ON THE PROBABILITY OF ATTAINING A GIVEN STANDARD DEVIATION RATIO IN AN INFINITE SERIES OF TRIALS

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Suppose an event with constant probability p of occurrence to be repeated an infinite number of times, and suppose the ratio of the deviation from the expected number of successes to the standard deviation $\sqrt{npq}$ to be recomputed after each trial. We are interested in the probability that this ratio will at some time equal or exceed some positive number k. It is not difficult to show that the value of this probability is unity, but as the fact has not, to our knowledge, been previously pointed out in the literature, we give the following proof.

Let x_n denote the number of successes obtained in the first n trials, let

$$t_n = \frac{x_n - np}{\sqrt{npq}},$$

and let P denote the probability that, for some n, $t_n \geqq k$. We shall prove that $P = 1$. To do this, let the infinite series of trials be subdivided into consecutive, mutually exclusive subseries of finite length, and let m_i denote the number of trials in the i-th subseries. Let $N_i = \sum_{j=1}^{i-1} m_j$ for $i \geqq 2$, while $N_1 = 0$. Let k' be any number greater than k, and let m_i be so chosen that

$$m_i \geqq \frac{k'^2 p}{q} \qquad \text{for every } i, \tag{1}$$

and

$$\sqrt{m_i}\left(k' - k\sqrt{\frac{N_i}{m_i} + 1}\right) \geqq N_i \sqrt{\frac{p}{q}} \qquad \text{for } i \geqq 2. \tag{2}$$

It follows from (1) that

$$m_i \geqq m_i p + k'\sqrt{m_i pq} \qquad \text{for every } i. \tag{3}$$

It follows from (2) that

$$m_i p + k'\sqrt{m_i pq} \geqq (N_i + m_i)p + k\sqrt{(N_i + m_i)pq} \qquad \text{for every } i. \tag{4}$$

Let y_i denote the number of successes in the i-th subseries. It is evident from (4) that if

$$y_i \geqq m_i p + k'\sqrt{m_i pq} \tag{5}$$

for any i, then

$$t_{N_i + m_i} \geqq k.$$

Hence P is at least equal to the probability that (5) holds for some i.

Let p_i denote the probability that (5) holds for a particular i. It follows from (3) that, for every i, $p_i > 0$. Moreover, there exists a positive integer M

and a number $h > 0$, such that if $m_i \geqq M$, $p_i \geqq h$.[1] Since there is but a finite number of possible values of m_i less than M, there is a number $p_0 > 0$ such that $p_i \geqq p_0$ for every i. Hence the probability that (5) holds for no value of i is at most

$$\lim_{s \to \infty} (1 - p_0)^s = 0 .$$

Therefore, the probability that (5) holds for some i is unity.

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[1] Uspensky, J. V., *Introduction to Mathematical Probability*, p. 129.