073 095	–.(5)399 7797 905	.(7)388 9448 970
11.0	.(3)355 9882 993	–.(5)291 5234 609	.(7)259 2965 980
11.5	.(3)286 8401 475	–.(5)215 5402 971	.(7)175 9512 629
12.0	.(3)233 2288 596	–.(5)161 3897 791	.(7)121 3456 986
12.5	.(3)191 2235 292	–.(5)122 2557 913	.(8)849 4198 899
13.0	.(3)157 9915 515	–.(6)936 0842 049	.(8)602 8141 154
13.5	.(3)131 4646 693	–.(6)723 8747 984	.(8)433 2726 455
14.0	.(3)110 1142 734	–.(6)564 9425 143	.(8)315 1073 785
14.5	.(4)927 9784 865	–.(6)444 6919 780	.(8)231 6966 018
15.0	.(4)786 5252 285	–.(6)352 8408 251	.(8)172 1174 757
15.5	.(4)670 2028 598	–.(6)282 0575 132	.(8)129 0881 067
16.0	.(4)573 9480 513	–.(6)227 0555 132	.(9)976 8829 699
16.5	.(4)493 8316 295	–.(6)183 9826 864	.(9)745 5159 507
17.0	.(4)426 7803 550	–.(6)150 0043 633	.(9)573 4738 083
17.5	.(4)370 3725 753	–.(6)123 0152 522	.(9)444 4422 014
18.0	.(4)322 6867 183	–.(6)101 4381 253	.(9)346 8817 182
18.5	.(4)282 1879 680	–.(7)840 8165 075	.(9)272 5499 214
19.0	.(4)247 6427 884	–.(7)700 3899 143	.(9)215 5045 890
19.5	.(4)218 0539 471	–.(7)586 1479 930	.(9)171 4241 049
20.0	.(4)192 6107 529	–.(7)492 7218 558	.(9)137 1392 839
20.5	.(4)170 6506 766	–.(7)415 9387 633	.(9)110 3076 849
21.0	.(4)151 6295 539	–.(7)352 5353 142	.(10)891 8493 673
21.5	.(4)135 0983 037	–.(7)299 9440 718	.(10)724 6276 109
22.0	.(4)120 6846 287	–.(7)256 1336 780	.(10)591 5327 436
22.5	.(4)108 0785 483	–.(7)219 4882 245	.(10)485 0568 497
23.0	.(5)970 2089 990	–.(7)188 7156 473	.(10)399 4585 821
23.5	.(5)872 9415 063	–.(7)162 7777 261	.(10)330 3215 199
24.0	.(5)787 1501 985	–.(7)140 8362 750	.(10)274 2291 863
24.5	.(5)711 2852 655	–.(7)122 2115 427	.(10)228 5243 219
25.0	.(5)644 0316 483	–.(7)106 3498 773	.(10)191 1294 329

TABLE V

(The numbers in parentheses denote the number of ciphers between the decimal point and the first significant figure.)

ρ	A'	B'	C'
2.5	2.755 1030 816	− 1.695 6163 194	.200 8159 722
3.0	1.170 5555 556	− .456 9444 444	.(1)363 8888 889
3.5	.658 2671 327	− .181 6553 455	.(1)104 8944 979
4.0	.418 6208 236	−.(1)868 9782 440	.(2)382 4786 325
4.5	.286 4246 324	−.(1)466 0481 012	.(2)162 0459 402
5.0	.206 0275 835	−.(1)270 7119 270	.(3)763 8888 889
5.5	.153 7850 411	−.(1)166 9293 642	.(3)390 4384 270
6.0	.118 1431 967	−.(1)107 8702 752	.(3)212 7325 289
6.5	.(1)928 9213 334	−.(2)724 0120 171	.(3)122 1004 174
7.0	.(1)744 5226 377	−.(2)501 4851 420	.(4)731 8541 452
7.5	.(1)606 4555 217	−.(2)356 7180 657	.(4)455 0831 382
8.0	.(1)500 8805 006	−.(2)259 6021 548	.(4)292 0668 942
8.5	.(1)418 6842 624	−.(2)192 7113 758	.(4)192 6724 344
9.0	.(1)353 6828 101	−.(2)145 5695 568	.(4)130 2141 757
9.5	.(1)301 5707 729	−.(2)111 6692 311	.(5)899 1006 061
10.0	.(1)259 2832 573	−.(3)868 5125 822	.(5)632 8140 010
10.5	.(1)224 5954 598	−.(3)683 8979 612	.(5)453 1298 476
11.0	.(1)195 8642 756	−.(3)544 5806 418	.(5)329 5585 675
11.5	.(1)171 8573 720	−.(3)438 0711 370	.(5)243 1030 464
12.0	.(1)151 6376 163	−.(3)355 6755 034	.(5)181 6613 061
12.5	.(1)134 4833 894	−.(3)291 2423 953	.(5)137 3671 277
13.0	.(1)119 8326 681	−.(3)240 3543 455	.(5)105 0128 024
13.5	.(1)107 2431 572	−.(3)199 7956 319	.(6)810 9219 640
14.0	.(2)963 6345 096	−.(3)167 1959 380	.(6)632 0799 809
14.5	.(2)869 1189 432	−.(3)140 7878 615	.(6)496 9749 385
15.0	.(2)786 6089 931	−.(3)119 2394 389	.(6)393 9212 946
15.5	.(2)714 2517 761	−.(3)101 5368 865	.(6)314 6050 442
16.0	.(2)650 5281 970	−.(4)869 0145 535	.(6)253 0429 506
16.5	.(2)594 1846 661	−.(4)747 2977 719	.(6)204 8829 224
17.0	.(2)544 1803 654	−.(4)645 5059 265	.(6)166 9275 426
17.5	.(2)499 6461 807	−.(4)559 9303 930	.(6)136 8055 923
18.0	.(2)459 8524 699	−.(4)487 6318 743	.(6)112 7430 005
18.5	.(2)424 1835 784	−.(4)426 2652 106	.(7)934 0140 025
19.0	.(2)392 1175 513	−.(4)373 9474 673	.(7)777 6318 298
19.5	.(2)363 2098 774	−.(4)329 1578 113	.(7)650 4887 492
20.0	.(2)337 0803 860	−.(4)290 6609 405	.(7)546 5721 085
20.5	.(2)313 4026 161	−.(4)257 4481 065	.(7)461 2129 835
21.0	.(2)291 8951 572	−.(4)228 6913 803	.(7)390 7629 285
21.5	.(2)272 2750 483	−.(4)203 7079 595	.(7)332 3537 754
22.0	.(2)254 4494 241	−.(4)181 9321 409	.(7)283 7178 436
22.5	.(2)238 1156 895	−.(4)162 8931 866	.(7)243 0524 324
23.0	.(2)223 1524 850	−.(4)146 1977 446	.(7)208 9170 140
23.5	.(2)209 4188 368	−.(4)131 5158 166	.(7)180 1547 432
24.0	.(2)196 7908 418	−.(4)118 5694 971	.(7)155 8321 715
24.5	.(2)186 2576 776	−.(4)107 7593 614	.(7)135 9946 487
25.0	.(2)174 4277 221	−.(5)969 7978 581	.(7)117 6202 932

TABLE V—(Continued)

ρ	D'	E'	C'
2.5	1.151 0416 667	− .140 9722 222	.(1)175 0000 000
3.0	.203 1250 000	−.(1)170 1388 889	.(2)145 8333 333
3.5	.(1)579 2905 012	−.(2)355 2350 427	.(3)224 3589 744
4.0	.(1)210 1544 289	−.(3)988 2478 632	.(4)480 7692 308
4.5	.(2)887 9662 005	−.(3)331 1965 812	.(4)128 2051 282
5.0	.(2)417 9414 336	−.(3)126 8696 582	.(5)400 6410 256
5.5	.(2)213 4163 324	−.(4)538 1158 874	.(5)141 4027 149
6.0	.(2)116 2108 929	−.(4)247 4547 511	.(6)549 8994 470
6.5	.(3)666 7389 843	−.(4)121 5567 199	.(6)231 5366 092
7.0	.(3)399 5246 884	−.(5)630 9372 602	.(6)104 1914 742
7.5	.(3)248 3850 322	−.(5)343 1703 316	.(7)496 1498 770
8.0	.(3)159 3881 480	−.(5)194 3253 685	.(7)248 0749 385
8.5	.(3)105 1352 375	−.(5)113 9706 601	.(7)129 4304 027
9.0	.(4)710 4822 114	−.(6)689 3966 587	.(8)701 0813 479
9.5	.(4)490 5434 324	−.(6)428 5943 973	.(8)392 6055 548
10.0	.(4)345 2433 425	−.(6)273 0621 968	.(8)226 5032 047
10.5	.(4)247 2044 428	−.(6)177 8469 607	.(8)134 2241 213
11.0	.(4)179 7851 483	−.(6)118 1651 639	.(9)814 9321 650
11.5	.(4)132 6177 560	−.(7)799 4765 550	.(9)505 8199 645
12.0	.(5)990 9817 818	−.(7)549 9387 059	.(9)320 3526 442
12.5	.(5)749 3410 036	−.(7)384 0787 078	.(9)206 6791 253
13.0	.(5)572 8402 600	−.(7)272 0198 696	.(9)135 6331 760
13.5	.(5)442 3498 419	−.(7)195 1610 699	.(10)904 2211 731
14.0	.(5)344 7903 612	−.(7)141 7056 417	.(10)611 6790 289
14.5	.(5)271 0905 849	−.(7)104 0436 901	.(10)419 4370 484
15.0	.(5)214 8754 286	−.(8)771 8806 793	.(10)291 2757 280
15.5	.(5)171 6092 639	−.(8)578 2216 817	.(10)204 6802 413
16.0	.(5)138 0280 549	−.(8)437 1000 417	.(10)145 4306 978
16.5	.(5)111 7576 435	−.(8)333 2476 075	.(10)104 4117 830
17.0	.(6)910 5379 546	−.(8)256 1134 028	.(11)756 9854 269
17.5	.(6)746 2299 456	−.(8)198 3240 275	.(11)553 8917 758
18.0	.(6)614 9748 289	−.(8)154 6720 805	.(11)408 8248 821
18.5	.(6)509 4718 636	−.(8)121 4431 743	.(11)304 2417 727
19.0	.(6)424 1700 765	−.(9)959 6292 582	.(11)228 1813 296
19.5	.(6)354 8175 306	−.(9)762 8862 452	.(11)172 4036 712
20.0	.(6)298 1344 148	−.(9)609 9716 846	.(11)131 1767 064
20.5	.(6)251 5739 152	−.(9)490 3776 023	.(11)100 4757 751
21.0	.(6)213 1458 799	−.(9)396 2862 254	.(12)774 5007 663
21.5	.(6)181 2857 886	−.(9)321 8393 150	.(12)600 6332 473
22.0	.(6)154 7566 781	−.(9)262 6168 770	.(12)468 4939 329
22.5	.(6)132 5752 232	−.(9)215 2622 450	.(12)367 4462 219
23.0	.(6)113 9556 568	−.(9)177 2103 622	.(12)289 7172 134
23.5	.(7)982 6695 408	−.(9)146 4894 049	.(12)229 5872 257
24.0	.(7)849 9994 800	−.(9)121 5749 393	.(12)182 8194 575
24.5	.(7)741 7936 742	−.(9)101 8827 904	.(12)147 1231 630
25.0	.(7)641 5689 752	−.(10)846 8458 443	.(12)117 5267 941

TABLE VI

(The numbers in parentheses denote the number of ciphers between the decimal point and the first significant figure.)

ρ	A	B
3.0	1.000 0000 000	-1.361 1111 111
3.5	.745 3327 179	-.617 5664 266
4.0	.619 2696 193	-.362 6910 127
4.5	.536 5078 449	-.238 2141 749
5.0	.475 9358 289	-.167 1854 290
5.5	.428 9313 952	-.122 7793 517
6.0	.391 0671 372	-.(1)932 5031 693
6.5	.359 7514 629	-.(1)727 0467 546
7.0	.333 3333 333	-.(1)578 9958 809
7.5	.310 6966 019	-.(1)469 2645 603
8.0	.291 0527 351	-.(1)386 0218 617
8.5	.273 8253 011	-.(1)321 6238 375
9.0	.258 5812 357	-.(1)270 9606 497
9.5	.244 9877 912	-.(1)230 5173 392
10.0	.232 7844 334	-.(1)197 8161 991
10.5	.221 7638 626	-.(1)171 0732 926
11.0	.211 7588 265	-.(1)148 9801 195
11.5	.202 6327 307	-.(1)130 5610 007
12.0	.194 2728 127	-.(1)115 0776 715
12.5	.186 5850 870	-.(1)101 9640 947
13.0	.179 4905 415	-.(2)907 8107 302
13.5	.172 9222 326	-.(2)811 8410 815
14.0	.166 8230 401	-.(2)729 0028 821
14.5	.161 1439 098	-.(2)657 1143 441
15.0	.155 8424 640	-.(2)594 4165 421
15.5	.150 8818 928	-.(2)539 4804 055
16.0	.146 2300 606	-.(2)491 1364 663
16.5	.141 8587 811	-.(2)448 4212 624
17.0	.137 7432 239	-.(2)410 5360 674
17.5	.133 8614 260	-.(2)376 8148 358
18.0	.130 1938 866	-.(2)346 6991 057
18.5	.126 7232 293	-.(2)319 7181 992
19.0	.123 4339 187	-.(2)295 4734 917
19.5	.120 3120 213	-.(2)273 6258 312
20.0	.117 3450 034	-.(2)253 8854 120
20.5	.114 5607 471	-.(2)236 0035 778
21.0	.111 8314 598	-.(2)219 7661 473
21.5	.109 2654 341	-.(2)204 9879 526
22.0	.106 8150 524	-.(2)191 5083 451
22.5	.104 4726 355	-.(2)179 1874 832
23.0	.102 2311 723	-.(2)167 9032 479
23.5	.100 0842 483	-.(2)157 5486 730
24.0	.(1)980 2598 324	-.(2)148 0297 913
24.5	.(1)960 5097 234	-.(2)139 2638 202
25.0	.(1)941 5425 829	-.(2)131 1776 633

TABLE VI—(Continued)

ρ	C	D
3.0	.388 8888 889	–.(1)277 7777 778
3.5	.116 7093 913	–.(2)581 8684 896
4.0	.(1)498 5754 986	–.(2)185 1851 852
4.5	.(1)251 6301 473	–.(3)727 3356 120
5.0	.(1)140 7742 584	–.(3)326 7973 856
5.5	.(2)846 3824 237	–.(3)161 6301 360
6.0	.(2)537 1649 335	–.(4)859 9931 201
6.5	.(2)355 7417 128	–.(4)484 8904 080
7.0	.(2)243 8852 284	–.(4)286 6643 734
7.5	.(2)172 0852 322	–.(4)176 3237 847
8.0	.(2)124 4257 604	–.(4)112 1730 157
8.5	.(3)918 7769 007	–.(5)734 6824 363
9.0	.(3)690 9857 765	–.(5)493 5612 689
9.5	.(3)528 1236 437	–.(5)339 0842 014
10.0	.(3)409 4730 527	–.(5)237 6406 110
10.5	.(3)321 5757 191	–.(5)169 6421 007
11.0	.(3)255 4794 154	–.(5)122 9175 574
11.5	.(3)205 1024 695	–.(6)904 2245 370
12.0	.(3)166 2345 586	–.(6)674 0640 244
12.5	.(3)135 9108 167	–.(6)508 6263 021
13.0	.(3)112 0109 002	–.(6)388 0974 686
13.5	.(4)929 9691 087	–.(6)299 1919 424
14.0	.(4)777 3890 832	–.(6)232 8584 812
14.5	.(4)653 9677 383	–.(6)182 8395 204
15.0	.(4)513 3898 749	–.(6)144 7498 667
15.5	.(4)470 8598 806	–.(6)115 4775 918
16.0	.(4)402 7015 521	–.(7)927 8837 607
16.5	.(4)346 0719 079	–.(7)750 6043 467
17.0	.(4)298 7541 988	–.(7)611 0454 032
17.5	.(4)259 0066 153	–.(7)500 4028 978
18.0	.(4)225 4506 124	–.(7)412 1003 883
18.5	.(4)196 9877 124	–.(7)341 1837 940
19.0	.(4)172 7369 850	–.(7)283 8913 786
19.5	.(4)151 9876 810	–.(7)237 3452 480
20.0	.(4)134 1630 697	–.(7)199 3279 892
20.5	.(4)118 7926 279	–.(7)168 1195 507
21.0	.(4)105 4905 051	–.(7)142 3771 352
21.5	.(5)939 3873 877	–.(7)121 0460 765
22.0	.(5)838 7409 116	–.(7)103 2932 157
22.5	.(5)750 7766 502	–.(8)884 5674 819
23.0	.(5)673 6666 593	–.(8)760 0821 534
23.5	.(5)605 8783 224	–.(8)655 2351 718
24.0	.(5)546 1216 848	–.(8)566 6066 961
24.5	.(5)493 3070 093	–.(8)491 4263 585
25.0	.(5)446 5105 454	–.(8)427 4401 392

TABLE VI—(Continued)

ρ	E	F
3.0	2.988 9351 852	−.948 1481 481
3.5	.875 4725 025	−.189 2894 604
4.0	.370 6973 366	−.(1)590 7882 241
4.5	.186 3394 083	−.(1)229 8587 984
5.0	.104 0300 478	−.(1)102 7515 921
5.5	.(1)624 7216 370	−.(2)506 6826 853
6.0	.(1)396 1994 936	−.(2)269 0942 361
6.5	.(1)262 2649 076	−.(2)151 5410 433
7.0	.(1)179 7450 169	−.(3)895 1744 340
7.5	.(1)126 8009 428	−.(3)550 2997 477
8.0	.(2)916 6922 412	−.(3)349 9462 528
8.5	.(2)676 8238 027	−.(3)229 1312 716
9.0	.(2)508 9783 157	−.(3)153 8970 152
9.5	.(2)388 9905 578	−.(3)105 7119 802
10.0	.(2)301 5839 811	−.(4)740 7668 221
10.5	.(2)236 8373 742	−.(4)528 4390 327
11.0	.(2)188 1526 456	−.(4)383 0865 153
11.5	.(2)151 0481 065	−.(4)281 7940 194
12.0	.(2)122 4214 851	−.(4)210 0557 062
12.5	.(2)100 0884 111	−.(4)158 4944 724
13.0	.(3)824 6707 418	−.(4)120 9320 228
13.5	.(3)684 8389 962	−.(5)932 2623 816
14.0	.(3)572 4725 971	−.(5)725 5546 509
14.5	.(3)481 5811 670	−.(5)569 6913 075
15.0	.(3)407 5132 846	−.(5)451 0040 878
15.5	.(3)346 7368 814	−.(5)359 7939 891
16.0	.(3)296 5444 286	−.(5)289 0976 344
16.5	.(3)254 8421 133	−.(5)233 8608 712
17.0	.(3)219 9973 645	−.(5)190 3777 170
17.5	.(3)190 7274 117	−.(5)155 9046 664
18.0	.(3)166 0170 278	−.(5)128 3924 319
18.5	.(3)145 0572 508	−.(5)106 2973 088
19.0	.(3)127 1993 361	−.(6)884 4715 492
19.5	.(3)111 9198 707	−.(6)739 4524 672
20.0	.(4)987 9413 925	−.(6)621 0067 139
20.5	.(4)874 7564 250	−.(6)523 7749 416
21.0	.(4)776 8023 679	−.(6)443 5733 082
21.5	.(4)691 6366 980	−.(6)377 1157 568
22.0	.(4)617 6240 310	−.(6)321 8064 139
22.5	.(4)552 8493 126	−.(6)275 5833 129
23.0	.(4)496 0674 883	−.(6)236 7999 628
23.5	.(4)446 1499 515	−.(6)204 1350 268
24.0	.(4)402 1467 932	−.(6)176 5230 172
24.5	.(4)365 3958 471	−.(6)154 0028 718
25.0	.(4)328 7959 685	−.(6)133 1661 235

TABLE VI—(Continued)

ρ	G	H
3.0	.(1)703 2407 407	.311 9212 963
3.5	.(2)996 0214 120	.(1)431 6767 940
4.0	.(2)233 6419 753	.(1)100 2196 106
4.5	.(3)711 2449 363	.(2)303 4820 991
5.0	.(3)256 2636 166	.(2)109 0241 118
5.5	.(3)104 2151 284	.(3)442 5904 068
6.0	.(4)464 8740 588	.(3)197 2080 911
6.5	.(4)223 1993 819	.(4)946 1621 940
7.0	.(4)113 8216 820	.(4)482 2607 431
7.5	.(5)610 4831 370	.(4)258 5707 479
8.0	.(5)341 8161 061	.(4)144 7403 415
8.5	.(5)198 6271 380	.(5)840 9223 280
9.0	.(5)119 2274 520	.(5)504 7006 725
9.5	.(6)736 4557 705	.(5)311 7158 600
10.0	.(6)466 6351 290	.(5)197 4943 272
10.5	.(6)302 4952 338	.(5)128 0172 024
11.0	.(6)200 1684 248	.(6)847 0785 482
11.5	.(6)134 9506 134	.(6)571 0651 132
12.0	.(7)925 4160 570	.(6)391 5919 775
12.5	.(7)644 5499 552	.(6)272 7356 553
13.0	.(7)455 3925 745	.(6)192 6912 065
13.5	.(7)326 0189 201	.(6)137 9465 952
14.0	.(7)236 2653 096	.(7)999 6815 163
14.5	.(7)173 1717 709	.(7)732 7121 006
15.0	.(7)128 2726 524	.(7)542 7324 021
15.5	.(8)959 5450 485	.(7)405 9879 530
16.0	.(8)724 4284 089	.(7)306 5066 664
16.5	.(8)551 6645 046	.(7)233 4084 850
17.0	.(8)423 5236 197	.(7)179 1912 403
17.5	.(8)327 6400 420	.(7)138 6225 823
18.0	.(8)255 2963 323	.(7)108 0139 227
18.5	.(8)200 2846 930	.(8)847 3860 468
19.0	.(8)158 1422 442	.(8)669 0830 603
19.5	.(8)125 6316 099	.(8)531 5324 962
20.0	.(8)100 3843 545	.(8)424 7132 721
20.5	.(9)806 5374 215	.(8)341 2348 327
21.0	.(9)651 4166 128	.(8)275 6048 185
21.5	.(9)528 7636 968	.(8)223 7117 748
22.0	.(9)431 2539 387	.(8)182 4566 123
22.5	.(9)353 3296 685	.(8)149 4878 961
23.0	.(9)290 7474 080	.(8)123 0102 298
23.5	.(9)240 2476 147	.(8)101 6445 207
24.0	.(9)199 3121 346	.(9)843 2535 668
24.5	.(9)166 9628 406	.(9)706 3889 111
25.0	.(9)138 7382 906	.(9)586 9755 711

TABLE VI—(Continued)

ρ	I	J
3.0	–.(1)234 9537 037	.(2)178 2407 407
3.5	–.(2)232 9282 407	.(3)127 3148 148
4.0	–.(3)408 9506 173	.(4)169 7530 864
4.5	–.(4)972 9456 019	.(5)318 2870 370
5.0	–.(4)282 5435 730	.(6)748 9106 754
5.5	–.(5)947 9582 728	.(6)208 0307 432
6.0	–.(5)355 3443 795	.(7)656 9391 889
6.5	–.(5)145 5344 260	.(7)229 9287 161
7.0	–.(6)641 0133 904	.(8)875 9189 186
7.5	–.(6)300 1017 659	.(8)358 3304 667
8.0	–.(6)148 0060 623	.(8)155 7958 551
8.5	–.(7)763 5619 773	.(9)714 0643 358
9.0	–.(7)409 7430 989	.(9)342 7508 812
9.5	–.(7)227 6566 932	.(9)171 3754 406
10.0	–.(7)130 4646 031	.(10)888 6133 957
10.5	–.(8)768 7010 766	.(10)476 0428 905
11.0	–.(8)464 4029 703	.(10)262 6443 534
11.5	–.(8)287 0085 966	.(10)148 8318 003
12.0	–.(8)181 0859 646	.(11)864 1846 467
12.5	–.(8)116 4409 022	.(11)513 1096 340
13.0	–.(9)761 8900 625	.(11)310 9755 357
13.5	–.(9)506 5928 672	.(11)192 0731 250
14.0	–.(9)341 8901 625	.(11)120 7316 786
14.5	–.(9)233 9443 041	.(12)771 3412 798
15.0	–.(9)162 1522 356	.(12)500 3294 788
15.5	–.(9)113 7486 502	.(12)329 1641 308
16.0	–(10)806 9508 539	.(12)219 4427 539
16.5	–(10)578 5246 623	.(12)148 1238 589
17.0	–(10)418 8850 767	.(12)101 1577 573
17.5	–(10)306 1363 264	.(13)698 4702 287
18.0	–(10)225 7107 282	.(13)487 3048 107
18.5	–(10)167 8017 503	.(13)343 3283 894
19.0	–(10)125 7344 857	.(13)244 1446 325
19.5	–(11)949 1786 023	.(13)175 1472 363
20.0	–(11)721 6269 402	.(13)126 7022 561
20.5	–(11)552 3276 496	.(14)923 8706 171
21.0	–(11)425 4604 167	.(14)678 7620 861
21.5	–(11)329 7379 935	.(14)502 2839 437
22.0	–(11)257 0422 413	.(14)374 2507 816
22.5	–(11)201 4893 910	.(14)280 6880 862
23.0	–(11)158 7837 578	.(14)211 8400 650
23.5	–(11)125 7671 107	.(14)160 8415 309
24.0	–(11)100 1019 200	.(14)122 8244 418
24.5	–(12)805 1862 542	(15)948 6730 535
25.0	–(12)642 9736 393	.(15)728 0195 608

TABLE VII
(The numbers in parentheses denote the number of ciphers between the decimal point and the first significant figure.)

ρ	A'	B'
3.5	28.751 2015 275	-2.018 4913 853
4.0	1.362 9280 045	-.639 9553 571
4.5	.826 9536 776	-.286 6103 525
5.0	.557 2123 756	-.150 4344 206
5.5	.399 2748 295	-.(1)869 6309 477
6.0	.298 3433 421	-.(1)537 5591 718
6.5	.229 9463 348	-.(1)349 3835 704
7.0	.181 5696 107	-.(1)236 2100 221
7.5	.146 2040 440	-.(1)164 8961 618
8.0	.119 6560 886	-.(1)118 2279 748
8.5	.(1)992 8681 536	-.(2)867 1239 512
9.0	.(1)833 6752 283	-.(2)648 5413 101
9.5	.(1)707 2793 855	-.(2)493 4138 461
10.0	.(1)605 5349 149	-.(2)381 0899 486
10.5	.(1)522 6404 211	-.(2)298 3079 274
11.0	.(1)454 3790 237	-.(2)236 3306 957
11.5	.(1)397 6190 688	-.(2)189 2706 474
12.0	.(1)350 0205 660	-.(2)153 0798 651
12.5	(1)309 7896 316	-.(2)124 9247 269
13.0	.(1)275 5429 712	-.(2)102 7891 576
13.5	.(1)246 1999 134	-.(3)852 1744 589
14.0	.(1)220 9073 683	-.(3)711 4425 518
14.5	.(1)198 9854 583	-.(3)597 8026 293
15.0	.(1)179 8875 910	-.(3)505 3397 258
15.5	.(1)163 1707 725	-.(3)429 5745 974
16.0	.(1)148 4732 835	-.(3)367 0820 942
16.5	.(1)135 4977 156	-.(3)315 2193 163
17.0	.(1)123 9979 553	-.(3)271 9296 956
17.5	.(1)113 7691 065	-.(3)235 6002 537
18.0	.(1)104 6396 241	-.(3)204 9565 434
18.5	.(2)964 6512 317	-.(3)178 9845 869
19.0	.(2)891 2347 318	-.(3)156 8723 569
19.5	.(2)842 5297 627	-.(3)137 9655 374
20.0	.(2)765 3875 595	-.(3)121 7338 175
20.5	.(2)711 3114 773	-.(3)107 7450 175
21.0	.(2)662 2270 210	-.(4)956 4508 777
21.5	.(2)617 5695 829	-.(4)851 4293 652
22.0	.(2)576 8498 581	-.(4)759 9627 856
22.5	.(2)539 6419 780	-.(4)680 0595 916
23.0	.(2)505 5744 586	-.(4)610 0441 976
23.5	.(2)474 3260 497	-.(4)548 5160 065
24.0	.(2)445 5992 272	-.(4)494 2969 537
24.5	.(2)419 1587 794	-.(4)446 3969 384
25.0	.(2)394 7661 411	-.(4)403 9598 359

TABLE VII—(Continued)

ρ	C'	D'
3.5	.357 6197 452	–.(1)173 0659 191
4.0	.(1)798 5119 048	–.(2)282 3837 868
4.5	.(1)269 7269 968	–.(3)732 8051 663
5.0	.(1)111 5137 722	–.(3)241 1381 219
5.5	.(2)523 4528 750	–.(4)925 0558 091
6.0	.(2)268 7939 211	–.(4)396 2769 318
6.5	.(2)147 7366 521	–.(4)184 7282 763
7.0	.(3)856 9155 732	–.(5)921 2018 141
7.5	.(3)519 3903 212	–.(5)485 5539 780
8.0	.(3)326 6117 660	–.(5)268 1183 076
8.5	.(3)211 9254 433	– (5)154 0554 574
9.0	.(3)141 2859 194	–.(6)916 1750 131
9.5	.(4)964 5009 167	–.(6)561 5350 601
10.0	.(4)672 3458 517	–.(6)353 4749 049
10.5	.(4)477 5025 292	–.(6)227 8602 485
11.0	.(4)344 8411 857	–.(6)150 0558 578
11.5	.(4)252 8236 420	–.(6)100 7434 807
12.0	.(4)187 9176 343	–.(7)688 3248 748
12.5	.(4)141 4318 835	–.(7)477 8804 517
13.0	.(4)107 6721 454	–.(7)336 6794 370
13.5	.(5)828 3962 809	–.(7)240 4245 028
14.0	.(5)643 5779 618	–.(7)173 8435 375
14.5	.(5)504 5238 369	–.(7)127 1625 885
15.0	.(5)398 8437 502	–.(8)940 2155 070
15.5	.(5)317 7726 592	–.(8)702 1757 322
16.0	.(5)255 0351 049	–.(8)529 3336 632
16.5	.(5)206 0875 950	–.(8)402 5511 467
17.0	.(5)167 6061 262	–.(8)308 6648 353
17.5	.(5)137 1349 784	–.(8)238 5149 470
18.0	.(5)112 8433 182	–.(8)185 6576 541
18.5	.(6)933 5432 357	–.(8)145 5130 248
19.0	.(6)776 2421 574	–.(8)114 7942 689
19.5	.(6)648 5562 423	–.(9)911 2100 394
20.0	.(6)544 3500 271	–.(9)727 5436 127
20.5	.(6)458 8701 286	–.(9)584 1368 200
21.0	.(6)388 4099 727	–.(9)471 4844 101
21.5	.(6)330 0128 092	–.(9)382 4793 533
22.0	.(6)281 5293 946	–.(9)311 7704 559
22.5	.(6)240 9925 109	–.(9)255 3016 408
23.0	.(6)206 9976 250	–.(9)209 9789 105
23.5	.(6)178 3794 880	–.(9)173 4277 959
24.0	.(6)154 1991 740	–.(9)143 8154 281
24.5	.(6)133 6979 531	–.(9)119 7203 889
25.0	.(6)116 2537 745	–.(9)100 0289 166

TABLE VII—(Continued)

ρ	E'	F'
3.5	1.579 8913 122	−.291 1769 387
4.0	.344 9276 620	−.(1)455 1504 630
4.5	.115 4442 217	−.(1)115 8781 297
5.0	.(1)475 1410 972	−.(2)377 5757 988
5.5	.(1)222 4862 588	−.(2)144 0406 919
6.0	.(1)114 0808 001	−.(3)614 9607 748
6.5	.(2)626 4468 334	−.(3)286 0508 258
7.0	.(2)363 1390 089	−.(3)142 4423 327
7.5	.(2)220 0141 501	−.(4)750 0507 440
8.0	.(2)138 3131 064	−.(4)413 8794 788
8.5	.(3)897 2722 669	−.(4)237 6853 941
9.0	.(3)598 0994 308	−.(4)141 2990 508
9.5	.(3)408 2508 260	−.(5)865 7917 011
10.0	.(3)284 5633 018	−.(5)544 8786 925
10.5	.(3)202 0841 107	−.(5)351 1847 710
11.0	.(3)145 9325 974	−.(5)231 2393 114
11.5	.(3)106 9873 276	−.(5)155 2312 455
12.0	.(4)795 1834 088	−.(5)106 0518 396
12.5	.(4)598 4598 613	−.(6)736 2301 182
13.0	.(4)455 5973 535	−.(6)518 6638 399
13.5	.(4)350 5159 780	−.(6)370 3629 282
14.0	.(4)272 3103 032	−.(6)267 7874 363
14.5	.(4)213 4710 252	−.(6)195 8739 375
15.0	.(4)168 7544 633	−.(6)144 8213 551
15.5	.(4)134 4512 878	−.(6)108 1535 449
16.0	.(4)107 9058 431	−.(7)815 2967 606
16.5	.(5)871 9546 641	−.(7)620 0115 559
17.0	.(5)709 1357 523	−.(7)475 4002 735
17.5	.(5)580 2104 892	−.(7)367 3519 091
18.0	.(5)477 4317 347	−.(7)285 9398 751
18.5	.(5)394 9737 353	−.(7)224 1091 135
19.0	.(5)328 4199 721	−.(7)176 7967 053
19.5	.(5)274 3965 893	−.(7)140 3360 406
20.0	.(5)230 3075 611	−.(7)112 0487 209
20.5	.(5)194 1416 638	−.(8)899 6217 202
21.0	.(5)164 3306 063	−.(8)726 1234 346
21.5	.(5)139 6233 668	−.(8)589 0458 851
22.0	.(5)119 1105 861	−.(8)480 1471 084
22.5	.(5)101 9599 333	−.(8)393 1799 685
23.0	.(6)875 7714 928	−.(8)323 3791 671
23.5	.(6)754 6921 922	−.(8)267 0876 406
24.0	.(6)652 3888 702	−.(8)221 4825 166
24.5	.(6)565 6513 791	−.(8)184 3745 724
25.0	.(6)491 8478 631	−.(8)154 0485 226

TABLE VII—(Continued)

ρ	G'	H'
3.5	.(1)143 3986 442	.(1)545 8043 981
4.0	.(2)165 3852 513	.(2)616 8981 481
4.5	.(3)325 3719 022	.(2)120 1878 234
5.0	.(4)847 0633 624	.(3)311 2518 155
5.5	.(4)264 7923 510	.(4)969 9931 081
6.0	.(5)944 9259 723	.(4)345 4902 916
6.5	.(5)373 3302 227	.(4)136 3314 118
7.0	.(5)160 0116 167	.(5)583 8397 737
7.5	.(6)733 3626 200	.(5)267 4300 767
8.0	.(6)355 6040 392	.(5)129 6221 514
8.5	.(6)180 9471 484	.(6)659 3767 449
9.0	.(7)960 0363 156	.(6)349 7616 94[illegible]
9.5	.(7)528 3674 230	.(6)192 4628 423
10.0	.(7)300 3768 815	.(6)109 4008 386
10.5	.(7)175 7761 725	.(7)640 1325 618
11.0	.(7)105 5698 126	.(7)384 4277 647
11.5	.(8)649 1137 736	.(7)236 3567 152
12.0	.(8)407 7298 452	.(7)148 4557 926
12.5	.(8)261 1497 436	.(8)950 8148 611
13.0	.(8)170 2826 250	.(8)619 9571 695
13.5	.(8)112 8752 998	.(8)410 9394 710
14.0	.(9)759 6818 749	.(8)276 5670 073
14.5	.(9)518 5449 520	.(8)188 7758 141
15.0	.(9)358 6183 357	.(8)130 5524 791
15.5	.(9)251 0639 560	.(9)913 9675 989
16.0	.(9)177 7849 669	.(9)647 1965 291
16.5	.(9)127 2480 223	.(9)463 2203 741
17.0	.(10)919 9584 852	.(9)334 8890 426
17.5	.(10)671 4105 266	.(9)244 4091 600
18.0	.(10)494 3960 450	.(9)179 9705 267
18.5	.(10)367 1241 847	.(9)133 6401 141
19.0	.(10)274 7920 816	.(9)100 0289 750
19.5	.(10)207 2366 516	.(10)754 3731 214
20.0	.(10)157 4099 489	.(10)572 9940 142
20.5	.(10)120 3776 663	.(10)438 1898 319
21.0	.(11)926 5399 250	.(10)337 2711 214
21.5	.(11)717 5522 633	.(10)261 1964 982
22.0	.(11)558 9721 931	.(10)203 4711 966
22.5	.(11)437 8836 453	.(10)159 3934 600
23.0	.(11)344 8668 374	.(10)125 5342 827
23.5	.(11)273 0031 904	.(11)993 7516 417
24.0	.(11)217 1767 993	.(11)790 5382 130
24.5	.(11)173 5819 878	.(11)631 8492 407
25.0	.(11)139 3623 589	.(11)507 2869 968

TABLE VII—(Continued)

ρ	I'	J'
3.5	–.(2)270 9986 772	.(3)135 1095 994
4.0	–.(3)227 3478 836	.(5)844 4349 962
4.5	–.(4)343 6965 064	.(6)993 4529 367
5.0	–.(5)713 2482 623	.(6)165 5754 895
5.5	–.(5)182 5352 461	.(7)348 5799 778
6.0	–.(6)544 3210 423	.(8)871 4499 445
6.5	–.(6)182 6693 153	.(8)248 9856 984
7.0	–.(7)674 0025 445	.(9)792 2272 223
7.5	–.(7)268 9333 213	.(9)275 5572 947
8.0	–.(7)114 6212 362	.(9)103 3339 855
8.5	–.(8)516 9083 906	(10)413 3359 421
9.0	–.(8)244 8220 580	.(10)174 8728 986
9.5	–.(8)121 0509 065	.(11)777 2128 825
10.0	–.(9)621 7703 060	.(11)360 8488 383
10.5	–.(9)330 4159 767	.(11)174 2028 875
11.0	–.(9)181 0370 007	.(12)871 0144 373
11.5	–.(9)101 9713 676	.(12)449 5558 386
12.0	–.(10)588 9829 883	.(12)238 8265 393
12.5	–.(10)348 0938 563	.(12)130 2690 214
13.0	–.(10)210 1044 796	.(13)727 9739 432
13.5	–.(10)129 2993 719	.(13)415 9851 104
14.0	–.(11)810 1043 368	.(13)242 6579 811
14.5	–.(11)516 0618 886	.(13)144 2831 239
15.0	–.(11)333 8664 755	.(14)873 2925 919
15.5	–.(11)219 1292 642	.(14)537 4108 258
16.0	–.(11)145 7726 865	.(14)335 8817 661
16.5	–.(12)982 0445 539	.(14)212 9981 931
17.0	–.(12)669 4697 055	.(14)136 9274 099
17.5	–.(12)461 4992 604	.(15)891 6203 434
18.0	–.(12)321 4946 024	.(15)587 6588 627
18.5	–.(12)226 1959 235	.(15)391 7725 751
19.0	–.(12)160 6464 099	.(15)264 0206 484
19.5	–.(12)115 1112 743	.(15)179 7587 394
20.0	–.(13)831 8162 819	.(15)123 5841 333
20.5	–.(13)605 9221 411	.(16)857 5225 577
21.0	–.(13)444 7507 764	.(16)600 2657 904
21.5	–.(13)328 8288 593	.(16)423 7170 285
22.0	–.(13)244 8106 616	.(16)301 4909 626
22.5	–.(13)183 4686 278	.(16)216 1633 317
23.0	–.(13)138 3685 504	.(16)156 1179 618
23.5	–.(13)104 9876 917	.(16)113 5403 358
24.0	–.(14)801 2235 814	.(17)831 2774 587
24.5	–.(14)614 8704 752	.(17)612 5263 839
25.0	–.(14)474 3701 124	.(17)454 1098 277