COMBINED EXPANSIONS OF PRODUCTS OF SYMMETRIC POWER SUMS AND OF SUMS OF SYMMETRIC POWER PRODUCTS WITH APPLICATION TO SAMPLING[1]

By Paul S. Dwyer

PREFACE

This article is divided into two parts. Part I has for its title "Combined Expansions of Products of Symmetric Power Sums and of Sums of Symmetric Power Products" and develops the general mathematical theory which is applied in Part II to "The Fundamentals of Sampling." Part II will appear in a latter issue of this journal.

Each part is treated as an organic unit and has its own introduction and bibliography. Each article is assigned a given number and each book is given a letter so that references can be indicated concisely in the body of the dissertation.

Each part is divided into chapters and sections. Braces are used to indicate the important formulas.

PART I. COMBINED EXPANSIONS OF PRODUCTS OF SYMMETRIC POWER SUMS AND OF SUMS OF SYMMETRIC POWER PRODUCTS

Introduction

The mathematical material which is presented here has proved useful in generalizing that portion of the fundamental theory of sampling in which relations are established between the moments of the sample and the moments of the parent population. It is the purpose to establish the theorems in algebraic form since they constitute an extension of partition and symmetric function theory and may be of value to someone not necessarily interested in sampling.

A great deal of work has been done in symmetric function theory but not much of this is of present value to the statistician. His problem deals with the "power sum" while the classical theory, for the most part, deals with the interrelations of elementary symmetric functions and monomial symmetric functions. Only one phase of the reasoning developed in this investigation seems to have received extensive consideration previously and that is the subject covered in Chapter III.

Previous authors have noted that much of symmetric function theory reduces, with a proper choice of notation, to partition theory. It is the plan of this treatise to present in Chapter I an outline of new partition theory which

---

[1] A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the University of Michigan.

shows how the parts of one partition are combined to form the parts of another partition, and which serves as a means of expressing the main result of Chapters II, III, IV, V.

Chapter II shows how the formulas of Chapter I are applicable to the problem of finding products of power sums. The multiplication theorem for power sums, a generalization of the multinomial theorem, is stated in terms of power product sums and appropriate special cases are indicated.

Chapter III deals with the expansion of power product sums in terms of power sums and shows how the formulas of Chapter I may be used.

Chapter IV is the key chapter of the paper. The problem is to expand products of power sums in terms of power product sums, to multiply each power product sum by a quantity which is a uniquely defined function of the quantities composing the power product sum, and then to expand back in terms of all possible power sums. It is shown that the results can be written in a compact form which also utilizes the results of Chapter I. This result, as is shown in Part II, is directly applicable to the sampling problem of finding the moments of the sample moments in terms of the moments of the universe.

Extension is made to multivariate distributions in Chapter V.

Chapter I. The Combination of the Parts of Partitions

It is the purpose of this chapter to provide a precise notation which shows how the parts of one partition of r may be combined to form the parts of another partition of r. For example, 2111, a four part partition of 5, can be made into 32, a two part partition of 5, by combining the three unit parts into a new part or by combining the 2 with one of the unit parts to form the 3 and the other two unit parts to form the 2. This last formation can be made in three different ways since anyone of the unit parts might be combined with the 2. The combination of the parts of the partition 2111 to form the parts of the partition 32 is to be indicated symbolically by $P_{31} + 3P_{22}$ where the subscripts indicate the number of parts collected and the coefficients indicate the number of ways in which an equivalent collection can be made.

1. **Definitions and Notation.** a. *Partition* [G; 105] [K; I; 1] [16; 105]. We consider the integer r to be composed of r unit indistinguishable parcels and define the partitions of r to be all those different groupings into new parcels, each new parcel containing one or more unit parcels, such that each resultant grouping of parcels contains exactly all the original r unit parcels. For example the partitions of 4 are

4 ; 31 ; 22 ; 211 ; 1111

b. *Parts of Partitions.* The numbers of the grouped unit parcels indicate the parts of the partition. Thus the partitions of 4 above

4 ; 31 ; 22 ; 211 ; 1111 have respectively

1 ; 2 ; 2 ; 3 ; 4 parts.

The pattern 22 may also appear as 2^2. In general a ρ part partition of r is to be designated by

$p_1 p_2 p_3 \cdots p_\rho$ where the p's may or may not be equal and where $p_1 + p_2 + p_3 + \cdots + p_\rho = r$ or by

$$p_1^{\pi_1} \cdots p_s^{\pi_s} \text{ where } \begin{cases} p_1 \gtrless p_2 \gtrless p_3 \gtrless p_4 \gtrless \cdots \gtrless p_s \\ p_1\pi_1 + p_2\pi_2 + \cdots + p_s\pi_s = r \\ \pi_1 + \pi_2 + \cdots + \pi_s = \rho \end{cases}$$

c. *Order of Partitions.* When the parts of a partition are arranged in descending order we say that the partition is ordered. Thus

$$p_1p_2 \cdots p_\rho \text{ is ordered if } p_1 \geq p_2 \geq p_3 \geq \cdots \geq p_\rho$$

and $$p_1^{\pi_1} \cdots p_s^{\pi_s} \text{ is ordered if } p_1 > p_2 > p_3 > \cdots > p_s .$$

For example, 21^2 is an ordered partition while 312 is not. Unless otherwise specified it is hereafter assumed that all partitions are ordered.

It is sometimes convenient to refer to the order of the partition which is the size of the largest part, p_1, when the partition is ordered. Thus the two part partition, 31, is of the order 3, while the four part partition, 1111, is of order 1. The set of the numbers $p_1^{\pi_1} \cdots p_s^{\pi_s}$ is to be known as the complete order.

These definitions of order and part are consistent with the usual definitions. [16; 105–106] [*K*; *I*; 1] [*G*; 100]. The concept of complete order, as far as I know, is not found in the literature.

d. *Weight of Partitions. Isobaric Partitions.* The weight of any partition is defined to be the sum of all the parts of the partition. Thus the weight of $p_1^{\pi_1} \cdots p_s^{\pi_s}$ is $p_1\pi_1 + p_2\pi_2 + \cdots + p_s\pi_s = r$. Partitions having the same weight are called isobaric. Thus 4 and 211 are isobaric partitions.

e. *Algebraic Partitions.* If the r original units are composed of a_1, a_2, a_3, $\cdots a_r$ nonseparable primary units, then the result of combining these in any possible way is to be called an algebraic partition since the r original units are now replaced by the r algebraic quantities a_1, a_2, $\cdots a_r$. Thus a_1, a_2, a_3 may be combined to form

$$a_1 + a_2 + a_3;\ \overline{a_1 + a_2}\cdot a_3;\ \overline{a_1 + a_3}\cdot a_2;\ \overline{a_2 + a_3}\cdot a_1;\ a_1\cdot a_2\cdot a_3$$

which are the algebraic partitions of $a_1 + a_2 + a_3$.

The parts of the algebraic partitions are the resulting combinations while the order and complete order, which indicate the numbers of algebraic expressions combined, agree with the order and complete order of the partitions in which the a's are unity. The weight, which is equal to the sum of the parts, is indicated by $w = a_1 + a_2 + \cdots + a_r$. Thus if $a_1 = 5$, $a_2 = 4$, and $a_3 = 3$, $w = 12$. It is to be noted that the algebraic partitions are formed by combining the parts 5, 4, 3 and not by combining all parts of 12.

Now $a_1\, a_2 \cdots a_r$ is itself a partition of weight $w = a_1 + a_2 + \cdots + a_r$. If groups of the a's are alike it may be written

$$a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_h^{\alpha_h} \text{ where}$$

$$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_h\alpha_h = w$$

$$\alpha_1 + \alpha_2 + \cdots + \alpha_h = r.$$

Algebraic partitions having the same weight are called isobaric.

f. *Partition Combination Notation.* Let $\begin{pmatrix} 1^r \\ p_1^{\pi_1} \cdots p_s^{\pi_s} \end{pmatrix}$ indicate the number of different ways the r units, ordinary or algebraic, can be collected to form the partition. Thus $\begin{pmatrix} 1^5 \\ 32 \end{pmatrix}$ indicates the number of ways in which the five units can be collected to form a partition with three units in one part and two units in the other. Since the three units forming the first part can be selected in ${}_5C_3$ ways and since this selection automatically indicates the other two units forming the second part, it follows that

$$\begin{pmatrix} 1^5 \\ 32 \end{pmatrix} = {}_5C_3 = {}_5C_2 = 10. \quad \text{It is to be noted that } \begin{pmatrix} 1^4 \\ 22 \end{pmatrix} = 3 \neq {}_4C_2 = 6$$

for if the four unit parts are a_1, a_2, a_3, a_4, then the three 22 partitions are

$$\overline{a_1 + a_2} \cdot \overline{a_3 + a_4};\ \overline{a_1 + a_3} \cdot \overline{a_2 + a_4};\ \overline{a_1 + a_4} \cdot \overline{a_2 + a_3}$$

since

$$\overline{a_3 + a_4} \cdot \overline{a_1 + a_2};\ \overline{a_2 + a_4} \cdot \overline{a_1 + a_3};\ \overline{a_2 + a_3} \cdot \overline{a_1 + a_4}$$

are essentially the same groupings as the first three indicated.

2. **Formula for** $\begin{pmatrix} 1^r \\ p_1^{\pi_1} \cdots p_s^{\pi_s} \end{pmatrix}$. In establishing this formula we first take the case in which no part is repeated. i.e. $\pi_1 = \pi_2 \cdots \pi_3 = 1$ and $p_1 + p_2 + p_3 \cdots + p_s = r$. In this case the formula becomes

$$\begin{pmatrix} 1^r \\ p_1 p_2 \cdots p_s \end{pmatrix} = \frac{r!}{p_1!\, p_2! \cdots p_s!}$$

This results from the fact that the p_1 units can be grouped in ${}_rC_{p_1}$ different ways. The p_2 units then in ${}_{r-p_1}C_{p_2}$ different ways, the p_3 units then in ${}_{r-p_1-p_2}C_{p_3}$ different ways etc. So that

$$\begin{pmatrix} 1^r \\ p_1 p_2 \cdots p_s \end{pmatrix} = {}_rC_{p_1} \cdot {}_{r-p_1}C_{p_2} \cdot {}_{r-p_1-p_2}C_{p_3} \cdots {}_{r-p_1-p_2 \cdots p_{s-1}}C_{p_s}$$

$$= \frac{r!}{p_1!\,p_2! \cdots p_s!} \qquad \text{Compare [B; 49][19; 12]}$$

If however $p_1 = p_2 = \cdots = p_s$, then the same partition has been used $s!$ different times since $p_1, p_2, \cdots, p_s$ may be interchanged in $s!$ different ways, so that

$$\binom{1^r}{p_1^s} = \frac{r!}{(p_1!)^s s!}$$

By similar reasoning

$$\binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} = \frac{r!}{(p_1!)^{\pi_1} (p_2!)^{\pi_2} \cdots (p_s!)^{\pi_s} \pi_1! \pi_2! \cdots \pi_s!} \qquad \{1\}$$

Compare [19; 12, 13] [I; II, 252]

3. **Values of** $\binom{a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_h^{\alpha_h}}{p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s}}$. The number of ways in which the r parts of $a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_h^{\alpha_h}$ may be collected to form the ρ parts of $p_1^{\pi_1} \cdots p_s^{\pi_s}$ may be indicated by

$$\binom{a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_h^{\alpha_h}}{p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s}}. \quad \text{Thus} \quad \binom{2111}{32} = 4 \quad \text{and} \quad \binom{1111}{22} = 3$$

Formulas useful is evaluating this expression can be worked out from the results of this paper. A table of values of this expression for $w \leq 8$ has been given by the author [19; 29–32].

4. **Notation for Combining the Parts of a Partition. Table** I. We wish to indicate not only the number of ways in which a given r part partition of weight w can be grouped to form a ρ part partition of weight w, but also the number of parts of the r part partition grouped to form each of the ρ parts of the ρ part partition. As indicated in the opening paragraph, $P_{31} + 3P_{22} = P\binom{2111}{32}$ serves this purpose for the case in which the parts 2111 are collected to form 32. $P\binom{a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_h^{\alpha_h}}{p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s}}$ serves this purpose in the more general case. Its expansion gives sums of P functions whose subscripts are the numbers of parts combined and whose coefficients are the number of ways of forming the partitions from the parts. For example

$$P\binom{31}{4} = P_1; \qquad P\binom{111}{21} = 3P_{21}; \text{ etc.}$$

The use of $P\binom{a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_h^{\alpha_h}}{p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s}}$ is so fundamental to the present approach that a table is provided showing the different values when $w \leq 6$.

Table I gives values of the function when $w = 1, 2, 3, 4, 5, 6$. The values $a_1^{\alpha_1} \cdots a_h^{\alpha_h}$ are given in the left hand columns and the values of $p_1^{\pi_1} \cdots p_s^{\pi_s}$ in the top row. The partitions are ordered from the top and from the left. To

TABLE I

Values of $P\begin{pmatrix} a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_r^{\alpha_r} \\ p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s} \end{pmatrix}$ *when* $W \leqq 6$.

$W = 1$

$a \backslash p$	1
1	P_1

$W = 2$

$a \backslash p$	2	11
2	P_1	
11	P_2	P_{11}

$W = 6$

$a \backslash p$	6	51	42	33	411	321	222	31^3	2^21^2	21^4	1^6
6	P_1										
51	P_2	P_{11}									
42	P_2		P_{11}								
33	P_2			P_{11}							
411	P_3	$2P_{21}$	P_{12}		P_{111}						
321	P_3	P_{21}	P_{21}	P_{12}		P_{111}					
222	P_3		$3P_{21}$				P_{111}				
31^3	P_4	$3P_{31}$	$3P_{22}$	P_{13}	$3P_{211}$	$3P_{121}$		P_{1111}			
2^21^2	P_4	$2P_{31}$	$2P_{31}$ P_{22}	$2P_{22}$	P_{211}	$4P_{211}$	P_{112}		P_{1111}		
21^4	P_5	$4P_{41}$	P_{41} $6P_{32}$	$4P_{23}$	$6P_{311}$	$4P_{311}$ $12P_{221}$	$3P_{122}$	$4P_{2111}$	$6P_{1211}$	P_{11111}	
1^6	P_6	$6P_{51}$	$15P_{42}$	$10P_{33}$	$15P_{411}$	$60P_{321}$	$15P_{222}$	$20P_{3111}$	$45P_{2211}$	$15P_{21111}$	P_{111111}

$W = 3$

$a \backslash p$	3	21	111
3	P_1		
21	P_2	P_{11}	
111	P_3	$3P_{21}$	P_{111}

$W = 4$

$a \backslash p$	4	31	22	211	1111
4	P_1				
31	P_2	P_{11}			
22	P_2		P_{11}		
211	P_3	$2P_{21}$	P_{12}	P_{111}	
1111	P_4	$4P_{31}$	$3P_{22}$	$6P_{211}$	P_{1111}

$W = 5$

$a \backslash p$	5	41	32	311	221	2111	11111
5	P_1						
41	P_2	P_{11}					
32	P_2		P_{11}				
311	P_3	$2P_{21}$	P_{12}	P_{111}			
221	P_3	P_{21}	$2P_{21}$		P_{111}		
2111	P_4	$3P_{31}$	P_{31} $3P_{22}$	$3P_{211}$	$3P_{121}$	P_{1111}	
11111	P_5	$5P_{41}$	$10P_{32}$	$10P_{311}$	$15P_{221}$	$10P_{2111}$	P_{11111}

find a given value, say $P\binom{2111}{22}$ we note that $w = 5$, look for 2111 on the left and 32 at the top. The result is $P_{31} + 3P_{22}$. In the table the order of the subscripts is important in indicating the number of parts collected to form the respective parts of the ordered $p_1^{\pi_1} \cdots p_s^{\pi_s}$.

The values in the table previously mentioned [19; 29–32] may be obtained when $w \leqq 6$ by placing every P in Table I equal to unity.

5. **Value of** $P(a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_h^{\alpha_h})$. The parts of the partition $a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_h^{\alpha_h}$ may be collected to form a large number of partitions of the type $p_1^{\pi_1} \cdots p_s^{\pi_s}$. Thus the parts of the partition 2111 may be collected to form 5, 41, 32, 311, 221, 2111. We denote by $P(2111)$ the values

$$P\binom{2111}{5} 5 + P\binom{2111}{41} 41 + P\binom{2111}{32} 32 + P\binom{2111}{311} 311 + P\binom{2111}{221} 221$$

$$+ P\binom{2111}{2111} 2111 = P_4 5 + 3P_{31} 41 + [P_{31} + 3P_{22}]\, 32 + 3P_{211} 311$$

$$+ 3P_{211} 221 + P_{1111} 2111$$

and in general

$$P(a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_h^{\alpha_h}) = \sum P\binom{a_1^{\alpha_1} \cdots a_h^{\alpha_h}}{p_1^{\pi_1} \cdots p_s^{\pi_s}} p_1^{\pi_1} \cdots p_s^{\pi_s} \quad \cdots\cdots\cdots \{2\}$$

where the summation holds for every partition $p_1^{\pi_1} \cdots p_s^{\pi_s}$ which can be formed by combining parts of $a_1^{\alpha_1} \cdots a_h^{\alpha_h}$. The values of $P(a_1^{\alpha_1} \cdots a_h^{\alpha_h})$ for $w \leqq 6$ are given in the rows of Table I. Thus the value of $P(2111)$ above is found along the row 2111 where $w = 5$.

6. **Values of** $P(1^r)$ **and** $P(a^r)$. When $a_1 = 1$ and $\alpha_1 = r$ we have

$$P(1^r) = \sum P\binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} p_1^{\pi_1} \cdots p_s^{\pi_s}$$

and since there are $\binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ different ways of forming $p_1^{\pi_1} \cdots p_s^{\pi_s}$ from the r units and each way is indicated by $P_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ we have

$$P(1^r) = \sum \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} P_{p_1^{\pi_1} \cdots p_s^{\pi_s}} p_1^{\pi_1} \cdots p_s^{\pi_s}. \qquad \{3\}$$

When $r = 2, 3, 4$, etc., we get

$$P(1^2) = P_2\, 2 + P_{11}\, 1^2$$

$$P(1^3) = P_3\, 3 + 3P_{21}\, 21 + P_{111}\, 1^3$$

$$P(1^4) = P_4\, 4 + 4P_{31}\, 31 + 3P_{22}\, 22 + 6P_{211}\, 211 + P_{1111}\, 1^4$$

etc.

as indicated in Table I.

Similarly when $a_1 = a$ and $\alpha_1 = r$ {2} becomes

$$P(a^r) = \sum \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} P_{p_1^{\pi_1} \cdots p_s^{\pi_s}} (ap_1)^{\pi_1} (ap_2)^{\pi_2} \cdots (ap_s)^{\pi_s} \qquad \{3'\}$$

since there are $\binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ different ways of forming the partition $(ap_1)^{\pi_1} (ap_2)^{\pi_2} \cdots (ap_s)^{\pi_s}$ from the r equal a's and each way is indicated by $P_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$. For example

$$P(a) = P\binom{a}{a} a = P_1 a$$

$$P(a^2) = P\binom{a^2}{2a} + P\binom{a^2}{aa} = P_2 2a + P_{11} a^2$$

$$P(a^3) = P\binom{a^3}{3a} + P\binom{a}{2a \cdot a} + P\binom{a}{a \cdot a \cdot a} = P_3 3a + 3P_{21} 2a \cdot a + P_{111} a^3$$

$$P(a^4) = P_4 4a + 4P_{31} 3a \cdot a + 3P_{22} 2a \cdot 2a + 6P_{211} 2a \cdot a^2 + P_{1111} a^4.$$

7. **Values of** $P(a_1 a_2 \cdots a_r)$. From the definition

$$P(a_1) = P\binom{a_1}{a_1} a_1 = P_1 a_1$$

$$P(a_1 a_2) = P\binom{a_1 a_2}{a_1 + a_2} + P\binom{a_1 a_2}{a_1 a_2}$$

$$= P_2 \overline{a_1 + a_2} + P_{11} a_1 a_2$$

$$P(a_1 a_2 a_3) = P\binom{a_1 a_2 a_3}{a_1 + a_3 + a_3}(a_1 + a_2 + a_3) + P\binom{a_1 a_2 a_3}{\overline{a_1 + a_2} \cdot a_3}(\overline{a_1 + a_2} \cdot a_3)$$

$$+ P\binom{a_1 a_2 a_3}{\overline{a_1 + a_3} \cdot a_2}(\overline{a_1 + a_3} \cdot a_2) + P\binom{a_1 a_2 a_3}{\overline{a_2 + a_3} \cdot a_1}(\overline{a_2 + a_3} \cdot a_1)$$

$$+ P\binom{a_1 a_2 a_3}{a_1 a_2 a_3}(a_1 a_2 a_3) = P_3(a_1 + a_2 + a_3)$$

$$+ P_{21}\{(\overline{a_1 + a_2} \cdot a_3) + (\overline{a_1 + a_3} \cdot a_2) + (\overline{a_2 + a_3} \cdot a_1)\} + P_{111}(a_1 a_2 a_3)$$

etc.

Now if complete order of the general partition indicates the number of a's collected to form the partition, the subscripts of the P's are the respective complete orders. If we indicate the sum of partitions having the same complete order by the term "partition type" and indicate the partition type composed of all terms having the same complete order

$$p_1^{\pi_1} \cdots p_s^{\pi_s} \text{ by } T_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$$

then

$$P(a_1) = P_1 T_1$$
$$P(a_1 a_2) = P_2 T_2 + P_{11} T_{11}$$
$$P(a_1 a_2 a_3) = P_3 T_3 + P_{21} T_{21} + P_{111} T_{111}$$
$$P(a_1 a_2 a_3 a_4) = P_4 T_4 + P_{31} T_{31} + P_{22} T_{22} + P_{211} T_{211} + P_{1111} T_{1111}$$

etc.

and in general

$$P(a_1 a_2 \cdots a_r) = \sum P_{p_1^{\pi_1} \cdots p_s^{\pi_s}} T_{p_1^{\pi_1} \cdots p_s^{\pi_s}} \qquad \{4\}$$

This formula can be used in writing the formula of Table II or formulas of weight greater than 6. Thus

$P(543)$ is given by $P_3 T_3 + P_{21} T_{21} + P_{111} T_{111}$

where

$$T_3 = 12, \; T_{21} = 9 \cdot 3 + 8 \cdot 4 + 7 \cdot 5, \text{ and } T_{111} = 5 \cdot 4 \cdot 3$$

where the dots do not indicate multiplication, but merely the separation of the parts.

In general $T_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ is composed of $\binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ partitions since this is the number of ways in which the a's can be combined to form partitions having the same complete order, $p_1^{\pi_1} \cdots p_s^{\pi_s}$.

Formula $\{3'\}$ is a special case of this formula. If the a's are all equal, the $\binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ partitions are equal so that $T_{p_1^{\pi_1} \cdots p_s^{\pi_s}} = \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} (ap_1)^{\pi_1} \cdots (ap_s)^{\pi_s}$. Substitution in $\{4\}$ gives $\{3'\}$. Similarly $\{4\}$ gives $\{3\}$ when all the a's are unity.

8. **Generalization from Symmetry.** The function $P(a_1 a_2 \cdots a_r)$ is a symmetric function of the parts $a_1, a_2, \cdots a_r$, i.e., the interchange of any two of the parts does not change the value of the function. It is possible to use this fact as a basis of generalization and to derive $\{4\}$ from $\{3\}$ by its use. From $\{3\}$ we have

$$P(1^r) = \sum \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} P_{p_1^{\pi_1} \cdots p_s^{\pi_s}} p_1^{\pi_1} \cdots p_s^{\pi_s} \qquad \{3\}$$

where $\binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ is the number of the equivalent partitions which can be formed from the r units. In case the r units are replaced by the r different a's, there will result $\binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ different partitions having the same complete order. These $\binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ different partitions defined by $T_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ replace the $\binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ equivalent partitions of {2} and we have

$$P(a_1 a_2 \cdots a_r) = \sum P_{p_1^{\pi_1} \cdots p_s^{\pi_s}} T_{p_1^{\pi_1} \cdots p_s^{\pi_s}} \qquad \{4\}$$

9. **The Recursion Rule.** It is possible to establish a recursion property by which the value of $P(a_1 a_2 \cdots a_r a_{r+1})$ can be obtained from the value of $P(a_1 a_2 \cdots a_r)$. We note, from the results of Table I or by {4} that

$$P(3) = P_1 3$$

$$P(32) = P_2 5 + P_{11} 32$$

$$P(321) = P_3 6 + P_{21} 51 + P_{21} 42 + P_{12} 33 + P_{111} 111$$

$P(32)$ is obtained from $P(3)$ by symbolic multiplication of its expansion, $P_1(3)$, by the expansion of $P(2)$, $P_1(2)$. This symbolic multiplication is accomplished by adding the 2 to the 3 and also suffixing the 2 to the 3. If the 2 is added, the subscripts of the P's are added while if suffixed, the subscripts of the P's are suffixed.

More generally if $P(a_1) = P_1(a_1)$ and $P(a_2) = P_1(a_2)$, then the result $P(a_1 a_2) = P_2(\overline{a_1 + a_2}) + P_{11}(a_1)(a_2)$ is obtained by multiplying $P_1(a_1)$ by $P_1(a_2)$ [or $P_1(a_2)$ by $P_1(a_1)$] symbolically if the subscripts are added when the a's are added and suffixed when the a's are suffixed. Similarly $P(a_1 a_2) = P_2(\overline{a_1 + a_2}) + P_{11}(a_1)(a_2)$ when multiplied by $P(a_3) = P_1(a_3)$ gives

$$P(a_1 a_2 a_3) = P_3(\overline{a_1 + a_2 + a_3}) + P_{21}(\overline{a_1 + a_2} \cdot a_3) + P_{21}(\overline{a_1 + a_3} \cdot a_2) + P_{12}(a_1 \cdot \overline{a_2 + a_3}) + P_{111}(a_1 a_2 a_3)$$

when the rule of multiplication is the adding of a_3 in turn to every part of every partition with the appropriate adding of subscripts and the suffixing of a_3 to every partition with the corresponding suffixing of subscripts. It is important to note that the P coefficient of $a_1 \cdot \overline{a_2 + a_3}$ is P_{12} and not P_{21} although the term could be written $P_{21} \overline{a_2 + a_3} \cdot a_1$. The applications do not demand the retention of a given order of subscripts though the continued application of the recursion rule does demand it.

In general the value of $P(a_1 \cdots a_r a_{r+1})$ can be obtained from the value of $P(a_1 a_2 \cdots a_r)$ by the symbolic multiplication of the expansion $P(a_1 a_2 \cdots a_r)$ by $P_1(a_{r+1})$ since all possible algebraic partitions of $a_1 + a_2 + \cdots + a_r + a_{r+1}$ are obtained from all possible algebraic partitions of $a_1 + a_2 + \cdots + a_r$ by adding a_{r+1} in turn to each part of each partition and by suffixing it to each partition. The corresponding P subscript, indicating the number of a's collected, is increased by 1.

The recursion rule is useful in checking the entries of Table I. As a matter of fact Table I was computed with its use and the order of the subscripts is that which results from its use. The rule is also useful in finding values when $w > 6$. For example, since

$$P(321) = P_3 6 + P_{21} 51 + P_{21} 42 + P_{12} 33 + P_{111} 321$$

$$P(3221) = P_4 8 + P_{31} 62 + P_{31} 71 + P_{22} 53 + P_{211} 512 + P_{31} 62 + P_{22} 44$$
$$+ P_{211} 422 + P_{22} 53 + P_{13} 35 + P_{121} 332 + P_{211} 521 + P_{121} 341 + P_{112} 323$$
$$+ P_{1111} 3212 = P_4 8 + P_{31} 71 + 2 P_{31} 62 + (P_{31} + 2P_{22}) 53 + P_{22} 44$$
$$+ 2P_{211} 521 + P_{211} 431 + P_{211} 422 + 2P_{211} 332 + P_{1111} 3221.$$

A useful check is based on the fact that the sum of the P coefficients of $P(a_1 \cdots a_r)$ should equal the sum of the coefficients of $P(1^r)$. In the above illustration the sum of the coefficients is $P_4 + 4P_{31} + 3P_{22} + 6P_{211} + P_{1111}$ as desired.

10. **Use of the P Function Formulas.** The P function formulas, as defined, represent concisely the ways in which the parts of a given partition may be combined to get the parts of other partitions. They are also useful in writing expansions of certain partition functions whose expanded values are expressed in terms of other partition functions. They are used, in this paper, in expressing the multinomial theorem, the multiplication theorem for power sums, the expansions of power product sums in terms of power sums, expansions of monomial symmetric functions in terms of power sums, the double expansion theorem itself, the coefficients in the double expansion theorem as well as the sampling laws of Part II. They are also useful in representing the expansions of different moment functions and can be associated with important concepts of mathematics and statistics such as, for example, the differences of 0. Such applications, however, are not pertinent to the line of reasoning which is developed in Chapters II, III, IV, V.

Chapter II

It is the purpose of this chapter to obtain formulas for the expansion of power sums.

11. **Definitions.** a. *Power Sum.* Let x be a variable which is restricted to the N variates, $x_1, x_2, x_3, \cdots, x_N$. Then the a-th power sum of the variable indicated by (a) is defined to be

$$(a) = x_1^a + x_2^a + \cdots + a_N^a = \sum_{i=1}^{N} x_i^a \qquad \{5\}$$

It is assumed for the purposes of this paper that a is a positive integer or 0.

b. *Power Product Sum.* The expression $\sum_{i \neq j} x_i^{a_1} x_j^{a_2}$ is to be called a power product sum since it is composed of the sum of products of the powers of the variates. It is to be denoted by $(a_1 \cdot a_2)$ or $(a_1 a_2)$. Thus $\sum_{i \neq j} x_i^3 x_j^2 = (3 \cdot 2)$ or (32). The value $(a \cdot a) = (a^2) = \sum_{i \neq j} x_i^a x_j^a$ is a special case of $(a_1 a_2)$ where $a_2 = a_1 = a$. In general the power product sum is defined by the right hand member and indicated by the left hand member of

$$(a_1 a_2 \cdots a_r) = \sum_{i_1 \neq i_2 \neq i_3 \neq \cdots \neq i_r} x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_p}^{a_p} \qquad \{6\}$$

If $i_1 = i_2$, the power product sum becomes

$$(\overline{a_1 + a_2} \cdot a_3 \cdot a_4 \cdots a_r) = \sum_{i_1 = i_2 \neq i_3 \neq i_4 \neq \cdots \neq i_r} x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_p}^{a_p} \qquad \{7\}$$

There are many different definitions since there are many different ways of indicating equality relations among the i's. Each results in a unique power product sum which is to be called, for brevity, a power product. If the a's are all unity, there are many duplicates. Thus for the grouping $p_1^{\pi_1} \cdots p_s^{\pi_s}$, there are $\binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ equal power products $(p_1^{\pi_1} \cdots p_s^{\pi_s})$. In the more general case we can let $T_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ represent the $\binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ different power products having the same complete order, $p_1^{\pi_1} \cdots p_s^{\pi_s}$. We may represent any one of these forms having this complete order by

$$(q_1 q_2 q_3 \cdots q_\rho)$$

where $\quad q_1 + q_2 + q_3 + \cdots + q_\rho = w$

or by $\quad (q_1^{x_1} q_2^{x_2} \cdots q_t^{x_t})$*

where $\quad q_1 x_1 + q_2 x_2 + \cdots + q_t x_t = w$

and $\quad x_1 + x_2 + \cdots + x_t = \rho.$

c. *Symmetric Functions.* Both the power sum and the power product are symmetric functions of the variates since the interchange of any x_i with any x_j does not change the value of the function. Also the powder product having ρ parts is composed of $N^{(\rho)}$ products of powers since the first group of equal i's may be selected in N ways, the next group in $N - 1$ ways etc.

d. *Monomial Symmetric Function.* It is customary to use the monomial symmetric function which is defined as

$$\sum_{i_1 < i_2 < i_3 < \cdots < i_\rho} x_{i_1}^{q_1} x_{i_2}^{q_2} \cdots x_{i_\rho}^{q_\rho}$$

* "It was intended that the letter representing the exponents of the q's should be the Greek 'chi,' and not the English 'x.'"

and which we designate by $M(q_1 \cdots q_\rho)$ or by $M(q_1^{x_1} \cdots q_t^{x_t})$. This function is not useful for our purposes since the number of terms in its expansion varies with the number of repeated q's. For example if $N = 3$ and $q_1 \neq q_2$; $M(q_1 q_2) = x_1^{q_1}x_2^{q_2} + x_1^{q_1}x_3^{q_2} + x_2^{q_1}x_3^{q_2} + x_2^{q_1}x_1^{q_2} + x_3^{q_1}x_1^{q_2} + x_3^{q_1}x_2^{q_2} = (q_1 q_2)$ while if $q_1 = q_2 = q$

$$M(q^2) = x_1^q x_2^q + x_1^q x_3^q + x_2^q x_3^q = \frac{(q^2)}{2!}.$$

The monomial symmetric function keeps the number of product terms a minimum by eliminating all repeated terms while the power product sum keeps the number of product terms the same by the use of repeated terms, when some of the parts are alike.

12. The Formula Connecting $(q_1^{x_1}q_2^{x_2} \cdots q_t^{x_t})$ and $M(q_1^{x_1} \cdots q_t^{x_t})$. The power product is composed of $N^{(\rho)}$ products, each of which is repeated $x_1!x_2! \cdots x_t!$ times. The monomial symmetric function is composed of the $\frac{N^{(\rho)}}{x_1!x_2!\cdots x_t!}$ different products which, when repeated $x_1!x_2! \cdots x_t!$ times, gives the $N^{(\rho)}$ terms above. Hence

$$(q_1^{x_1} \cdots q_t^{x_t}) = x_1!x_2! \cdots x_t!\, M(q_1^{x_1} \cdots q_t^{x_t}) \qquad \{8\}$$

$$M(q_1^{x_1} \cdots q_t^{x_t}) = \frac{1}{x_1!x_2!\cdots x_t!}(q_1^{x_1} \cdots q_t^{x_t}) \qquad \{9\}$$

In the special case in which $q_1 = 1$ and $x_1 = \rho$

$$(1^\rho) = \rho!\,M(1^\rho) \quad \text{and} \quad M(1^\rho) = \frac{(1^\rho)}{\rho!} \qquad \{10\}$$

The function, $M(1^\rho)$ is commonly called an elementary symmetric function. We refer to the corresponding (1^ρ) as the unitary power product sum.

13. Correspondence of Partitions and Power Products. To each power product $(q_1^{x_1} \cdots q_t^{x_t})$ there corresponds an algebraic partition $q_1^{x_1} \cdots q_t^{x_t}$ having ρ parts and weight $w = a_1 + a_2 + \cdots + a_r$.

This follows at once from the definitions and notation. Thus if $w = a_1 + a_2 + a_3$, the power product

$$\sum_{i_1 = i_2 \neq i_3} x_{i_1}^{a_1} x_{i_2}^{a_2} x_{i_3}^{a_3} = \sum_{i_1 \neq i_2} x_{i_1}^{a_1+a_2} x_{i_2}^{a_3} = (\overline{a_1 + a_2}\cdot a_3)$$

is, by notation, associated with the partition $\overline{a_1 + a_2}\cdot a_3$. Conversely each algebraic partition, when enclosed in parentheses, represents a power product sum.

This proposition is useful in that it enables one to establish a relationship between the theory of power product sums and the theory of partitions. Earlier writers have used a similar correspondence in relating the theory of monomial symmetric functions to that of partitions. See for instance [3; 106], [4; 5] [5; I; 7].

Due to this correspondence we do not hesitate to apply such terms as part, order, complete order, similar, etc. to the power product as well as to the algebraic partition. Also the sum of all power products $(q_1^{x_1} \cdots q_t^{x_t})$ having the same complete order is represented by $T(p_1^{\pi_1} \cdots p_s^{\pi_s})$. This represents the sum of $\binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ similar power products.

14. **The Multiplication Theorem for Power Sums.** The correspondence property enables us to derive a theorem, to be known as the multiplication theorem, which expresses products of power sums in terms of power products. The type of argument is introduced by establishing simple cases of the theorem

$$(a_1)(a_2) = \Big(\sum_{i=1} x_i^{a_1}\Big)\Big(\sum_{i=1} x_i^{a_2}\Big) = \sum_{\substack{i_1=1\\ i_2=1}}^{N} x_{i_1}^{a_1} x_{i_2}^{a_2}$$

$$= \sum_{i_1=i_2} x_{i_1}^{a_1} x_{i_2}^{a_2} + \sum_{i_1\neq i_2} x_1^{a_1} x_2^{a_2} = (a_1 + a_2) + (a_1 a_2)$$

$$(a_1)(a_2)(a_3) = \Big(\sum x_i^{a_1}\Big)\Big(\sum x_i^{a_2}\Big)\Big(\sum x_i^{a_3}\Big) = \sum_{i_1,i_2,i_3} x_{i_1}^{a_1} x_{i_2}^{a_2} x_{i_3}^{a_3}$$

$$= (a_1 + a_2 + a_3) + (\overline{a_1 + a_2}\cdot a_3) + (\overline{a_1 + a_3}\cdot a_2) + (a_1\cdot \overline{a_2 + a_3}) + (a_1\cdot a_2\cdot a_3)$$

since the value $\sum_{i_1,i_2,i_3}$ is broken into

$$\sum_{i_1=i_2=i_3}, \quad \sum_{i_1=i_2\neq i_3}, \quad \sum_{i_1=i_3\neq i_2}, \quad \sum_{i_1\neq i_2=i_3}, \quad \sum_{i_1\neq i_2\neq i_3}.$$

In general, when $r \leqq N$

$$(a_1)(a_2) \cdots (a_r) = \sum_{i_1,i_2,\cdots i_r} x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_r}^{a_r}$$

and this can be broken into summations featuring different equality relations. These summations define all the different power product sums of weight $w = a_1 + a_2 + \cdots + a_r$. The different algebraic partitions of $a_1 + a_2 + a_3 + \cdots + a_r$ correspond to the different power product sums. It follows at once that the value of $(a_1)(a_2) \cdots (a_r)$ is obtained by writing each algebraic partition of $a_1 + a_2 + \cdots + a_r$, enclosing it in parentheses to represent a power product, and adding. More symbolically we have

$$(a_1)(a_2) \cdots (a_r) = \sum (q_1^{x_1} \cdots q_t^{x_t}) \tag{11}$$

where $q_1^{x_1} \cdots q_t^{x_t}$ represents any algebraic partition of $a_1 + a_2 + \cdots + a_r$. and the summation holds for all such partitions or by

$$(a_1)(a_2) \cdots (a_r) = \sum T(p_1^{\pi_1} \cdots p_s^{\pi_s}) \tag{12}$$

where $T(p_1^{\pi_1} \cdots p_s^{\pi_s})$ represents the $\binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ similar power products and the summation holds for each different complete order.

For example $(a_1)(a_2)(a_3) = T(3) + T(21) + T(111)$ and $T(3) = (a_1 + a_2 + a_3)$, $T(21) = (\overline{a_1 + a_2} \cdot a_3) + (\overline{a_1 + a_3} \cdot a_2) + (\overline{a_2 + a_3} \cdot a_1)$, and $T(111) = (a_1 \cdot a_2 \cdot a_3)$.

The theorem has been established on the assumption that $r \leqq N$. If such is not the case it is possible to satisfy the assumption by adding additional variates, $x_{N+1}, x_{N+2}, \cdots x_r$, all 0, without changing the value of the power sums or of the product of the power sums since the added terms are always 0. Thus

$$(x_1^a + x_2^a)(x_1^b + x_2^b)(x_1^c + x_2^c) = (x_1^a + x_2^a + x_3^a)(x_1^b + x_2^b + x_3^b)(x_1^c + x_2^c + x_3^c)$$

when $x_3 = 0$

Then

$$(a)(b)(c) = (a + b + c) + (\overline{a + b} \cdot c) + (\overline{a + c} \cdot b) + (\overline{b + c} \cdot a) + (a \cdot b \cdot c)$$

which is

$$\sum x_i^{a+b+c} + \sum_{i \neq j} x_i^{a+b} x_j^c + \sum_{i \neq j} x_i^{a+c} x_j^b + \sum_{i \neq j} x_i^{b+c} x_j^a + \sum_{i \neq j \neq k} x_i^a x_j^b x_k^c$$

The term $\sum_{i \neq j \neq k} x_i^a x_j^b x_k^c = 0$ since every product composing it contains an $x_3 = 0$. The other power product sums are to be applied to the original variates only since the terms involving x_3 are 0 in every case.

In general, if $r > N$, it is only necessary to write out the power product sums having N or less parts since all those having more than N parts will be 0.

15. The Multiplication Theorem Using the Results of Chapter I. Comparison of {12} with {4} shows that {12} can be obtained from {4} by placing $P(a_1 \cdots a_r) = (a_1)(a_2) \cdots (a_r)$, $T_{p_1^{\pi_1} \cdots p_s^{\pi_s}} = T(p_1^{\pi_1} \cdots p_s^{\pi_s})$ and $P_{p_1^{\pi_1} \cdots p_s^{\pi_s}} = 1$. Since this can be done for all values of a and r it follows at once that the entire theory of Chapter I is applicable to the present problem. For example Table I shows that

$$P(321) = P_3 6 + P_{21} 51 + P_{21} 42 + P_{12} 33 + P_{111} 321$$

and it follows that

$$(3)(2)(1) = (6) + (51) + (42) + (33) + (321)$$

It should be noted that it is possible to use the table previously published [19; 29–32] since the entries in this table are the values obtained when $P_{p_1^{\pi_1} \cdots p_s^{\pi_s}} = 1$. The value $(3)(2)(1)$ may also be checked from this table.

16. The Multinomial Theorem. The multinomial theorem is a special case of the multiplication theorem for power sums in which the power sums are all equal. If $a_1 = a_2 = \cdots = a_r = 1$,

$$T(p_1^{\pi_1} \cdots p_s^{\pi_s}) = \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} (p_1^{\pi_1} \cdots p_s^{\pi_s})$$

and {12} becomes

$$(1)^r = \sum \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} (p_1^{\pi_1} \cdots p_s^{\pi_s}) \qquad \{13\}$$

which is the multinomial theorem in terms of power product sums. Special cases are

$$(1)^2 = (2) + (11)$$
$$(1)^3 = (3) + 3(21) + (111)$$
$$(1)^4 = (4) + 4(31) + 3(22) + 6(211) + (1111)$$

etc.

The result of {13} may also be obtained immediately from {3} by placing $P(1^r) = (1)^r$, $P_{p_1^{\pi_1} \cdots p_s^{\pi_s}} = 1$, and $p_1^{\pi_1} \cdots p_s^{\pi_s} = (p_1^{\pi_1} \cdots p_s^{\pi_s})$.

A more general form of the multinomial theorem is that in which $a_1 = a_2 = \cdots = a_r = a$. In this case

$$(x_1^a + x_2^a + \cdots + x_N^a)^r = (\mathrm{a})^r$$

and {12} gives

$$(a)^r = \sum \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} ((ap_1)^{\pi_1} \cdots (ap_s)^{\pi_s}) \qquad \{14\}$$

where $((ap_1)^{\pi_1} \cdots (ap_s)^{\pi_s})$ has parts $ap_1, \cdots, ap_s$. Thus

$$(a)^3 = (3a) + 3(2a \cdot a) + (a^3)$$

so that

$$(2)^3 = (6) + 3(4 \cdot 2) + (2^3).$$

The result {14} may also be obtained immediately from {3′} by placing $P(a^r) = (a)^r$, $P_{p_1^{\pi_1} \cdots p_s^{\pi_s}} = 1$, and $(ap_1)^{\pi_1} \cdots (ap_s)^{\pi_s} = ((ap_1)^{\pi_1} \cdots (ap_s)^{\pi_s})$. When $N = 2$, {13} gives the binomial theorem

$$(1)^r = \sum \binom{1^r}{p_1 p_2} (p_1 p_2)$$

special cases of which are

$$(1)^2 = (2) + (11)$$
$$(1)^3 = (3) + 3(21)$$
$$(1)^4 = (4) + 4(31) + 3(22)$$
$$(1)^5 = (5) + 5(41) + 10(32)$$

etc.

These can be readily translated to the usual form. Thus

$$(a + b)^4 = a^4 + b^4 + 4(a^3b + b^3a^3) + 3(a^2b^2 + b^2a^2).$$

In a similar manner the trinomial theorem appears as

$$(1)^r = \sum \binom{1^r}{p_1 p_2 p_3}(p_1 p_2 p_3)$$

A special case of the multinomial theorem {13} is also useful in writing N^r in terms of sums of $N^{(\rho)}$. When the variates are all unity the power sums are all N, and the power product sums are the number of terms in the partition representing it. If a partition has ρ parts the number of terms in it is $N^{(\rho)}$. We then have

$$N^r = \sum \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} N^{(\rho)} \qquad \{15\}$$

Special cases are

$$N^2 = N + N^{(2)}$$

$$N^3 = N + 3N^{(2)} + N^{(3)}$$

$$N^4 = N + 4N^{(2)} + 3N^{(2)} + 6N^{(3)} + N^{(4)} = N + 7N^{(2)} + 6N^{(3)} + N^{(4)}$$

etc.

17. **The Use of Monomial Symmetric Functions.** It is possible to express the results in terms of the monomial symmetric functions by means of {8}. Thus

$$(2)(2)(2) = (6) + 3(42) + (222)$$

$$= M(6) + 3M(42) + 6M(222).$$

In general, Table I may be used to express products of power sums in terms of monomial symmetric functions. It is only necessary to place every $P_{p_1^{\pi_1} \cdots p_s^{\pi_s}} = 1$ and to multiply by the factorials indicating the repeated entries at the head of each column. The table [19; 29–32] may be used similarly.

The multinomial theorem in terms of monomial symmetric functions becomes

$$(1)^r = \sum \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} \pi_1! \pi_2! \cdots \pi_s! M(p_1^{\pi_1} \cdots p_s^{\pi_s})$$

and by {1}

$$(1)^r = \sum \frac{r!}{(p_1!)^{\pi_1}(p_2!)^{\pi_2} \cdots (p_s!)^{\pi_s}} M(p_1^{\pi_1} \cdots p_s^{\pi_s}) \qquad \{16\}$$

as it is conventionally stated.

18. **The Multiplication Theorem from the Multinomial Theorem.** It is possible to use generalization from symmetry in deriving the multiplication theorem from the multinomial theorem though this can not well be done from its conventional statement (16). The monomial symmetric function does not have the property that $M(a \cdot b) = M(a \cdot a)$ when $b = a$ while $(a \cdot b)$ does become $(a \cdot a)$ when $b = a$. The first step then is to reduce {16} to power product sums by means of {9}. We then have

$$(1)^r = \sum \frac{r!}{(p_1!)^{\pi_1}(p_2!)^{\pi_2} \cdots (p_s!)^{\pi_s} \pi_1!\pi_2! \cdots \pi_s!} (p_1^{\pi_1} \cdots p_s^{\pi_s})$$

Next it is necessary to introduce the factor $\begin{pmatrix} 1^r \\ p_1^{\pi_1} \cdots p_s^{\pi_s} \end{pmatrix}$ for there are many equal terms for each value $p_1^{\pi_1} \cdots p_s^{\pi_s}$ when the a's are all unity. This is very easy in this case since the value of the coefficient of $(p_1^{\pi_1} \cdots p_s^{\pi_s})$ is $\begin{pmatrix} 1^r \\ p_1^{\pi_1} \cdots p_s^{\pi_s} \end{pmatrix}$. It follows at once that

$$(1)^r = \sum \begin{pmatrix} 1^r \\ p_1^{\pi_1} \cdots p_s^{\pi_s} \end{pmatrix} (p_1^{\pi_1} \cdots p_s^{\pi_s}).$$

Suppose that the r units are replaced by $a_1 a_2 \cdots a_r$. Then the $\begin{pmatrix} 1^r \\ p_1^{\pi_1} \cdots p_s^{\pi_s} \end{pmatrix}$ power products, $(p_1^{\pi_1} \cdots p_s^{\pi_s})$ will be replaced by the $\begin{pmatrix} 1^r \\ p_1^{\pi_1} \cdots p_s^{\pi_s} \end{pmatrix}$ different power products composing $T(p_1^{\pi_1} \cdots p_s^{\pi_s})$. It follows at once that

$$(a_1)(a_2) \cdots (a_r) = \sum T(p_1^{\pi_1} \cdots p_s^{\pi_s}).$$

19. **The Determination of the Coefficient of a Given Power Product in the Expansion of a Product of Power Sums.** In some cases we wish to determine the coefficient of a given power product without computing the complete expansion. This is given by $P \begin{pmatrix} a_1^{\alpha_1} \cdots a_h^{\alpha_h} \\ q_1^{\pi_1} \cdots q_t^{\pi_t} \end{pmatrix}$ where the P coefficients are unity. Thus the coefficient of (32) in the expansion of (2)(1)(1)(1) is found from $P \begin{pmatrix} 2111 \\ 32 \end{pmatrix} = P_{31} + 3P_{22}$ and is 4.

20. **Relation to Previous Results.** The multiplication theorem may be viewed as a generalization of the multinomial theorem. A more general proof, applicable to multivariate problems, could be presented with the use of more involved notation. It seems wise rather to present the simpler one variate case and to emphasize the principle of generalization from symmetry which will enable us to write the multivariate laws with relative ease.

The general problem discussed here seems to have received a very small amount of consideration as much of the extensive classical theory of symmetric functions is limited to the interrelations of the elementary symmetric functions and the monomial symmetric functions.

A monumental work on symmetric functions not subject to this limitation is the Combinatory Analysis of MacMahon [*K*]. MacMahon provided a technique for multiplying power sums in many variables as a special case of a more general theory. [*K*; II, 321].

Some of the work on alternants is closely related to the problem of products of power sums although the alternant, as usually defined, is limited to the case in which $r = N$ [I; II, 446]. For an example the reader is referred to a development by Muir [L; 335–6].

Thiele (1889) gave tables[2] of products of power sums in terms of monomial symmetric functions for partition products of weight $\leqq 8$ [H; 114–117]. J. R. Roe has later given one for $w \leqq 10$ [N; Plates 17, 18]. Statisticians have sometimes stated the results in nontabular form. See for example, the multiplication formulas of Church [13; 81–83] [14; 370–1], whose results may not at first appear to agree with those above since Church has used a less compact notation and, of course, the monomial symmetric function.

The chief contributions of the present attack are

1. The use of the formulas and tables of Chapter I in writing expansions of products of power sums.
2. The use of power product sums in place of monomial symmetric functions which makes feasible.
3. Generalization from symmetry.

Chapter III

It is the purpose of this chapter to establish formulas giving the expansion of power products in terms of products of power sums.

21. **The Binet (Waring) Identities.** It is customary to introduce this subject with formulas for $M(a\cdot b)$, $M(a\cdot b\cdot c)$, etc. so we first derive the formulas for $(a\cdot b)$, $(a\cdot b\cdot c)$, etc. We may use the results of Chapter II since the problem here is the inverse of the multiplication problem. By the multiplication theorem

$$(a)(b) = (a + b) + (a\cdot b)$$

$$(a)(b)(c) = (a + b + c) + (\overline{a + b}\cdot c) + (\overline{a + c}\cdot b) + (\overline{b + c}\cdot a) + (a\cdot b\cdot c)$$

$$(a + b)(c) = (c + b + c) + (\overline{a + b}\cdot c)$$

[2] These tables are not accessible to me, but Thiele refers to them in his "Theory of Observations."

so we get

$$(a \cdot b) = (a)(b) - (a + b) \qquad \{17\}$$

$$(a \cdot b \cdot c) = (a)(b)(c) - (a + b)(c) - (a + c)(b) - (b + c)(a) + 2(a + b + c) \qquad \{18\}$$

Similarly

$$\begin{aligned}(a \cdot b \cdot c \cdot d) = {} & (a)(b)(c)(d) - (a + b)(c)(d) - (a + c)(b)(d) \\ & - (a + d)(b)(c) - (b + c)(a)(d) - (b + d)(a)(c) \\ & - (c + d)(a)(b) - (a + b)(c + a) - (a + c)(b + d) \\ & - (a + d)(b + c) + 2(a + b + c)(d) + 2(a + b + d)(c) \\ & + 2(a + c + d)(b) + 2(b + c + d)(a) - 6(a + b + c + d) \\ & \ldots\ldots\ldots\ldots \qquad \{19\}\end{aligned}$$

When $a \not\doteq b \not\doteq c \not\doteq d$, {18}, {19}, {20} are also the formulas for $M(ab)$, $M(abc)$, $M(abcd)$. These formulas are quite commonly attributed to Binet who gave them in 1812 in connection with certain proofs of determinant theory [1; 284] [I; I; 81]. Waring should be given credit (see Miscellanea Analytica 1762). Binet gave no proof. The reader is also referred to the earlier work of Paoli [A; section 28].

A much more adequate treatment was given by Hirsch in the early 19th century [B; 35–38]. He wrote, out the terms for $M(a \cdot b \cdot c \cdot d \cdot e)$ and indicated a scheme for extending the results. More than this he proved that any "numerical expression"—his term for monomial symmetric function—can be reduced to numerical expression having one less part [B; 26]. The continued application of this theorem leads eventually to numerical expressions having only one part, i.e. to power sums. Hence all numerical expressions can be reduced to power sums [B; 27, 32].

Recent authors give essentially the same proof. See for example Bocher [J; 241–242] who states the theorem, "Every symmetric polynomial is a linear combination with constant coefficients of a certain number of the Σ's." See also O'Toole [16; 114] and Burnside and Panton [E; 167]. Thus modern authors provide a proof of the fact that $M(a_1 \cdots a_r)$ can be expanded in terms of power sums but most of them fail to provide, a formula giving this precise expansion. Even MacMahon after writing the values of $M(\lambda\mu)$, $M(\lambda\mu r)$, $M(\lambda^2)$, $M(\lambda^3)$ avoids the immediate generalization by stating [K; I; 7], "In actual practice there are easier ways of calculating the many part functions and the general formula is of little importance."

While MacMahon's statement has a certain amount of truth in that any given monomial symmetric function may be computed from others having one less part by the recursion property described by Hirsch, yet there are many cases in which a definite formula, rather than a method, is desirable. A formula particularly is demanded by the statistician who is working with a large number of monomial symmetric functions simultaneously. See for example the remarks and efforts of Carver [15; 103–104, 119–120], Church [14; 373, 377–378], and O'Toole [16; 115].

Some authors have provided solutions and it appears that statisticians are not entirely familiar with all the work which has previously been done. It is the aim of the remainder of this chapter to suggest references which make previous work available to statisticians as well as to present a logical and quite complete development. The main results are not essentially new although their explicit statement in the language of power products is necessary for the development of the next chapter. The argument features the easy generalization from symmetry. The value of (1^r) is expressed in such a form that the value $(a_1 \cdots a_r)$ may be obtained immediately from it.

22. **The Value of (1^r) from Waring's Expansion for the Elementary Symmetric Function.** We first derive the formula (1^r) from the conventional Waring's expression for p_m in terms of the power sums. Burnside and Panton [E; II; 92] give this as

$$p_m = \sum \frac{(-1)^{r_1+r_2+\cdots+r_m} s_1^{r_1} s_2^{r_2} \cdots s_m^{r_m}}{\Gamma(r_1+1)\,\Gamma(r_2+1)\cdots\Gamma(r_m+1)\,2^{r_2}3^{r_3}\cdots m^{r_m}} \qquad \{20\}$$

where $p_m = (-1)^m M(1^m)$ and where $S_1^{r_1} \cdots S_m^{r_m}$ is any $r_1 + r_2 + \cdots + r_m$ part partition of m. When $m = r$ and $p_1^{\pi_1} \cdots p_s^{\pi_s}$ is any ρ part partition of m, {20} becomes

$$(-1)^r M(1^r) = \sum \frac{(-1)^{\rho}(p_1)^{\pi_1}\cdots(p_s)^{\pi_s}}{p_1!^{\pi_1}\cdots p_s!^{\pi_s}\,\pi_1!\pi_2!\cdots\pi_s!} \qquad \{21\}$$

Dividing by $(-1)^r$ and noting that $(-1)^{\rho-r} = (-1)^{r-\rho}$ we have

$$M(1^r) = \sum \frac{(-1)^{r-\rho}(p_1)^{\pi_1}\cdots(p_s)^{\pi_s}}{p_1!^{\pi_1}\cdots p_s!^{\pi_s}\,\pi_1!\pi_2!\cdots\pi_s!} \qquad \{22\}$$

and hence that

$$(1^r) = r!M(1^r) = \sum \frac{(-1)^{r-\rho}r!\,(p_1)^{\pi_1}\cdots(p_s)^{\pi_s}}{p_1!^{\pi_1}\cdots p_s!^{\pi_s}\pi_1!\pi_2!\cdots\pi_s!} \qquad \{23\}$$

A second proof of {23}, given in the next sections, does not assume the formula {20} and develops by easy stages. Although somewhat longer than the method above, it contacts much of the work that has been done in this field. It also provides two useful arithmetic checks dealing with the coefficients which the more analytic method above does not provide. Those who are familar with {20} above and are interested in the immediate development of the argument with the use of {23} should turn to the equivalent {38} of section 28.

23. **The Newtonian Formulas.** The development begins with the well known formulas connecting the power sums and the elementary symmetric functions which appeared in Newton's Arithmetica Universalis. These formulas are given by Bocher (J; 244) as follows

$$S_k - p_1 S_{k-1} + \cdots + (-1)^{k-1} p_{k-1} S_1 + (-1)^k k p_k = 0 \quad k = 1, 2, \cdots \qquad \{24\}$$

where S_k is the sum of the k-th powers and p is the i-th elementary symmetric function.

So many proofs of this theorem are accessible that a repetition here is hardly justifiable. A proof using calculus was given by Bocher (J; 243). Proofs using algebra only were given by Hirsch (B; 16) and Chrystal (F; I, 437). Muirhead (9; 66–70) gave three proofs of which the second is perhaps best adapted to the present development.

24. **The Determinant Equivalent of (1^r).** It is usual to solve the Newtonian equations for the power sums (J; 244) but our objective is the solution in terms of the power sums. The equations are

$$\begin{aligned}
&p_1 &&= (1)\\
&(1)p_1 - 2p_2 &&= (2)\\
&(2)p_2 - (1)p_2 + 3p_3 &&= (3)\\
&(3)p_1 - (2)p_2 + (1)p_3 - 4p_4 &&= (4)\\
&\cdots\cdots\cdots\cdots\cdots\cdots &&
\end{aligned}$$

whence

$$p_r = \frac{\begin{vmatrix}
1 & 0 & 0 & \cdots\cdots & 0 & (1)\\
(1) & -2 & 0 & \cdots\cdots & 0 & (2)\\
(2) & -(1) & 3 & \cdots\cdots & 0 & (3)\\
\cdot & \cdot & \cdot & \cdots\cdots & \cdot & \cdot\\
(r-2) & -(r-3) & (r-4) & \cdots\cdots & (-1)^{r-2}(r-1) & (r-1)\\
(r-1) & -(r-2) & (r-3) & \cdots\cdots & (-1)^{r-2}(1) & (r)
\end{vmatrix}}{\begin{vmatrix}
1 & 0 & 0 & \cdots\cdots & 0 & 0\\
(1) & -2 & 0 & \cdots\cdots & 0 & 0\\
(2) & -(1) & 3 & \cdots\cdots & 0 & 0\\
\cdot & \cdot & \cdot & \cdots\cdots & \cdot & \cdot\\
(r-2) & -(r-3) & (r-4) & \cdots\cdots & (-1)^{r-2}(r-1) & 0\\
(r-1) & -(r-2) & (r-3) & \cdots\cdots & (-1)^{r-2}(1) & (-1)^{r-1}r
\end{vmatrix}}$$

Next, factor out all the negative signs in the even numbered columns in each determinant. The number of these columns of negative signs is the same as the number in the denominator if r is odd. If r is even, there is one more in the denominator. Hence the negative signs may be dropped in both determinants if $(-1)^{r-1}$ is inserted in the numerator. Furthermore the value of the determinant in the denominator is $r!$ Next, change the numerator by moving the r-th column to the first column position and inserting the compensating factor $(-1)^{r-1}$. If Δ_r represents the resulting numerator determinant, the value of p_r becomes

$$p_r = \frac{(-1)^{r-1}(-1)^{r-1}}{r!}\,\Delta_r$$

and

$$\Delta_r = r!p_r = (1^r).$$

We have then

$$(1^r) = \Delta_r = \begin{vmatrix} (1) & 1 & 0 & \cdots\cdots & 0 & 0 \\ (2) & (1) & 2 & \cdots\cdots & 0 & 0 \\ (3) & (2) & (1) & \cdots\cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots\cdots & \cdot & \cdot \\ (r-1) & (r-2) & (r-3) & \cdots\cdots & (1) & r-1 \\ (r) & (r-1) & (r-2) & \cdots\cdots & (2) & (1) \end{vmatrix} \qquad \{25\}$$

The determinant has received the attention of earlier writers {19; 3}. Generalizations of it will be mentioned at the close of the chapter. Its expansion in terms of power sums is known and may be written

$$\Delta_r = \sum \frac{(-1)^{r-\rho} r!\,(p_1)^{\pi_1} \cdots (p_s)^{\pi_s}}{p_1!^{\pi_1} \cdots p_s!^{\pi_s}\,\pi_1!\,\pi_2! \cdots \pi_s!} \qquad \{26\}$$

where $$p_1\pi_1 + p_2\pi_2 + \cdots + p_s\pi_s = r$$

and $$\pi_1 + \pi_2 + \cdots + \pi_s = \rho.$$

See for example O'Toole (16; 113).

It is at once evident that {26} is equivalent to {23}. Those who are familiar with the expansion of Δ_r above may wish to turn immediately to {38} of section 28 since the intervening sections are devoted to a rather detailed and rigorous expansion of the determinant. This development follows, in a general way, that given by Mola (5; 190–195).

25. **The Expansion of $(1^r) = \Delta_r$.** The determinant, Δ_r, is a special type of determinant which is known as a recurrent. There is a simple recursion property which is useful in its expansions in terms of products of power sums.

$$\Delta_{r+1} = \begin{vmatrix} (1) & 1 & 0 & \cdots\cdots\cdots\cdots & 0 & 0 \\ (2) & (1) & 2 & \cdots\cdots\cdots\cdots & 0 & 0 \\ (3) & (2) & (1) & \cdots\cdots\cdots\cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots\cdots\cdots\cdots & \cdot & \cdot \\ (r) & (r-1) & (r-2) & & (1) & r \\ (r+1) & (r) & (r-1) & & (2) & (1) \end{vmatrix}$$

If we expand Δ_{r+1} in terms of the $(r + 1)$st column we have

$$\Delta_{r+1} = (1)\Delta_r - r\blacktriangle_r \qquad \{27\}$$

where $\blacktriangle_r$ represents the determinant Δ_r with every power sum in the r-th row increased by unity. It is only necessary to arrive at some method of designating these terms if the above recurrence formula is to be applied. This can be done by inserting the power sum (1) before the other power sums which it is to multiply. Also in forming $\blacktriangle_r$ add unity to the first power sums in the expansion of Δ_r being careful to retain the previous order. Thus

$$\left.\begin{aligned} \Delta_1 &= (1) \\ \Delta_2 &= (1)(1) - 1(1 + 1) = (1)(1) - (2) \\ \Delta_3 &= (1)[(1)(1) - (2)] - 2[(2)(1) - (3)] = (1)(1)(1) - (1)(2) \\ &\quad - 2(2)(1) + 2(3) \\ \Delta_4 &= (1)(1)(1)(1) - (1)(1)(2) - 2(1)(2)(1) + 2(1)(3) - 3(2)(1)(1) \\ &\quad +3(2)(2) + 6(3)(1) - 6(4) \\ \Delta_5 &= (1)(1)(1)(1)(1) - (1)(1)(1)(2) - 2(1)(1)(2)(1) + 2(1)(1)(3) \\ &\quad -3(1)(2)(1)(1) + 3(1)(2)(2) + 6(1)(3)(1) - 6(1)(4) \\ &\quad - 4(2)(1)(1)(1) + 4(2)(1)(2) + 8(2)(2)(1) - 8(2)(3) \\ &\quad + 12(3)(1)(1) - 12(3)(2) - 24(4)(1) + 24(5) \\ &\quad \text{etc.} \end{aligned}\right\} \{28\}$$

By collection of repeated terms and recalling that $(1^r) = \Delta_r$, the expansion becomes

$$\left.\begin{aligned} (1) &= (1) \\ (1^2) &= (1)^2 - (2) \\ (1^3) &= (1)^3 - 3(2)(1) + 2(3) \\ (1^4) &= (1)^4 - 6(2)(1)^2 + 3(2)^2 + 8(3)(1) - 6(4) \\ (1^5) &= (1)^5 - 10(2)(1)^3 + 15(2)(2)(1) + 20(3)(1)(1) - 20(3)(2) \\ &\quad -30(4)(1) + 24(5). \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \{29\}$$

It is possible to write values of (1^r) in terms of power sums though the practical difficulty increases as r increases. Also continued use of the recursion formula {27} is apt to lead to error. Two simple checks are available. If D_r represents the sum of the coefficients of the expansion of (1^r) and $|D_r|$ represents the sum of the absolute values of these coefficients, then

$$D_r = 0 \qquad \text{when } r > 1 \qquad \{30\}$$

$$|D_r| = r! \qquad \{31\}$$

The proof of {30} and {31} follows directly from {27} since the coefficients of Δ_r and $\blacktriangle_r$ are the same. Thus $D_{r+1} = (1 - r)D_r$ and $|D_{r+1}| = (1 + r)|D_r|$. Since $D_2 = 0$ it follows that $D_3, D_4, \cdots, D_r = 0$ and since $|D_2| = 2!$ it follows that $|D_3|, |D_4|, \cdots, |D_r|$ are $3!, 4!, r!$ respectively.

26. **Determination of the Coefficient of Any Ordered Product of Power Sums in the Expansion of Δ_r.** We next attempt to revise the process outlined above so as to get the formulas {29} without going through the work of writing out {28}. We note first that every product of power sums in the expansion of (1^r) in {28} has been obtained from (1) by a succession of $r - 1$ operations which were either prefixes (when the (1) was prefixed) or raises (when the (1) was added). Also the order of the power sums in a given term indicates which operations have been prefixes and which raises. For example (1)(1)(1)(1)(1) results from 4 prefixes while (5) results from 4 raises. The term (3)(2) results from 1 raise, 1 prefix, and 2 raises respectively, while the term (2)(3) results from 2 raises, a prefix, and a raise. The product $(p_4)(p_3)(p_2)(p_1)$ results from prefixes when $r = p_1$, $r = p_1 + p_2$, $r = p_1 + p_2 + p_3$ and raises at all other times.

The sign of the coefficient of $(p_4)(p_3)(p_2)(p_1)$ can be determined when we recall that each raise is accompanied by a multiplication by $-r$ while each prefix is accompanied by no change in the coefficient. There have been $p_1 - 1 + p_2 - 1 + p_3 - 1 + p_4 - 1 = p_1 + p_2 + p_3 + p_4 - 4$ raises so the sign is $(-1)^{r-\rho}$ where $p_1 + p_2 + p_3 + p_4 = r$. More generally if $(p_\rho) \cdots (p_3)(p_2)(p_1)$ is a term in the expansion of (1^r) where $p_\rho + \cdots + p_3 + p_2 + p_1 = r$ the number of changes in the sign is $p_1 - 1 + p_2 - 1 + \cdots + p_\rho - 1 = p_1 + p_2 + \cdots + p_\rho - p = r - \rho$. It follows at once that those products of power sums in the expansion of (1^r) which have the same number of factors, ρ, also have the same sign and that this sign is $(-1)^{r-\rho}$.

In determining the numerical part of the coefficient we note that each prefix is accompanied by a multiplication by unity which can be written in the form $\frac{r}{r}$. Each raise is accompanied by a multiplication by r so there appears in the numerator the product of all possible values of r and in the denominator the product of those values of r corresponding to each prefix. For example the numerical coefficient of $(p_4)(p_3)(p_2)(p_1)$ is

$$\frac{(p_4 + p_3 + p_2 + p_1 - 1)!}{(p_3 + p_2 + p_1)(p_2 + p_1)(p_1)} = \frac{(p_4 + p_3 + p_2 + p_1)!}{(p_4 + p_3 + p_2 + p_1)(p_3 + p_2 + p_1)(p_2 + p_1)(p_1)}$$

Similarly the coefficient, without sign of $(p_\rho)(p_{\rho-1}), \cdots (p_3)(p_2)(p_1)$ in the expansion of (1^r) is

$$\frac{(p_\rho + p_{\rho-1} + \cdots + p_3 + p_2 + p_1)!}{(p_\rho + p_{\rho-1} + \cdots + p_3 + p_2 + p_1)(p_{\rho-1} + \cdots + p_3 + p_2 + p_1) \cdots (p_3 + p_2 + p_1)(p_2 + p_1)(p_1)} \qquad \{32\}$$

The denominator of {32} has a certain resemblance to a factorial. Thus $4! = (1 + 1 + 1 + 1)\ (1 + 1 + 1)\ (1 + 1)\ (1)$ in which the successive factors are found by dropping the first unit. The corresponding algebraic expression $(p_4 + p_3 + p_2 + p_1)\ (p_3 + p_2 + p_1)\ (p_2 + p_1)\ (p_1)$ is found in the same way and might be called an "algebraic factorial." It might be designated by

$$(p_4 + p_3 + p_2 + p_1)\text{¡}$$

It should be noted that the order of the terms in the algebraic factorial is significant. Thus $(p_2 + p_1)\text{¡} \neq (p_1 + p_2)\text{¡}$ unless $p_1 = p_2$.

The coefficient of $(p_\rho)(p_{\rho-1}) \cdots (p_2)(p_1)$ in the expansion of (1^r) may now be written

$$(-1)^{r-\rho} \frac{r!}{(p_\rho + \cdots + p_2 + p_1)\text{¡}} \qquad \{33\}$$

For example the coefficient

$$(2)(1)(2) \text{ is } (-1)^{5-3} \frac{5!}{5 \cdot 3 \cdot 2} = 4$$

$$(1)(2)(2) \text{ is } (-1)^{5-3} \frac{5!}{5 \cdot 4 \cdot 2} = 3$$

$$(2)(2)(1) \text{ is } (-1)^{5-3} \frac{5!}{5 \cdot 3 \cdot 1} = 8$$

and the total coefficient of all terms involving $(2)(2)(1)$ is 15.

With a less formal notation we might designate the sum of the $\rho!$ "algebraic factorials" which can be formed from $p_\rho, p_{\rho-1}, \cdots p_2, p_1$ by

$$\sum (p_\rho + p_{\rho-1} + \cdots + p_2 + p_1)\text{¡}$$

and the sum of their reciprocals by

$$\sum \frac{1}{(p_\rho + p_{\rho-1} + \cdots + p_2 + p_1)\text{¡}}$$

This notation calls for the inclusion of all the $\rho!$ algebraic factorials even though some of them may be alike. If

$$\pi_1, \pi_2, \cdots, \pi_s$$

indicate the numbers of repeated p's

$$\sum \frac{1}{(p_\rho + p_{\rho-1} + \cdots + p_2 + p_1)_1^!} = \pi_1!\pi_2! \cdots \pi_s! \sum{}' \frac{1}{(p_\rho + p_{\rho-1} + \cdots + p_2 + p_1)_1^!} \qquad \{34\}$$

where $\sum'$ holds for the $\frac{r!}{\pi_1!\pi_2!\cdots\pi_s!}$ non-repeated terms.

In general the total coefficient of $(p_1)(p_2) \cdots (p_\rho)$ in the expansion of (1^r) is obtained by adding all possible terms {33} in which the same p's occur in different positions in the product. Every possible different position grouping of the p's is present but once since it is dependent solely on the unique order in which prefixes and raises have been combined to produce that particular position grouping. The number of these position groupings varies with the number of repeated p's. The sum of the coefficients of these position groupings of the same p's, i.e. the total coefficient of $(p_1)(p_2) \cdots (p_\rho)$ is then given by

$$(-1)^{r-\rho} r! \sum{}' \frac{1}{(p_\rho + \cdots + p_1)_1^!}$$

which can be written by means of {34}

$$(-1)^{r-\rho} \frac{r!}{\pi_1!\pi_2! \cdots \pi_s!} \sum \frac{1}{(p_\rho + \cdots + p_1)_1^!}.$$

The formula for (1) may be written

$$(1^r) = \sum (-1)^{r-\rho} \frac{r!}{\pi_1!\pi_2! \cdots \pi_s!} \sum \frac{1}{(p_\rho + \cdots + p_1)_1^!} (p_1)(p_2) \cdots (p_\rho) \qquad \{35\}$$

27. **Theorem on Algebraic Factorials.** The result {35} can be further simplified by the theorem

$$\sum \frac{1}{(p_\rho + \cdots + p_2 + p_1)_1^!} = \frac{1}{p_\rho p_{\rho-1} \cdots p_2 p_1} \qquad \{36\}$$

which is proved by mathematical induction.

A. It is true when $\rho = 2$, since

$$\sum \frac{1}{(p_2 + p_1)_1^!} = \frac{1}{(p_2 + p_1)_1^!} + \frac{1}{(p_1 + p_2)_1^!} = \frac{1}{p_2 + p_1}\left[\frac{1}{p_1} + \frac{1}{p_2}\right] = \frac{1}{p_1 p_2}$$

B. If it is true for $\rho = k$, it is true for $\rho = k + 1$ since

$$\sum \frac{1}{(p_{k+1} + p_k + \cdots + p_2 + p_1)_1^!} = \frac{1}{p_{k+1} + \cdots + p_2 + p_1}\left[\sum \frac{1}{(p_k + \cdots + p_1)_1^!} + \sum_{p_{k+1}} \sum \frac{1}{(p_k + p_{k-1} + \cdots + p_1)_1^!}\right]$$

where $\sum_{p_{k+1}}$ gives the k terms in which p_{k+1} replaces $p_k, p_{k-1}, \cdots, p_2, p_1$ respectively. Now if {36} is true when $\rho = k$

$$\sum \frac{1}{(p_{k+1} + \cdots + p_1)!} = \frac{1}{p_{k+1} + \cdots + p_1}\left[\frac{p_{k+1} + p_k + p_{k-1} + \cdots + p_2 + p_1}{p_{k+1}p_k p_{k-1} \cdots p_2 p_1}\right]$$

$$= \frac{1}{p_{k+1}p_k \cdots p_2 p_1}.$$

C. Hence it is true when $k = 2, 3, 4 \cdots$.

28. **Formulas for (1^r).** Formula {35} may now be written

$$(1^r) = \sum (-1)^{r-\rho} \frac{r!}{\pi_1! \cdots \pi_s!} \frac{1}{p_1 p_2 \cdots p_\rho} (p_1)(p_2) \cdots (p_\rho) \qquad \{37\}$$

or if the p's are ordered it may be written as

$$(1^r) = \sum (-1)^{r-\rho} \frac{r!}{\pi_1! \cdots \pi_s!} \frac{1}{p_1^{\pi_1} \cdots p_s^{\pi_s}} (p_1)^{\pi_1} \cdots (p_s)^{\pi_s} \qquad \{38\}$$

which is the formula previously given as {23} and {26}. In addition the check formulas {30} and {31} become

$$\sum (-1)^{r-\rho} \frac{r!}{p_1^{\pi_1} \cdots p_s^{\pi_s}\pi_1! \cdots \pi_s!} = 0 \qquad \{39\}$$

$$\sum \frac{r!}{p_1^{\pi_1} \cdots p_s^{\pi_s}\pi_1! \cdots \pi_s!} = r! \qquad \{40\}$$

These relations {39} and {40} correspond to statements of Cauchy (2), (I; 1; 252–3) and to later remarks of Cayley (D; 577). By dividing by $r!$, they become

$$\sum \frac{(-1)^{r-\rho}}{p_1^{\pi_1} \cdots p_s^{\pi_s} \pi_1! \cdots \pi_s!} = 0 \qquad \{41\}$$

$$\sum \frac{1}{p_1^{\pi_1} \cdots p_s^{\pi_s} \pi_1! \cdots \pi_s!} = 1 \qquad \{42\}.$$

The formula {38} is easily applied. Thus

$$(1^5) = \frac{5!}{5}(5) - \frac{5!}{4}(4)(1) - \frac{5!}{32}(3)(2) + \frac{5!}{32!}(3)(1)^2 + \frac{5}{222!}(2)^2(1)$$

$$- \frac{5!}{23!}(2)(1)^3 + \frac{5!}{5!}(1)^5$$

$$= 24(5) - 30(4)(1) - 20(3)(2) + 20(3)(1)^2 + 15(2)^2(1)$$

$$- 10(2)(1)^3 + (1)^5.$$

with

$$24 - 30 - 20 + 20 + 15 - 10 + 1 = 0$$

$$24 + 30 + 20 + 20 + 15 + 10 + 1 = 5!$$

We next write the formula (1^r) in such form that we use the principle of generalization from symmetry. If we multiply numerator and denominator of {38} by $(p_1 - 1)!^{\pi_1}(p_2 - 1)!^{\pi_2} \cdots (p_s - 1)!^{\pi_s}$ we get

$$(1^r) = \sum (-1)^{r-\rho}(p_1 - 1)!^{\pi_1}(p_2 - 1)!^{\pi_2} \cdots (p_s - 1)!^{\pi_s} \frac{r!\,(p_1)^{\pi_1} \cdots (p_s)^{\pi_s}}{(p_1!)^{\pi_1}(p_2!)^{\pi_2} \cdots (p_s!)^{\pi_s}\,\pi_1! \cdots \pi_s!}$$

which immediately becomes

$$(1^r) = \sum (-1)^{r-\rho}(p_1 - 1)!^{\pi_1}(p_2 - 1)!^{\pi_2} \cdots (p_s - 1)!^{\pi_s} \binom{1^r}{p_1^{\pi_1} \cdots p_2^{\pi_2}} (p_1)^{\pi_1} \cdots (p_s)^{\pi_s} \quad \{43\}.$$

This somewhat formidable appearing formula is easy to apply. For example, in finding the value of (1^5) we write in one row all possible partitions of 5. In the next row we place the well known values of $\binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}}$. In the next row we place the indicated products with proper signs. Thus

1^5	21^3	2^21	31^2	32	41	5
1	10	15	10	10	5	1
1	-1	$+1$	$+2$	-2	-6	$+24$

results in

$$(1^5) = (1)^5 - 10(2)(1)^3 + 15(2)^2(1) + 20(3)(1)^2 - 20(3)(2) - 30(4)(1) + 24(5)$$

as indicated above.

It is immediately recognized that formula {43} can be obtained from formula {3} by placing $P(1^r) = (1^r)$; $p_1^{\pi_1} \cdots p_s^{\pi_s}$ by $(p_1)^{\pi_1} \cdots (p_s)^{\pi_s}$ and $P_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ by $(-1)^{r-\rho}(p_1 - 1)!^{\pi_1} \cdots (p_s - 1)!^{\pi_s}$ and hence that formulas of Table I may be used in obtaining the values of (1^r).

29. **Values of** $(a_1 \cdots a_r)$. The form of {43} also permits generalization from symmetry since the $\binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ equal values $(p_1)^{\pi_1} \cdots (p_s)^{\pi_s}$ are replaced by the $\binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ different values composing $T(p_1)^{\pi_1} \cdots (p_s)^{\pi_s}$ when the r units are replaced by the a's. It follows at once that

$$(a_1 a_2 \cdots a_r) = \sum(-1)^{r-\rho}(p_1 - 1)!^{\pi_1} \cdots (p_s - 1)!^{\pi_s} T(p_1)^{\pi_1} \cdots (p_s)^{\pi_s} \quad \{44\}$$

where $$r = p_1\pi_1 + p_2\pi_2 + \cdots + p_s\pi_s$$

and $$\rho = \pi_1 + \pi_2 + \cdots + \pi_s .$$

As an illustration we write

$$(abc) = 2(a + b + c) - (a + b)(c) - (a + c)(b) - (b + c)(a) + (a)(b)(c)$$

as indicated earlier by {18} and

$$(a_1a_2a_3a_4) = -6T(4) + 2T(3)(1) + T(2)(2) - T(2)(1)^2 + T(1)^4$$

$$(a_1a_2a_3a_4a_5) = 24T(5) - 6T(4)(1) - 2T(3)(2) + 2T(3)(1)^2 + T(2)^2(1) - T(2)(1)^3 + T(1)^5$$

etc.

30. **Table of Values of $(a_1 \cdots a_r)$.** The values of the power products with $w \leqq 6$ are given in Table II which follows the general form of Table I. In fact Table II may be derived from Table I by placing every

$$P_{p_1^{\pi_1}\cdots p_s^{\pi_s}} = (-1)^{r-\rho}(p_1 - 1)!^{\pi_1} \cdots (p_s - 1)!^{\pi_s}$$

as indicated in the next section.

31. **Use of Partition Formulas.** By comparing {44} with {4} we see that {44} can be obtained from {4} by placing

$$P(a_1a_2 \cdots a_3) = (a_1a_2 \cdots a_r)$$

$$P_{p_1^{\pi_1}\cdots p_s^{\pi_s}} = (-1)^{r-\rho}(p_1 - 1)!^{\pi_1} \cdots (p_s - 1)!^{\pi_s}$$

and $$T_{p_1^{\pi_1}\cdots p_s^{\pi_s}} = T(p_1)^{\pi_1} \cdots (p_s)^{\pi_s}$$

It appears then that the values of any power product sum $(a_1a_2 \cdots a_r)$ can be obtained by writing the expansion of $P(a_1 \cdots a_r)$ and substituting as indicated. Thus since

$$P(321) = P_3 6 + P_{21} 51 + P_{21} 42 + P_{21} 33 + P_{111} 111$$

$$(321) = 2(6) - (5)(1) - (4)(2) - (3)(3) + (1)(1)(1).$$

It is also immediately apparent that Table II can be obtained from Table I by placing $P_{p_1^{\pi_1}\cdots p_s^{\pi_s}}$ equal to $(-1)^{r-\rho}(p_1 - 1)!^{\pi_1} \cdots (p_s - 1)!^{\pi_s}$ and that the main results of Chapter I, including the recursion rule, are applicable to the present problem.

32. **Coefficients of Given Terms in the Expansion of Product Power Sums.** The methods of the last section are also useful in finding the coefficient of any term in the expansion. For example we wish to find the coefficient of (3)(2) in the expansion of (2111). We note that $P\binom{2111}{32} = P_{31} + 3P_{22}$ and that

TABLE II

Power product sums in terms of products of power sums when $W \leqq 6$

$W = 6$

	6	51	42	33	411	321	222	31^3	2^21^2	21^4	1^6
6	1										
51	−1	1									
42	−1		1								
33	−1			1							
411	2	−2	−1		1						
321	2	−1	−1	−1		1					
222	2		−3				1				
31^3	−6	6	3	2	−3	−3		1			
2^21^2	−6	4	5	2	−1	−4	−1		1		
21^4	24	−24	−18	−8	12	20	3	−4	−6	1	
1^6	−120	144	90	40	−90	−120	−15	40	45	−15	1

$W = 1$

	1
1	1

$W = 2$

	2	11
2	1	
11	−1	1

$W = 3$

	3	21	111
3	1		
21	−1	1	
111	2	−3	1

$W = 5$

	5	41	32	311	221	2111	11111
5	1						
41	−1	1					
32	−1		1				
311	2	−2	−1	1			
221	2	−1	−2		1		
2111	−6	6	5	−3	−3	1	
11111	24	−30	−20	20	+15	−10	1

$W = 4$

	4	31	22	211	1111
4	1				
31	−1	1			
22	−1		1		
211	2	−2	−1	1	
1111	−6	8	3	−6	1

the coefficient of (3)(2) is $P_{31} + 3P_{22}$ where $P_{31} = (-1)^{4-2}2! = 2$ and $P_{22} = (-1)^{4-2} = 1$. Hence the coefficient is $1 \cdot 2 + 3 \cdot 1 = 5$.

33. **The Expansion of the Monomial Symmetric Function.** If $a_1 \neq a_2 \neq a_3 \neq \cdots \neq a_r$ then $M(a_1 \cdots a_r) = (a_1 \cdots a_r)$ and previous results are applicable. If however the product power sum is of the form

$$(a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_h^{\alpha_h})$$

then

$$M(a_1^{\alpha_1}a_2^{\alpha_2}\cdots a_h^{\alpha_h}) = \frac{1}{\alpha_1!\alpha_2!\cdots\alpha_h!}(a_1^{\alpha_1}\cdots a_h^{\alpha_h})$$

and

$$M(a_1^{\alpha_1}\cdots a_h^{\alpha_h})$$

$$= \frac{1}{\alpha_1!\alpha_2!\cdots\alpha_h!}\sum(-1)^{r-\rho}(p_1-1)!^{\pi_1}\cdots(p_s-1)!^{\pi_s}\,T(p_1)^{\pi_1}\cdots(p_s)^{\pi_s} \quad \{45\}$$

For example

$$M(421) = 2(7) - (6)(1) - (5)(2) - (4)(3) + (4)(2)(1)$$
$$M(322) = (7) - (5)(2) - \tfrac{1}{2}(4)(3) + \tfrac{1}{2}(3)(2)(2).$$
$$M(2^21^2) = -\tfrac{3}{2}(6) + (5)(1) + \tfrac{5}{4}(4)(2) + \tfrac{1}{2}(3)(3) - \tfrac{1}{4}(4)(1)(1) - (3)(2)(1) - \tfrac{1}{4}(2)(2)(2) + \tfrac{1}{4}(2)(2)(1)(1).$$

Study will show that the formula {45} is equivalent to one given by Faà de Bruno (C; 9) and later by Roe (7).

It is possible to use Table II in finding the expansion of the monomial symmetric functions. It is only necessary to multiply each term in the expansion of $(a_1 \cdots a_r)$ by $\dfrac{1}{\alpha_1! \cdots \alpha_h!}$.

The check formulas give, in the case of the monomial symmetric function:

The sum of the coefficients in the expansion is 0.

The sum of the absolute values of the coefficients is $\dfrac{r!}{\alpha_1!\alpha_2!\cdots\alpha_h!}$.

The reader might compare the second of these checks with the results of Faà de Bruno (C; 14).

Tables giving the expansion of monomial symmetric function have been given. One by J. R. Roe (12; plate 18) includes all cases of weight $\leqq 10$.

34. **Previous Results.** Previous authors have studied the monomial symmetric function. Gordan has deduced a monomial symmetric function formula which is recommended by J. R. Roe (M; 24–33). MacMahon has given a general formula (K; II; 320) for expanding any monomial symmetric function in terms of power sums together with an operational method for its evaluation. O'Toole also has given a differential operator and showed how it could be applied in obtaining expansions (16; 115–130). O'Toole has also given a method of expanding symmetric functions in many variables by means of differential operators, (17).

Another method of attack was based upon the close relation existing between the elementary symmetric function and the determinant of the power sums. This has resulted in the expression of the monomial symmetric function in determinant form. Brioschi appears to have been the first (1854) to see how a symbolic determinant could be used (3; 427) although he gave no proof. Bellavites tried in 1857, but obtained incorrect results (4). In 1876 Faà de Bruno made an attempt, but he too was in error (C; 10). In 1898 E. D. Roe, Jr. proved that Brioschi was right (7). Muir also gave a proof in 1908 (11; 5–9). The summation of determinants, rather than the symbolic determinant, was used by Hankel (6; 90–94) (L; III, 220).

The determinant of the power sums has been generalized in another way. A group of writers has studied the "immanents" of its matrix. D. E. Littlewood and A. R. Richardson have recently written a series of papers on this topic. One of these papers (18; 99–141) defined the term "immanents" and gave references to previous investigations dealing with this matrix.

It has been the aim of this chapter to present an easy development of the subject of the expansion of product power sums and monomial symmetric functions. This development is characterized by

1. The use of the formulas and tables of Chapter I in writing expansions of product power sums.
2. The use of product power sums in place of monomial symmetric functions which makes feasible
3. Generalization from symmetry.
4. References to previous work.

Chapter IV. The Double Expansion Theorem

In the present chapter we combine the multiplication expansion of Chapter II and the power product sum expansion of Chapter III into a new result which is to be known as the double expansion theorem. We show that this result may also be expressed in terms of the partition notation of Chapter I.

35. The Value of $K(a_1)(a_2)$. We know

$$(a_1)(a_2) = (a_1 + a_2) + (a_1 \cdot a_2)$$

and if we multiply $(a_1 + a_2)$ by k_2 and $(a_1 \cdot a_2)$ by k_{11} we have a new expression which we designate by $K(a_1)(a_2)$.

$$K(a_1)(a_2) = k_2(a_1 + a_2) + k_{11}(a_1 \cdot a_2) \qquad \{46\}$$

Since $\quad (a_1a_2) = (a_1)(a_2) - (a_1 + a_2)$

$$K(a_1)(a_2) = k_2(a_1 + a_2) + k_{11}[(a_1)(a_2) - (a_1 + a_2)]$$

$$K(a_1)(a_2) = (k_2 - k_{11})(a_1 + a_2) + k_{11}(a_1)(a_2)$$

which can be written

$$K(a_1)(a_2) = K_2(a_1 + a_2) + K_{11}(a_1)(a_2) \qquad \{47\}$$

if $k_2 - k_{11} = K_2$ and $K_{11} = k_{11}$

36. **The Value of $K(a_1)(a_2)(a_3)$.** We know from {12} that

$$(a_1)(a_2)(a_3) = T(3) + T(21) + T(111)$$

and we define $K(a_1)(a_2)(a_3) = k_3T(3) + k_{21}T(21) + k_{111}T(111)$. Inserting the values $T_3 = (a_1 + a_2 + a_3)$, $T_{21} = (\overline{a_1 + a_2}\cdot a_3) + (\overline{a_1 + a_3}\cdot a_2) + (\overline{a_2 + a_3}\cdot a_1)$, $T_{111} = (a_1a_2a_3)$ and reducing to power sums by {44}, we get

$$K(a_1)(a_2)(a_3) = (k_3 - 3k_{21} + 2k_{111})(a_1 + a_2 + a_3) + (k_{21} - k_{111})$$
$$\{(a_1 + a_2)(a_3) + (a_1 + a_3)(a_2) + (a_2 + a_3)(a_1)\} + k_{111}(a_1)(a_2)(a_3)$$

which may be written

$$K(a_1)(a_2)(a_3) = K_3(a_1 + a_2 + a_3) + K_{21}\{(a_1 + a_2)(a_3)$$
$$+ (a_1 + a_3)(a_2) + (a_2 + a_3)(a_1)\} + K_{111}(a_1)(a_2)(a_3) \qquad \{48\}$$

where $K_3 = k_3 - 3k_{21} + 2k_{111}$, $K_{21} = k_{21} - k_{111}$, $K_{111} = k_{111}$.

37. **Definition of $K(a_1)(a_2)\cdots(a_r)$.** We define

$$K(a_1)(a_2)\cdots(a_r) = \sum k_{p_1^{\pi_1}\cdots p_s^{\pi_s}}\, T(p_1^{\pi_1}\cdots p_s^{\pi_s}) \qquad \{49\}$$

where $T(p_1^{\pi_1}\cdots p_s^{\pi_s})$ is composed of $\begin{pmatrix} 1^r \\ p_1^{\pi_1}\cdots p_s^{\pi_s}\end{pmatrix}$ power product sums. We wish to find the value $K(a_1)(a_2)\cdots(a_r)$ in terms of power sums. This involves the expansion of each power product sum in terms of power sums and then the collection of the results. This algebraic process is to be called the double expansion process and the theorem which results, the double expansion theorem.

38. **Special Cases of the Theorem.** The results {47} and {48} are special cases of the double expansion theorem when $r = 2, 3$. When $r = 1$ it is evident that $K(a_1) = K_1(a_1) = k_1(a_1)$. {50}
The results {50}, {48}, and {49} may be written symbolically by

$$K(a_1) = K_1T(1)$$
$$K(a_1)(a_2) = K_2T(2) + K_{11}T(1)^2 \qquad \{51\}$$
$$K(a_1)(a_2)(a_3) = K_3T(3) + K_{21}T(2)(1) + K_{111}T(1)^3$$

It can also be shown, with a much more extensive use of the results of Chapters II and III, that

$$K(a_1)(a_2)(a_3)(a_4) = K_4T(4) + K_{31}T(3)(1) + K_{22}T(2)^2$$
$$+ K_{211}T(2)(1)^2 + K_{1111}T(1)^4 \qquad \{52\}$$
$$K(a_1)\cdots(a_5) = K_5T(5) + K_{41}T(4)(1) + K_{32}T(3)(2)$$
$$+ K_{311}T(3)(1)^2 + K_{221}T(2)^2(1) + K_{2111}T(2)(1)^3$$
$$+ K_{11111}T(1)^5 \qquad \{53\}$$

where

$$\left.\begin{aligned}
K_4 &= k_4 - 4k_{31} - 3k_{22} + 12k_{211} - 6k_{1111} \\
K_{31} &= k_{31} - 3k_{211} + 2k_{1111} \\
K_{22} &= k_{22} - 2k_{211} + k_{1111} \\
K_{211} &= k_{211} - k_{1111} \\
K_{1111} &= k_{1111}
\end{aligned}\right\} \quad \{54\}$$

and

$$\left.\begin{aligned}
K_5 &= k_5 - 5k_{41} - 10k_{32} + 20k_{311} + 30k_{221} - 60k_{2111} + 24k_{11111} \\
K_{41} &= k_{41} - 4k_{311} - 3k_{221} + 12k_{2111} - 6k_{11111} \\
K_{32} &= k_{32} - k_{311} - 3k_{221} + 5k_{2111} - 2k_{11111} \\
K_{311} &= k_{311} - 3k_{2111} + 2k_{11111} \\
K_{221} &= k_{221} - 2k_{2111} + k_{11111} \\
K_{2111} &= k_{2111} - k_{11111} \\
K_{11111} &= k_{11111}
\end{aligned}\right\} \quad \{55\}$$

We may say then that, for $r < 6$

$$K(a_1) \cdots (a_r) = \sum k_{p_1^{\pi_1} \cdots p_s^{\pi_s}} T(p_1^{\pi_1} \cdots p_s^{\pi_s})$$

$$= \sum k_{p_1^{\pi_1} \cdots p_s^{\pi_s}} T(p_1)^{\pi_1} \cdots (p_s)^{\pi_s} \quad \{56\}$$

where $K_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ is defined by the relations {47}, {48}, {54}, and {55}. In examining the value of K_r we note

$$\begin{aligned}
K_1 &= k_1 \\
K_2 &= k_2 - k_{11} \\
K_3 &= k_3 - 3k_{21} + 2k_{111} \\
K_4 &= k_4 - 4k_{31} - 3k_{22} + 12k_{211} - 6k_{1111} \\
K_5 &= k_5 - 5k_{41} - 10k_{32} + 20k_{311} + 30k_{221} - 60k_{2111} + 24k_{11111}
\end{aligned}$$

and that these are given, for $r < 6$ by

$$K_r = \sum (-1)^\rho (\rho - 1)! \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} k_{p_1^{\pi_1} \cdots p_s^{\pi_s}} \quad \{57\}$$

It is further to be noted that {57} can be obtained from {3} by placing $P(1^r) = K_r$, $p_1^{\pi_1} \cdots p_s^{\pi_s}$ by $k_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$, and $P_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ by $(-1)^\rho(\rho - 1)!$ Hence the last rows of Table I may be used in writing the values of K_r. Thus from

$$P(1^3) = P_3(3) + 3P_{21}(21) + P_{111}(1)^3$$

we get

$$K_3 = k_3 - 3k_{21} + 2k_{111}\,.$$

It is further evident that if $\overline{K_3K_2} = \overline{(k_3 - 3k_{21} + 2k_{111})(k_2 - k_{11})}$ indicates multiplication by suffixing of subscripts that $\overline{K_3K_2} = k_{32} - k_{311} - 3k_{221} + 5k_{2111} - 2k_{11111} = K_{32}$
and in general it can be shown that for $r < 6$

$$K_{r_1r_2} = \overline{K_{r_1}K_{r_2}}$$

$$K_{r_1r_2r_3} = \overline{K_{r_1}K_{r_2}K_{r_3}} \qquad \{58\}$$

etc.

so that all values $K_{p_1^{\pi_1}\cdots p_s^{\pi_s}}$ may be obtained by symbolic multiplication of equations {57}.

The method of this section can be used in demonstrating that the results {56}, {57}, and {58} hold also when $r = 6, 7, 8 \cdots$, but the amount of algebraic manipulation increases enormously with each increase in r. We establish these results, for all integral values of r, by a more general approach.

39. **A More General Definition.** We provide a more general definition of $K(a_1) \cdots (a_r)$ by letting the subscripts of the k's agree with parts of the given partition rather than with its complete order. Thus

$$K'(a_1)(a_2) = k_{a_1+a_2}(a_1 + a_2) + k_{a_1a_2}(a_1a_2)$$

and in general, if $q_1^{x_1} \cdots q_t^{x_t}$ represents any ρ part partition having complete order $p_1^{\pi_1} \cdots p_s^{\pi_s}$, then we may define

$$K'(a_1)(a_2) \cdots (a_r) = \sum k_{q_1^{x_1}\cdots q_t^{x_t}}\,(q_1^{x_1} \cdots q_t^{x_t}) \qquad \{59\}$$

where the summation holds, not only for every different complete order as does {49}, but for every possible partition. By {44} $(q_1^{x_1} \cdots q_t^{x_t})$ may be written as

$$(q_1^{x_1} \cdots q_t^{x_t}) = \sum(-1)^{\rho-g}(d_1 - 1)!(d_2 - 1)! \cdots (d_g - 1)!T(d_1) \cdots (d_g)$$

where $d_1 + d_2 + \cdots + d_g = \rho$ and where groups of the d's may be alike. If $(w_1)(w_2) \cdots (w_g)$ is one of the products of power sums having the complete order $(d_1 \cdots d_g)$ we may write

$$(q_1^{x_1} \cdots q_t^{x_t}) = \sum(-1)^{\rho-g}(d_1 - 1)! \cdots (d_g - 1)!(w_1)(w_2) \cdots (w_g) \qquad \{60\}$$

where

$$q_1x_1 + \cdots + q_tx_t = w = w_1 + w_2 + \cdots + w_g$$

and

$$x_1 + \cdots + x_t = \rho$$

and where the summation sign holds not only for every complete order $d_1 \cdots d_g$, but for all power sum partition products $(w_1)(w_2) \cdots (w_g)$.

The insertion of {60} in {59} gives

$$K'(a_1)(a_2) \cdots (a_r)$$
$$= \sum k_{q_1^{z_1} \cdots q_t^{z_t}} \sum (-1)^{\rho-g}(d_1 - 1)! \cdots (d_g - 1)!(w_1) \cdots (w_g) \quad \{61\}$$

40. **Value of K'_w.** The notation of K'_w is used to indicate the coefficient of the power sum $(w) = (a_1 + a_2 + \cdots + a_r)$ in the expansion of {61}. In this case $d_1 = \rho$ and $g = 1$ so that

$$K'_w = \sum k_{q_1^{z_1} \cdots q_s^{z_s}} (-1)^{\rho-1}(\rho - 1)! \quad \{62\}$$

which may be written more symbolically as

$$K'_w = \sum (-1)^{\rho-1}(\rho - 1)!\, k_{\pi_w} \quad \{63\}$$

where π_w represents any algebraic partition of $a_1 + \cdots + a_r$ and ρ indicates the number of its parts.

41. **Products of K''s.** The notation $\overline{K'_{w_1} K'_{w_2}}$ is used to indicate the product of K'_{w_1} by K'_{w_2} if the rule of multiplication is the suffixing of the subscripts of the k's in the expansion of K'_{w_1} and K'_{w_2}. Thus

$$\overline{K'_{a_1+a_2} \cdot K'_{a_3}} = \overline{(k_{a_1+a_2} - k_{a_1a_2})(k_{a_3})}$$
$$= k_{a_1+a_2 \cdot a_3} - k_{a_1a_2a_3}$$

More generally, if we write, from {63}

$$K'_{w_1} = \sum (-1)^{d_1-1}(d_1 - 1)!\, k_{\pi_{w_1}}$$
$$K'_{w_2} = \sum (-1)^{d_2-g}(d_2 - 1)!\, k_{\pi_{w_2}}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$K'_{w_g} = \sum (-1)^{d_g-1}(d_g - 1)!\, k_{\pi_{w_g}}$$

and use multiplication by suffixing of subscripts we have

$$\overline{K'_{w_1} K'_{w_2} \cdots K'_{m_g}} = \sum (-1)^{\rho-g}(d_1 - 1)! \cdots (d_g - 1)\, k_{\pi_{w_1} \cdots \pi_{w_g}} \quad \{64\}$$

where $\rho = d_1 + d_2 + \cdots + d_g$ and the summation holds for every partition which can be formed by combining any algebraic partition of w_1, any partition of w_2, $\cdots$, any partition of w_g.

42. **The Coefficient of $(w_1)(w_2) \cdots (w_g)$.** The coefficients of any specific product of power sums $(w_1)(w_2) \cdots (w_g)$ is from {61}

$$K'_{w_1 w_2 \cdots w_g} = \sum k_{q_1^{z_1} \cdots q_s^{z_s}} (-1)^{\rho-g}(d_1 - 1)! \cdots (d_g - 1)! \quad \{65\}$$

where the summation holds, not only for the partitions of $a_1 + a_2 + \cdots + a_r$, but for the partitions $\pi_{w_1}, \pi_{w_2}, \cdots, \pi_{w_g}$ since these partitions can be combined to form $(w_1)(w_2) \cdots (w_g)$. Hence {65} becomes

$$K'_{w_1 w_2 \cdots w_g} = \sum (-1)^{\rho - g}(d_1 - 1)! \cdots (d_g - 1)!\, k_{\pi_{w_1} \cdots \pi_{w_g}} \qquad \{66\}$$

and it is immediately seen that the right hand expressions of {66} and {64} are the same and hence that

$$K'_{w_1 w_2 \cdots w_g} = \overline{K'_{w_1} K'_{w_2} \cdots K'_{w_g}}$$

as expected from (58).

We can now say that

$$\begin{aligned} K'(a_1)(a_2) \cdots (a_r) &= \sum k_{q_1^{\pi_1} \cdots q_t^{\pi_t}} (q_1^{\pi_1} \cdots q_t^{\pi_t}) \\ &\doteq \sum K'_{w_1 w_2 \cdots w_r} (w_1)(w_2) \cdots (w_g) \qquad \{67\} \end{aligned}$$

where

$$K'_w = \sum (-1)^{\rho-1}(\rho - 1)!\, k_{\pi_w} \qquad \{68\}$$

and

$$K'_{w_1 w_2 \cdots w_g} = \overline{K'_{w_1} K'_{w_2} \cdots K'_{w_g}} \qquad \{69\}$$

Relations {67}, {68} and {69} constitute the general double expansion theorem.

43. **The Double Expansion Theorem.** The case of the double expansion theorem in which we are especially interested is that in which the coefficients of all similar power sum products are the same, i.e., $k_{q_1^{\pi_1} \cdots q_t^{\pi_t}}$ is a function of the complete order indicated by $k_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$. In this case {68} becomes

$$K_w = \sum (-1)^{\rho}(\rho - 1)! \begin{pmatrix} 1^r \\ p_1^{\pi_1} \cdots p_s^{\pi_s} \end{pmatrix} k_{p_1^{\pi_1} \cdots p_s^{\pi_s}} \qquad \{70\}$$

where the summation holds for all possible complete orders. Suppose now that the r algebraic expressions, $a_1, a_2, \cdots, a_r$ are all unity then {69} becomes

$$K_r = \sum (-1)^{\rho-1}(\rho - 1)! \begin{pmatrix} 1^r \\ p_1^{\pi_1} \cdots p_s^{\pi_s} \end{pmatrix} k_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$$

and we find that $K_w = K_r$. We may then write {67}, {68} and {69} as

$$K(a_1) \cdots (a_r) = \sum k_{p_1^{\pi_1} \cdots p_s^{\pi_s}} (p_1^{\pi_1} \cdots p_s^{\pi_s}) = \sum K_{r_1 \cdots r_g} T(r_1) \cdots (r_g) \qquad \{71\}$$

where

$$K_r = \sum (-1)^{\rho-1}(\rho - 1)! \begin{pmatrix} 1^r \\ p_1^{\pi_1} \cdots p_s^{\pi_s} \end{pmatrix} k_{p_1^{\pi_1} \cdots p_s^{\pi_s}} \qquad \{72\}$$

and

$$K_{r_1 r_2 \cdots r_g} = \overline{K_{r_1} K_{r_2} \cdots K_{r_g}} \qquad \{73\}$$

Now $r_1 r_2 \cdots r_g$ indicates any grouping of the a's, and hence any complete order of $a_1 + a_2 + \cdots + a_r$. So {71} may be written, with a slight change of notation as

$$K(a_1) \cdots (a_r) = \sum k_{p_1^{\pi_1} \cdots p_s^{\pi_s}} T(p_1^{\pi_1} \cdots p_s^{\pi_s})$$
$$= \sum K_{p_1^{\pi_1} \cdots p_s^{\pi_s}} T(p_1)^{\pi_1} \cdots (p_s)^{\pi_s} \quad \{74\}$$

The relations {74}, {72} and {73} are the desired generalizations of {56}, {57} and {58} and hold for all positive integral values of r.

The double expansion theorem provides a method of writing out the result of the double expansion process without going through the work involved in the process. Thus

$$\begin{aligned} K(3)(2)(1) &= K_3(6) + K_{21}\{(5)(1) + (4)(2) + (3)(3)\} \\ &+ K_{111}(3)(2)(1) = (k_3 - 3k_{21} + 2k_{111})(6) \qquad \{75\} \\ &+ (k_{21} - k_{111})\{(5)(1) + (4)(2) + (3)(3)\} + k_{111}(3)(2)(1) \end{aligned}$$

44. The Double Expansion Theorem and Partition Notation. It is immediately evident that {74} can be obtained from {4} if $P(a_1 \cdots a_r)$ is replaced by $K(a_1) \cdots (a_r)$, if $P_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ is replaced by $K_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$, and if $p_1^{\pi_1} \cdots p_s^{\pi_s}$ is replaced by $T(p_1)^{\pi_1} \cdots (p_s)^{\pi_s}$. It follows at once that the entire theory of Chapter I,—table, recursion formula, etc.—is applicable to double expansion theory. For example {75} above is obtained from

$$P(321) = P_3 6 + P_{21}\{51 + 42 + 33\} + P_{111} 321$$

simply by replacing the K's by the P's and enclosing the parts in parentheses. We can as well use P's as K's to represent the double expansion theorem and hence have available a list of double expansion formulas when $w \leqq 6$. We also have available a recursion property for writing double expansions beyond the scope of the table. Thus for example, the illustration at the end of section 9 may be interpreted as a statement of the double expansion theorem when $a_1 = 3$, $a_2 = 2$, $a_3 = 2$, $a_4 = 1$.

45. The Case of Equal Powers. In case $a_1 = a_2 = a_3 = \cdots = a$, {74} reduces to {3′} of Chapter I with

$$P_r = \sum (-1)^{\rho-1}(\rho - 1)! \binom{1'}{p_1^{\pi_1} \cdots p_s^{\pi_s}} k_{p_1^{\pi_1} \cdots p_s^{\pi_s}} \quad \{76\}$$

and

$$P_{r_1 r_2 \cdots r_g} = \overline{P_{r_1} P_{r_2} \cdots P_{r_g}}.$$

Formula {74} also reduces to {3} when $a_1 = a_2 = \cdots = a_r = 1$.

46. Special Values of $K_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$.

A. $k_{p_1^{\pi_1} \cdots p_s^{\pi_s}} = 1$. In this case the coefficients are all unity and

$$P(a_1)(a_2) \cdots (a_r) = (a_1)(a_2) \cdots (a_r)$$

It follows that $P_r = 0$ and that $P_{p_1^{\pi_1} \cdots p_s^{\pi_s}} = 0$ except that $P_r = 1$. Placing $P_r = 0$ and $k_{p_1^{\pi_1} \cdots p_s^{\pi_s}} = 1$ in {72} or its equivalent {76} we have, when $r > 1$

$$0 = \sum (-1)^{\rho-1}(\rho - 1)! \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} \qquad \{77\}$$

where the summation holds for every partition of r. This formula should be compared with {39} and {40}. When $r = 4$ and the partitions are

	4,	31,	22,	211,	1^4		
{77} gives	1	−4	−3	+12	−6	=	0
{39} gives	−6	+8	+3	−6	+1	=	0
{40} gives	6	+8	+3	+6	+1	=	4!

The equivalent of {77} was first given by Cayley (D; 576) who at the same time noted the similarity to {39}.

It follows immediately that the sum of the coefficients in the expansion of $P_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$, except P_{1^r}, is 0, for the sum of the coefficients of $P_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ is the sum of the coefficients of $(P_{p_1})^{\pi_1} \cdots (P_s)^{\pi_s}$ and is 0. For example the sum of the coefficient of $P_{32} = k_{32} - k_{311} - 3k_{221} + 5k_{2111} - 2k_{11111}$ is 0.

Since the coefficients of $(\mu_1')^{\pi_1} \cdots (\mu_s')^{\pi_s}$ (19; 25) in the expansion of Thiele half invariants are $(-1)^{\rho-1}(\rho - 1)! \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ it follows from {77} that the sum of these coefficients is 0.

B. $k_{p_1^{\pi_1} \cdots p_s^{\pi_s}} = \dfrac{n^{(\rho)}}{N^{(\rho)}}$. In this case all terms having the same number of parts, ρ, have the same coefficients. If we indicate $\dfrac{n^{(\rho)}}{N^{(\rho)}}$ by $\rho_1, \rho_2, \cdots$, when $\rho = 1, 2, \cdots$, {57}, {76} become

$$P_1 = \rho_1$$
$$P_2 = \rho_1 - \rho_2$$
$$P_3 = \rho_1 - 3\rho_2 + 2\rho_3$$
$$P_4 = \rho_1 - 7\rho_2 + 12\rho_3 - 6\rho_4$$
$$P_5 = \rho_1 - 15\rho_2 + 50\rho_3 - 60\rho_4 + 24\rho_5 .$$

etc.

which are the formulas which have been used by Carver (15) and O'Toole (16).

Many other additional cases can be obtained by giving different values to $k_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$, but a discussion of these is hardly justified here as the case in which $k_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ is a function of the number of parts, ρ, is to be used in Part II.

47. **Relation to Previous Results.** No general statement of the double expansion theorem has previously been given although the special case $K(a^r)$

has been developed by Carver (15) and O'Toole (16). Their results are further restricted to the special case (B) of section 46. The application of the double expansion theorem in this case is very useful in studying sampling from a finite universe as Carver has shown and as is demonstrated in Part II.

Most writers who have worked on the problem of moments of moments have gone through the double expansion process, but Carver was the first to note that the result of the process can be written in terms of the P polynomials above. It seems appropiate therefore to refer to these P polynomials of the coefficients as Carver polynomials.

Chapter V. The Multipartition and Multivariate Formulas

It is the purpose of this chapter to show how the results of Chapters I, II' III, and IV may be extended to the case of different variables.

48. **Multipartitions. Tables.** Formula $\{4\}$ is still applicable if we let the a_1 units be the units of one quantity, the a_2 units to be the units of a second quantity, etc. Thus for example the formula $P(a_1a_2a_3)$ may be used to represent the precise number of ways in which a_1 apples, a_2 pears and a_3 peaches can be formed into groups without breaking up the groups of apples, pears, and peaches.

Various conventions for representing multipartitions of this type have been used. We adopt the one in which the individual partitions are written in successive columns. The partitions of the first number are combined with the partitions of the second number to form all possible multipartitions. Thus the multipartite number 111 has the partitions

111	110	101	011	100
	001	010	100	010
				001

where the parts are given in the rows. It is desired to show the number of ways in which any one of those partitions may be combined to form partitions of fewer parts. Thus

$$P\begin{pmatrix}100\\010\\001\end{pmatrix} = P_3\,111 + P_{21}\begin{matrix}110\\001\end{matrix} + P_{21}\begin{matrix}101\\010\end{matrix} + P_{21}\begin{matrix}011\\100\end{matrix} + P_{111}\begin{matrix}100\\010\\001\end{matrix}$$

This is obtained from $P(a_1a_2a_3)$ by placing $a_1 = 1_1$, $a_2 = 1_2$, $a_3 = 1_3$, and could be written from $\{4\}$ as

$$P(1_1 1_2 1_3) = P_3(1_1 + 1_2 + 1_3) + P_{21}\{\overline{1_1 + 1_2}\cdot 1_3 + \overline{1_1 + 1_3}\cdot 1_2 + \overline{1_2 + 1_3}\cdot 1_1\} + P_{111}\, 1_1\cdot 1_2\cdot 1_3.$$

Similarly

$$P\begin{pmatrix}10\\10\\01\\01\end{pmatrix} = P_4\begin{matrix}2\\2\end{matrix} + 2P_{31}\begin{matrix}21\\01\end{matrix} + 2P_{31}\begin{matrix}12\\10\end{matrix} + P_{22}\begin{matrix}20\\02\end{matrix} + 2P_{11}\begin{matrix}11\\11\end{matrix} + P_{211}\begin{matrix}20\\01\\01\end{matrix} + P_{211}\begin{matrix}02\\10\\10\end{matrix}$$

$$+ 4P_{211}\begin{matrix}11\\10\\01\end{matrix} + P_{1111}\begin{matrix}10\\10\\01\\01\end{matrix}$$

is a special case of $P(a_1a_2a_3a_4)$ where $a_1 = 1_1$, $a_2 = 1_1$, $a_3 = 1_2$, $a_4 = 1_2$. Formula {4} is also true where the a_1 units are not of the same kind. Thus

$$P(a_1a_2) = P_2(a_1 + a_2) + P_{11}(a_1a_2)$$

gives

$$P\begin{pmatrix}11\\10\end{pmatrix} = P_2 21 + P_{11}\begin{matrix}11\\10\end{matrix} \qquad \text{when} \quad a_1 = 1_1 + 1_2 \quad \text{and} \quad a_2 = 1_1.$$

TABLE III

The Multipartite Number 11

	11	10 01
11	P_1	
10 01	P_2	P_{11}

The Multipartite Number 111

	111	110 001	101 010	011 100	100 010 001
111	P_1				
110 001	P_2	P_{11}			
101 010	P_2		P_{11}		
011 100	P_2			P_{11}	
100 010 001	P_3	P_{21}	P_{21}	P_{21}	P_{111}

TABLE III—*Continued*

The Multipartite Number 22

	22	21 01	12 10	20 02	11 11	20 01 01	02 10 10	11 10 01	10 10 01 01
22	P_1								
21 01	P_2	P_{11}							
12 10	P_2		P_{11}						
20 02	P_2			P_{11}					
11 11	P_2				P_{11}				
20 01 01	P_3	$2P_{21}$		P_{21}		P_{111}			
02 10 10	P_3		$2P_{21}$	P_{21}			P_{111}		
11 10 01	P_3	P_{21}	P_{21}		P_{21}			P_{111}	
10 10 01 01	P_4	$2P_{31}$	$2P_{31}$	P_{22}	$2P_{22}$	P_{211}	P_{211}	$4P_{211}$	P_{1111}

Tables can be made for the partitions of the various multipartite numbers. In Table III are presented values for the numbers 11, 111, 22.

When the units are indistinguishable 11 condenses to the $w = 2$ part of Table I.

When the units are indistinguishable 111 condenses to the $w = 3$ part of Table I.

When the units are alike 22 condenses to the $w = 4$ part of Table I.

49. **Multivariate Distributions.** The chief results of Chapters II, III, IV also hold for multivariate distributions. Some additional definitions are neces-

sary. We suppose that the N variates $x_1, x_2, \cdots, x_N$ are replaced by the Nr variates of the array

$$\begin{matrix} {}_1x_1, {}_1x_2, \cdots, {}_1x_N \\ {}_2x_1, {}_2x_2, \cdots, {}_2x_N \\ \cdots\cdots\cdots \\ {}_rx_1, {}_rx_2, \cdots, {}_rx_N \end{matrix} \qquad \{78\}$$

where the presubscript represents the variable. The power sums become

$$(a_1) = {}_1x_1^{a_1} + {}_1x_2^{a_1} + \cdots + x_N^{a_1} = \sum {}_1x_i^{a_1}$$

$$(a_2) = {}_2x_1^{a_2} + {}_2x_2^{a_2} + \cdots + {}_2x_N^{a_2} = \sum {}_2x_i^{a_2}$$

It is not necessary to utilize the presubscript since it is precisely the subscript of the a. That is the power sum (a_k) is defined by $\sum x_i^{a_k}$. Similarly $(a_1 a_2) = \sum_{i \neq j} {}_1x_i^{a_1}\, {}_2x_i^{a_2}$ can be written as $(a_1a_2) = \sum_{i \neq j} x_i^{a_1}\, x_j^{a_2}$ without introducing ambiguity. In general {6} as well as {4}, now holds for the multivariate case. It follows at once that the results of Chapters II, III, IV can be written for the multivariate case by means of the formulas of Chapter I as indicated by the previous section. Thus the formula for $P(1_11_11_21_2)$ may be written as [Table III]

$$P[\overline{10}\cdot\overline{10}\cdot\overline{01}\cdot\overline{01}] = P_4\overline{22} + 2P_{31}\overline{21}\cdot\overline{01} + 2P_{21}\overline{12}\cdot\overline{10} + P_{22}\overline{20}\,\overline{02} + 2P\,\overline{11}\,\overline{11}$$
$$+ P_{211}\overline{20}\,\overline{01}\,\overline{01} + P_{211}\overline{02}\,\overline{10}\,\overline{10} + 4P_{211}\overline{11}\,\overline{10}\,\overline{01} + P_{1111}\overline{10}\,\overline{10}\,\overline{01}\,\overline{01}$$

and can be interpreted as:

$$(10)^2(01)^2 = (\overline{22}) + 2(\overline{21}\cdot\overline{01}) + 2(\overline{12}\cdot\overline{10}) + (\overline{20}\cdot\overline{02}) + 2(\overline{11}\cdot\overline{11}) + (\overline{20}\cdot\overline{01}\cdot\overline{01})$$
$$+ (\overline{02}\cdot\overline{10}\cdot\overline{10}) + (\overline{11}\cdot\overline{10}\cdot\overline{01}) + (\overline{10}\cdot\overline{10}\cdot\overline{01}\cdot\overline{01})$$

by {12} of Chapter II. It may also be interpreted as

$$(\overline{10}\cdot\overline{10}\cdot\overline{01}\cdot\overline{01}) = -6(22) + 4(21)(01) + 4(12)(10) + (20)(02)$$
$$+ 2(11)(11) - (20)(01)(01) - (02)(10)(10)$$
$$- 4(11)(10)(01) + (10)(10)(01)(01)$$

by {44} of Chapter II. It can also be interpreted as a double expansion by means of section 44 where the values of the P's are given by the usual

$$P_r = \sum (-1)^{\rho-1}(\rho - 1)! \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} k_{p_1^{\pi_1}\cdots p_s^{\pi_s}}$$

$$P_{r_1\cdots r_q} = \overline{P_{r_1}P_{r_2}\cdots P_{r_q}}$$

50. **Summary.** It is apparent that {4} not only expresses (a) the number of ways in which the parts of one partition may be collected to form the parts

of another partition, (b) the formula for expanding products of power sums in terms of power product sums, (c) the formula expanding power product sums in terms of power sums, and (d) the formula for double expansions, but also that it can be used to make similar expansions in the case of multivariate distributions.

BIBLIOGRAPHY

(1) BINET: "Mémoire sur un système de formules analytiques etc." Journ. de l'Ec. Polyt., IX (1812), cah 16, pp. 280–302.

(2) CAUCHY: "Note sur la formation des fonctions alternées qui servent à résoudre le problème de l'élimination." Comptes Rendus, 12 (1841), pp. 414–426.

(3) BRIOSCHI: "Sulle funzioni simmetriche delle radici di una equazione." Annali di Tortolini, 5 (1854), pp. 422–428.

(4) BELLAVITIS: "Sposizione elementare della teorica dei determinante." Memorie .... Istituto Veneto, 7 (1857), pp. 67–144.

(5) MOLA: "Soluzione della quistione 5, 6, 7." Giornale di Matematiche, 3 (1865), pp. 190–201.

(6) HANKEL: "Darstellung symmetrischer Functionen durch die Potenzsummen." Crelle's Journ., 67 (1865), pp. 90–94.

(7) E. D. ROE, JR.: "Note on a formula of symmetric functions." American Mathematical Monthly, 5 (1898), pp. 161–164.

(8) E. D. ROE, JR.: "On the transcendental form of the resultant." American Mathematical Monthly, 7 (1900), pp. 59–66.

(9) MUIRHEAD: "Some proofs of Newton's theorems on the sums of powers of roots." Edinburgh Mathematical Society Proceedings, 23 (1905), pp. 66–70.

(10) MUIRHEAD: "A proof of Waring's expression for Σa^r in terms of the coefficients of the equation." Edinburgh Mathematical Society Proceedings, 23 (1905), pp. 71–74.

(11) MUIR: "Waring's expression for symmetric functions in terms of sums of like powers." Proceedings Edinburgh Math. Society, 27 (1909), pp. 5–9.

(12) A. A. TCHOUPROFF: "On the mathematical expectation of moments of frequency distributions." Biom., 12 (1918), pp. 140–169.

(13) A. E. R. CHURCH: "On the moments of the distribution of the squared standard deations, etc." Biom., 17 (1925), pp. 79–83.

(14) A. E. R. CHURCH: "On the means and squared standard deviations of small samples." Biom., 18 (1926), pp. 321–394.

(15) H. C. CARVER: "The fundamentals of sampling." Annals of Mathematical Statistics, 1 (1930), pp. 101–121.

(16) A. L. O'TOOLE: "On symmetric functions and symmetric functions of symmetric functions." Annals of Mathematical Statistics, 2 (1931), 101–149.

(17) A. L. O'TOOLE: "On symmetric functions of more than one variable and of frequency functions." Annals of Mathematical Statistics, 3 (1932), pp. 56–63.

(18) D. E. LITTLEWOOD AND A. R. RICHARDSON: "Group characters and algebra." Phil. Trans. Roy. Soc., A 233 (1934), pp. 99–141.

(19) P. S. DWYER: "Moments of any rational integral isobaric sample moment function." Annals of Mathematical Statistics, vol. viii, no. 1, Mar. 1937, pp. 21–65.

A. PAOLI: "Elementi di algebra." Supplemento Opuscolo II, 1804.

B. HIRSCH: "Examples, formulae, and calculations on the literal calculus and algebra." Translated from the German by J. A. Ross (1827).

C. FAÀ DE BRUNO: "Théorie des formes binaires" (1876).

D. CAYLEY: "Collected Mathematical Papers," VII (1894).

E. Burnside and Panton: "Theory of equations" (1881). Reference is to the seventh edition (1912).
F. Chrystal: "Algebra" (1886). Reference is to the fifth edition.
G. Whitworth: "Choice and chance," 4th edition (1886).
H. T. N. Thiele: "Almindelig Iagttagelseslaere" (1889).
I. Thomas Muir: "Theory of determinants."
J. M. Bocher: "Introduction to higher algebra" (1907).
K. MacMahon: "Combinatory analysis" (1915–16).
L. Thomas Muir: "Contributions to the history of determinants" (1900–1920).
M. Josephine Roe: "Interfunctional expressibility problems of symmetric functions" (1931).
N. Josephine Roe: "Interfunctional expressibility tables of symmetric functions." Distributed by Syracuse University (1931).