ON THE APPLICATION OF THE Z-TEST TO RANDOMIZED BLOCKS

By M. D. McCarthy

1. Introduction. When a series of experiments is performed with the object of measuring some quantity, it is implicitly postulated that the quantity in question has a "true value," which is theoretically obtainable as the result of an infinite repetition of the experiment under the standard conditions. In certain experiments, especially those of physical and chemical science, the materials and the methods employed are subject to such accurate control by the experimenter that he can repeat his experiment again and again with the "essential" factors kept constant, and with biassed errors eliminated. This repetition gives a series of observations of the "true value" in question subject only to random errors. All that is needed, usually, to increase the accuracy of the estimate of the "true value" is to continue the repetition of the experiment. Not only does such a repetition make the estimate more exact but it also provides an estimate of the degree of accuracy present, permits a comparison between different quantities and makes it possible to test various hypotheses as to their relative values.

In many cases which arise, notably in biological and social science and in dealing with data provided by modern mass-production methods, it is a practical impossibility to repeat an experiment under the same essential conditions. The material available is definitely non-homogeneous with regard to at least some of the qualities influencing the results. In testing, for instance, a number of varieties of some plant, to find which gives the best yield, it is possible to guarantee, that to a high degree of accuracy all the varieties are cultivated alike. If a relatively small area is covered by the experimental plots, it can be said that all the varieties experience the same climatic conditions and it is not difficult to ensure that they are all treated alike as to measurement of produce and so on. It is, however, practically impossible to make the plots, on which the varieties are grown, homogeneous as regards fertility of the soil and, even if this were possible, it would partially defeat the purpose of the experiment which is to test the varieties over a certain limited range of soil types. In a similar way in many other fields of biological or social experiments a similar non-homogeneity of the experimental material exists.

In experimenting with homogeneous materials, where the conditions of the whole series of experiments are the same, the differences which occur between the theoretical "true value" and the observations are explained as being due to a multiplicity of causes outside the control of the experimenter and of such a nature that their incidence varies "randomly" from experiment to experiment. It is a fact that certain fundamental factors influencing the results are definitely non-random in their incidence which differentiates the experiments with non-homogeneous material from the others and it is by *artifically* introducing randomization, as suggested by Fisher [1, 2, 3] that such experiments are made amenable to the usual error laws.

For convenience, in what follows, the word "variety" will be used when speaking of a single object of those under test, whether it actually be a variety of some plant, a manurial treatment, a method of feeding or anything else of the sort. For instance, if five varieties and three manurial treatments are being tested in the same experiment, a "variety" would be any one of the fifteen combinations of an actual variety under test with a manure. The word "plot" will be used for that portion of the non-homogeneous material which is required for the performance of an experiment on a single "variety," and the term "yield" will be applied to the value of the observed quantity obtained as the result of testing a "variety" on a single "plot." The plots are, of course, equalized with respect to "size," or whatever similar property would influence the test.

2. **Randomized Blocks.** Suppose that there are s varieties to be tested and that the necessary replication is attained by testing each variety on n separate plots. That the plots on which each variety is tested form a random sample of the material available is guaranteed by assigning each of the s varieties to n of the available ns plots *at random*, that is, as the result of a physical random experiment with cards, dice, or the like. This method of randomization may be so employed that no restrictions are put on the plots to which the varieties are assigned, or it may be further refined in different ways so that, while preserving the random nature of the assignment, certain restrictions may be placed on it. Such a method of randomization with restrictions is the method known as "randomized blocks."

The basic idea is that compact "blocks" of the non-homogeneous material are, probably, much more uniform than the material taken as a whole. Consequently, the material is first divided into n such "blocks," as compact and uniform as possible, each block containing s equal plots. Each of the s varieties under test is assigned to a single plot in every block and randomness is attained by making the assignment of the varieties to the plots in each block as the result of a separate random experiment. Thus the n plots to which each variety is assigned do actually form a random sample of the non-homogeneous material with the restriction that to each plot of any variety corresponds a plot of any other variety from the same block.

3. **Mathematical Formulation.** $X_{jl(k)}$ denotes the "true yield" of the kth variety which would be obtained by testing it on the lth plot in the jth block. $k = 1, 2, \cdots, s$ denotes the number by which the variety is known, $l = 1, 2, \cdots, s$ the order-number of the plot in the block and $j = 1, 2, \cdots, n$ the number of the block. Following Neyman [4, p. 110] we define the "true yield," again with particular reference to agricultural experiments, as:

"Suppose that the experiment is repeated indefinitely without any change of vegetative conditions or of arrangement so that the kth variety is always tested on the plot (j, l). The yields from this plot will form a population, say $\Pi_{jl(k)}$, and $X_{jl(k)}$ is defined as the mean of this population."

Thus, in any block, there are s^2 different possible populations with corresponding "true values," but in any single experiment on that block observations will be obtained from only s of the s^2 possible populations. To distinguish those populations for which an observation is available from those which are entirely hypothetical $X_{j(k)}$ will denote the "true yield," as already defined, of the kth variety *on the plot to which it has been assigned* in the jth block. Since this assignment has been carried out as the result of a random experiment the "true yield" is itself a *random variable*; $X_{j(k)}$ is randomly selected from the set of s possible values $X_{j1(k)}, X_{j2(k)}, \cdots X_{js(k)}$.

Using the dot notation to denote the mean of a quantity taken over all values of the letter replaced by the dot, it is clear that

$$\begin{aligned} X_{jl(k)} &= X_{\cdot\cdot(k)} + [X_{j\cdot(k)} - X_{\cdot\cdot(k)}] + [X_{jl(k)} - X_{j\cdot(k)}] \\ &= X_{\cdot\cdot(k)} + B_{jk} + u_{jl(k)}, \end{aligned}$$

and

$$\begin{aligned} X_{j(k)} &= X_{\cdot\cdot(k)} + [X_{j\cdot(k)} - X_{\cdot\cdot(k)}] + [X_{j(k)} - X_{j\cdot(k)}] \\ &= X_{\cdot\cdot(k)} + B_{jk} + \eta_{jk}, \end{aligned}$$

where

$$B_{jk} = X_{j\cdot(k)} - X_{\cdot\cdot(k)}, \qquad u_{jl(k)} = X_{jl(k)} - X_{j\cdot(k)}$$

and

$$\eta_{jk} = X_{j(k)} - X_{j\cdot(k)}.$$

Obviously

$$\sum_{j=1}^{n} B_{jk} = 0 \quad \text{and} \quad \sum_{l=1}^{s} u_{jl(k)} = 0$$

from their definitions, while η_{jk} is a random variable, with zero expectation, selected from the sequence $u_{j1(k)}, u_{j2(k)} \cdots u_{js(k)}$. Neyman (loc. cit.) calls η_{jk}, thus defined, the "soil error" of the kth variety when tested on its assigned plot in the jth block. The actual yield observed when the kth variety is tested on its assigned plot in the jth block is x_{jk} and the difference $x_{jk} - X_{j(k)} = \epsilon_{jk}$ is termed the "technical error." Clearly

$$x_{jk} = X_{\cdot\cdot(k)} + B_{jk} + \eta_{jk} + \epsilon_{jk}. \tag{1}$$

Both "soil error" and "technical error" enter into any comparisons which may be made and it is well known that the major source of error in, for instance, agricultural experiments is that due to the heterogeneity of the soil. As regards the relative magnitudes of the two errors, that of course depends on the experiment in question, but Fisher [5] has stated that in an agricultural uniformity trial (i.e. when the same variety is tested on all the plots) yields from plots of 1/40th of an acre frequently vary sufficiently among themselves, owing to soil heterogeneity, so as to give a standard deviation of ten per cent of the mean yield, while the inevitable random errors in treating the plots can be kept down to a much lower figure. By confining the randomization to a "block" of the material, which comprises only a relatively small compact portion of the whole material under test, the effects of soil heterogeneity may be much decreased. It appears, however, that it may very often be an unwarranted simplification to consider that the "true yield" of a variety is the same for all plots of a given block.

The two types of "error" are random variables of altogether different properties. Both have zero expectation and may be considered as independent of one another in the probability sense. It, therefore, appears reasonable to assume that ϵ_{jk} is independent both of the "technical error" in any other observation and of the η's. On the other hand η_{jk} is a random variable selected from the sequence

$$u_{j1(k)}\,,\ u_{j2(k)}\,,\ \cdots\,,\ u_{js(k)} \tag{2}$$

and since, if η_{jk} has the value $u_{jl(k)}$ and η_{jm} is free to assume any one of the values $u_{j1(m)}\,,\ u_{j2(m)}\,,\ \cdots\,,\ u_{js(m)}$ except $u_{jl(m)}$, it is clear that η_{jk} and η_{jm} are not independent. In the case of $\eta_{j'k}$ and $\eta_{j''m}$ where $j' \neq j''$, the random variables are drawn as the result of two separate random experiments from different sequences of the type (2). Obviously this means that the "soil errors" for different blocks are independent for either the same or different varieties. Writing E for the expected value, or the mean value in repeated experiments, since

$$\sum_{l=1}^{s} u_{jl(k)} = \sum_{l=1}^{s} u_{jl(m)} = 0,$$

the variance of η_{jk} is $\sigma^2_{\eta_{jk}}$ with

$$\sigma^2_{\eta_{jk}} = s^{-1}\cdot\sum_{l=1}^{s} u^2_{jl(k)} \tag{3}$$

and also

$$E\left[\eta_{jk}\eta_{jm}\right] = -\ \{s(s-1)\}^{-1}\sum_{l=1}^{s} u_{jl(k)}u_{jl(m)}\,. \tag{4}$$

Using (1), (3) and (4) it follows that

$$E[x_{jk}] = X_{..(k)} + B_{jk} = a_{jk}\,, \tag{5}$$

say,

$$E[(x_{jk} - a_{jk})^2] = E[(\eta_{jk} + \epsilon_{jk})^2] = \sigma^2_{\eta_{jk}} + \sigma^2_{\epsilon_j} \tag{6}$$

$$\begin{aligned} E[(x_{j'k} - a_{j'k})(x_{j''m} - a_{j''m})] &= E[(\eta_{j'k} + \epsilon_{j'k})(\eta_{j''m} + \epsilon_{j''m})] \\ &= E[\eta_{j'k}\eta_{j''m}]. \end{aligned}$$

The expectations of the various product terms on the right-hand sides of these equations vanish except in the case of the last one. If $j' \neq j''$ it too vanishes, whatever values of k and m, and it follows that the correlation of the observed yields of any two varieties, or of the same variety obtained from different blocks is zero. It is clear, however, that such is not the case when the yields are obtained from plots on the same block. Denoting by $\rho_{j(km)}$ the coefficient of correlation between x_{jk} and x_{jm} and using (4)

$$(7) \qquad \rho_{j(km)} = \rho_{j(mk)} = -\frac{\sum_{l=1}^{s} u_{jl(k)} u_{jl(m)}}{s(s-1)\{\sigma^2_{\eta_{jk}} + \sigma^2_{\epsilon_j}\}^{\frac{1}{2}}\{\sigma^2_{\eta_{jm}} + \sigma^2_{\epsilon_j}\}^{\frac{1}{2}}},$$

when $k \neq m$, while, of course, $\rho_{j(kk)} = 1$.

It may be noted that even when two sequences such as in (2) are identical the correlation $\rho_{j(km)}$ is not zero. In this case, when the varieties react in exactly the same way to variations in fertility within a block, $\sigma^2_{\eta_{jk}} = \sigma^2_{\eta_{jm}} = \sigma^2_{\eta_j}$, say, and

$$(7a) \qquad \rho_{j(km)} = -(s-1)^{-1}\{1 - \sigma^2_{\epsilon_j}/\sigma^2_{\eta_j}\}^{-1}.$$

Then the coefficient of correlation is negative and depends only on the relative magnitude of the technical and soil errors for the block in question, and on the number of plots in the block. In a given block it is greatest in absolute magnitude when the technical error is zero, or at any rate negligible with respect to the soil error which, of course, is usually uncontrollable. In order to have zero correlation between the yields of every pair of varieties it must be assumed either that (a) there is such a complete lack of relationship between the ways in which the various varieties react to the differences of fertility within a block that for each pair of varieties k and m all the products such as $\sum_{l=1}^{s} u_{jl(k)} u_{jl(m)}$ vanish identically even though the u's themselves are not zero, an assumption that lacks plausibility, or else that (b) all members of each sequence of the type (2) are zero. This latter assumption means that no allowance whatsoever is needed for variations of fertility within a block. Once variation of fertility within a block is admitted it appears only reasonable that it should be taken into account and the effect of the resulting correlations on any test concerning the yields of different varieties examined.

Cramér has shown [6, 7] that if the sum of two independent random variables be normally distributed each variable must itself follow the normal law. Strictly, therefore, it cannot be correct to apply normal theory to the random variables x_{jk} in the mathematical model elaborated above, for, though ϵ_{jk} may readily be assumed normally distributed, η_{jk} can obviously take only a finite number, s, of values and, consequently, as its distribution cannot be normal, it is impossible that x_{jk} can be exactly normally distributed either. However, as a first approximation, taking into account the correlations, it will be assumed that the yields from any block form a set of single observations of the variables in an s-variate normal distribution. Further, for the sake of simplicity, it will be assumed that the variances and covariances of the populations appropriate to the different blocks are the same. Dropping the distinguishing j's, the variances of the yields, as in (6) are defined by $\sigma_k^2 = \sigma_{\eta_{jk}}^2 + \sigma_{\epsilon_j}^2$ and ρ_{km} is written for $\rho_{j(km)}$ in (7). We define y_{jk} and A_{km} by

$$y_{jk} = x_{jk} - a_{jk} \tag{8}$$

and

$$A_{km} = \frac{\Delta_{km}}{2\sigma_k \sigma_m \Delta} = A_{mk} \tag{9}$$

where Δ is the s-rowed determinant $| \rho_{km} |$, symmetrical about its principal diagonal, Δ_{km} the cofactor of ρ_{km} in Δ and A is written for $| A_{km} |$ the determinant of the positive definite matrix $|| A_{km} ||$. Then since the interblock covariances are zero, the elementary probability law for the whole set of ns y's is given by

$$p\{y_{jk}\} = A^{\frac{1}{2}n} \pi^{-\frac{1}{2}ns} \exp \{ - \sum_j \sum_{k,m} A_{km} y_{jk} y_{jm} \}. \tag{10}$$

It may be noted that j and l where they occur run through all integral values from 1 to n while k and m take values from 1 to s. A sign such as $\sum_{k,m}$ means that the summation is taken over all the pairs of values of k and m the term (m, k) being taken as distinct from the term (k, m) and including the terms in which $k = m$. $\sum_{k \neq m}$ implies a similar summation with the omission of terms in which $k = m$.

The distribution law (10), or similar one substituting the x's for y's from (8), takes into account also cases in which, though the correlations may be zero, the variances of the different variety yields differ.

4. The Z-Test. If $\{x_q\}$, $q = 1, 2, \cdots f_1$, is a set of f_1 mutually independent random variables each of which follows the same normal law with zero mean and variance σ_1^2, and if $u_1 = \sum_{q=1}^{f_1} x_q^2$, then the distribution law for u_1 is, $u_1 \geq 0$,

$$p(u) = \{\alpha^{\frac{1}{2}f} / \Gamma(\tfrac{1}{2}f)\} u^{\frac{1}{2}f-1} e^{-\alpha u^2} \tag{11}$$

with $f = f_1$ and $\alpha^{-1} = 2\sigma_1^2$. If also $u_2 = \sum_{r=1}^{f_2} y_r^2$, where $\{y_r\}$, $r = 1, 2, \cdots, f_2$, is another set of mutually independent random variables each of which is normally distributed with zero mean and variance σ_2^2, then the distribution law of u_2 is (11) with $\alpha^{-1} = 2\sigma_2^2$ and $f = f_2$. If, in addition to the independence of the variables within each set, there is also independence between the sets, then u_1 is independent of u_2 and the distributions of different functions of u_1

and u_2 used as criteria may be obtained. The one originally proposed in this connection was z, defined by,

$$z = \tfrac{1}{2} \log_e (f_1 u_2 / f_2 u_1) - \log_e (\sigma_2/\sigma_1)$$

and its distribution law is [8, 9, 10]

$$p(z) = \frac{2 f_1^{\frac{1}{2}f_1} f_2^{\frac{1}{2}f_2} \Gamma[\frac{1}{2}(f_1 + f_2)] e^{f_2 z}}{\Gamma(\frac{1}{2}f_1)\Gamma(\frac{1}{2}f_2)(f_1 + f_2 e^{2z})^{\frac{1}{2}(f_1+f_2)}}. \tag{12}$$

Any other single-valued, monotone function of u_2/u_1 would when $\sigma_1 = \sigma_2$, as a criterion, be equivalent to z. $F = e^{2z} = f_1 u_2 / f_2 u_1$, $v = u_2/u_1$ and $w = u_2/(u_1 + u_2)$ have been adopted as criteria and their distribution laws are readily deduced from (12). All these criteria are equivalent in providing control of "errors of the first kind" [11, 12], that is, the risk of rejecting a hypothesis tested when true. As usual the procedure is to select arbitrarily in advance a certain "level of significance," say $\epsilon = 0.05$, 0.01 etc., and, assuming the hypothesis tested is true, to determine the value of the criterion, say the value z_0 of z, such that

$$P\{z \geq z_0 \mid H\} = \int_{z_0}^{\infty} p(z)\, dz = \epsilon. \tag{13}$$

If the sample of observations gives a value of $z \geq z_0$ H is rejected, if $z < z_0$, H is accepted. It is merely a matter of convenience which of the criteria z, F, u or w is used and tables are available to facilitate numerical work. Tables for z and F are given by Fisher [2], Fisher and Yates [13] and Snedecor [14], while for w *Tables of the Incomplete Beta Function* [15] may be used. Though no tables are directly available for v it is the simplest to use in theoretical discussion and in subsequent sections it is its distribution law, and not that of z, which will be considered. The latter may, of course, be readily deduced.

Considering the distribution law (10) with y_{jk} replaced by $x_{jk} - a_{jk}$ when $\rho_{km} = 0$ and $\sigma_k = \sigma_m = \sigma$, i.e., all the observations are normal and independent with the same variance. Writing

$$u_1 = \sum_{j,k} (x_{jk} - x_{\cdot k} - x_{j\cdot} + x_{\cdot\cdot})^2, \tag{14}$$

$$u_2 = \sum_{j,k} (x_{\cdot k} - x_{\cdot\cdot})^2 = n \sum_{k} (x_{\cdot k} - x_{\cdot\cdot})^2, \tag{15}$$

$$u_3 = \sum_{j,k} (x_{j\cdot} - x_{\cdot\cdot})^2 = s \sum_{j} (x_{j\cdot} - x_{\cdot\cdot})^2, \tag{16}$$

then it is readily seen that

$$u_1 + u_2 + u_3 = \sum_{j,k} (x_{jk} - x_{\cdot\cdot})^2.$$

Now if a_{jk} may be put in the form $M + B_j + V_k$ with $\sum_j B_j = \sum_k V_k = 0$ then u_1 is distributed as in (11) with $f = (n - 1)(s - 1)$. If, in addition to the additive assumption, $V_k = 0$ for all values of k then u_2 follows the same law, independently of u_1, and with $f = s - 1$. Similarly if $B_j = 0$, for all values of j, u_3 has the same distribution law with $f = n - 1$. It may be shown [16] that if $a_{jk} = M$ for all values of j and k the three quantities u_1, u_2 and u_3 follow independently the law (11) with suitable values for f, and then the corresponding values of z follow the law (12).

Making the assumption of additivity for a_{jk}, of which, incidentally, the correctness or adequacy cannot be tested without more than one set of ns observations of the variables, the z-test may be used to determine whether or not there is a "block effect" or a "variety effect," i.e., whether $B_j = 0$ or $V_k = 0$ for all values of j and k. For instance to test the hypothesis $V_k = 0$, $k = 1, 2, \cdots, s$, $z = \frac{1}{2} \log_e \{(n - 1)u_2/u_1\}$ is calculated from the observations and the hypothesis is rejected if $z \geq z_0$ where z_0 is found from Fisher's tables corresponding to a suitable value of ϵ in (13). Otherwise the hypothesis is accepted. This is the usual method of applying the z-test to randomized blocks.

The problem before us now is to consider what happens to such a test when $\sigma_k \neq \sigma_m$ and $\rho_{km} \neq 0$ in (10), and the hypotheses to be tested must be related to (1) and (5). As already stated this method of testing hypotheses controls, at a suitable level, the risk of rejecting the hypothesis when it is true. A complete examination of the application of any criterion as the test of a statistical hypothesis should involve, also, investigation of "errors of the second kind," i.e., the risk of accepting the hypothesis when some alternative is true. That is to say such an examination should involve a study of the "power function of the test" [17, 18, 19], and this would require a knowledge of the probability distribution of the criterion when the hypothesis tested is not true. In this paper, however, attention will be confined entirely to "errors of the first kind."

5. Hypotheses Tested.

In order that

$$u_1 = \sum_{j,k} (y_{jk} - y_{\cdot k} - y_{j\cdot} + y_{\cdot\cdot})^2 \tag{14a}$$

and

$$u_2 = n \sum_k (y_{\cdot k} - y_{\cdot\cdot})^2 \tag{15a}$$

may be true it is sufficient that

$$a_{jk} - a_{\cdot k} - a_{j\cdot} + a_{\cdot\cdot} = X_{j\cdot(k)} - X_{\cdot\cdot(k)} - X_{j\cdot(\cdot)} + X_{\cdot\cdot(\cdot)}$$

and

$$a_{\cdot k} - a_{\cdot\cdot} = X_{\cdot\cdot(k)} - X_{\cdot\cdot(\cdot)}$$

must both be zero in every case. It has been suggested by Neyman [4] that it would be desirable to test the hypothesis that $X_{\cdot\cdot(k)}$ is independent of k, i.e. that the average of the true yields over the whole field is the same for all varieties. He suggests that the variations in the responses of the different varieties within the field are relatively unimportant so that, while allowing for the effect of the variations in fertility within the field on the various distribution laws, it is the average over the whole field which should be tested. The functions u_1 and u_2 will not test this hypothesis for, in order that they may have the same expectation not only must $X_{\cdot\cdot(k)}$ be independent of k but also $X_{j\cdot(k)}$ must be independent of k for every j. Consequently one of the hypotheses tested here is that $X_{j\cdot(k)} = X_{j\cdot(\cdot)}$, and therefore, of course, $X_{\cdot\cdot(k)} = X_{\cdot\cdot(\cdot)}$, for every j and k, i.e. that the mean of the true yields over all the blocks is the same for all varieties while, by using (10), we make allowance for the variations in fertility over each block and for the resultant correlations introduced. We shall not consider u_3 from (16) as we are interested only in the presence or absence of a "variety effect."

It appears that two other hypotheses lead to results which are particular cases of the above. If we test whether the true yield on every plot is the same for all varieties, i.e. that $X_{jl(k)}$ is independent of k, then, assuming the hypothesis tested is true, the varieties all react in the same way to the variations of fertility within each block and in (10) $\sigma_k = \sigma_m = \sigma$, say, while $\rho_{km} = \rho$. On the other hand if we neglect all the variations in fertility within each block all the correlations vanish and $\sigma_k = \sigma_m = \sigma_\epsilon$. The hypothesis tested then is that either $X_{jl(k)}$ or, what is the same thing, $X_{j\cdot(k)}$ is independent of k.

It does not appear that the assumption of normality need cause any difficulty. E. S. Pearson [20] has examined the effect of skewness on the parent populations and by carrying out sampling experiments has concluded that even with skew populations "... it seems probable that the more elaborate forms of analysis of variance are also of fairly wide application, provided that the number of degrees of freedom apportioned to the residual variation is not too small." A further investigation by Eden and Yates [21] was also designed to test the effect of skewness, but the negative result there obtained was to be expected owing to the amalgamation of the observations into groups. It appears that the effect of skewness in the original populations will not have very much effect on the distribution of z.

Welch has examined [22] Randomized Blocks and Latin Square experiments from the "randomization" point of view. In the case of randomized blocks, in terms of the notation used above, he has taken $\epsilon_{jk} = 0$ or, expressed in another way, he has assumed that the actual observed yield in any plot is the "true yield" on that plot of the particular variety tested on the plot. The hypothesis he is then testing is that $X_{jl(k)}$, or, what is the same thing for him $x_{jl(k)}$ is independent of k. Taking the $(s!)^n$ different ways in which the varieties may be tagged on to the different yields he has considered the $(s!)^n$ different values of what we have called, w and he has compared the finite discrete distribution so obtained with that given by normal theory. Getting $E(w)$ and σ_w^2 from the finite distribution, he fitted a Pearson Type I curve in a number of examples and found that the 5 per cent and 1 per cent points in his fitted curves did not differ much from the corresponding points of the normal distribution of w. His theoretical discussion showed, however, that if there is too much discrepancy between the variancies in the different blocks the randomization test may seriously underestimate the significance of any differences between the varieties as compared with normal theory.

It was Neyman [4] who first pointed out that, when the variations of fertility within each block are taken into account, the correlations between the observed yields should be allowed for, and the method adopted here is a development of his point of view. A number of authors, however, while agreeing that such variations of fertility do occur, hold that this does not seriously affect the distribution of z.

6. Distribution of u_1 and u_2. As already stated, it is the distribution of $v = u_2/u_1$ which will be sought, not that of z, where u_1 and u_2 are defined by (14) and (15), or rather by (14a) and (15a), since the hypothesis tested is assumed true. Writing $i = \sqrt{-1}$, the characteristic function of the simultaneous distribution of u_1 and u_2, that is E [exp $\{i(t_1u_1 + t_2u_2)\}$], is found from (10).

From (14a) and (15a), by straightforward expansion, using the conventions already explained for $\sum_{k,m}$, $\sum_{k \neq m}$ etc., we get

$$u_1 = (ns)^{-1}\left[(n-1)(s-1)\sum_{j,k} y_{jk}^2 - (n-1)\sum_j \sum_{k \neq m} y_{jk}y_{jm} - (s-1)\sum_k \sum_{j \neq l} y_{jk}y_{lk} + \sum_{j \neq l}\sum_{k \neq m} y_{jk}y_{lm}\right],$$

$$u_2 = (ns)^{-1}\left[(s-1)\left\{\sum_{j,k} y_{jk}^2 + \sum_k \sum_{j \neq l} y_{jk}y_{lk}\right\} - \sum_j \sum_{k \neq m} y_{jk}y_{jm} - \sum_{j \neq l}\sum_{k \neq m} y_{jk}y_{lm}\right],$$

and using these expressions with (10) the characteristic function of u_1 and u_2 is

$$\begin{aligned} \varphi_{u_1,u_2}(t_1, t_2) &= A^{\frac{1}{2}n}\pi^{-\frac{1}{2}ns}\int_{-\infty}^{\infty} p\{y_{jk}\}\cdot\exp\{i(t_1u_1 + t_2u_2)\}\,dY \\ &= A^{\frac{1}{2}n}\pi^{-\frac{1}{2}ns}\int_{-\infty}^{\infty} \exp\left\{\sum_{j,l}\sum_{k,m} B_{jk,lm}y_{jk}y_{lm}\right\}dY \end{aligned}$$

where $dY = \prod_{j,k} dy_{jk}$ and the integral is an ns-fold one taken over the whole space of these variables. $B_{jk,lm}$ is defined by

$$B_{jk,lm} = \delta_{jl}A_{km} - i(s\delta_{km} - 1)[t_1(n\delta_{jl} - 1) + t_2]/ns \tag{17}$$

where the δ's have the usual meaning being equal to 1 when the suffixes are the same and equal to zero when the suffixes are different. This integral, since the real part of $B_{jk,lm}$ is positive definite, may readily be evaluated [23, 24] and gives

$$\varphi_{u_1,u_2}(t_1, t_2) = A^{\frac{1}{2}n}/B^{\frac{1}{2}} \tag{18}$$

B being the ns-rowed determinant $| B_{jk,lm} |$.

The determinant B may be written in the form

$$B = \begin{vmatrix} [P] & [Q] & \cdots & [Q] \\ [Q] & [P] & \cdots & [Q] \\ \cdots & \cdots & \cdots & \cdots \\ [Q] & [Q] & \cdots & [P] \end{vmatrix}$$

where $[P] = [p_{km}] = [B_{jk,jm}]$ and $[Q] = [q_{km}] = [B_{jk,lm}]$ and there are n^2 such arrays in B. This gives at once $B = | p_{km} + (n-1)q_{km} | \cdot | p_{km} - q_{km} |^{n-1}$, whence on substitution

$$B = | A_{km} - it_2(s\delta_{km} - 1)/s | \cdot | A_{km} - it_1(s\delta_{km} - 1)/s |^{n-1}. \tag{19}$$

The two determinants in (19) are identical, with t_1 and t_2 interchanged, and are readily reduced to symmetrical $(s-1)$-rowed determinants by: (a) Adding to the terms in the last row the corresponding terms in the other rows and repeating for columns, (b) Multiplying the terms in the last row successively by M_k/M $(k = 1, 2, 3, \cdots, s)$ and subtracting from the corresponding terms in each of the other rows, with

$$M_k = \sum_{m=1}^{s} A_{km} \quad \text{and} \quad M = \sum_{k=1}^{s} M_k = \sum_{k,m=1}^{s} A_{km}. \tag{20}$$

The following operations then reduce these $(s-1)$-rowed determinants to ones which are symmetrical and contain t's only in the diagonals: (i) To the terms in the last column add the corresponding terms in all the other columns and repeat for the rows, (ii) Multiply the terms in the last column by $(\sqrt{s}+1)^{-1}$ and add them to the corresponding terms in each of the other columns, repeat for the rows, (iii) From the terms in the last column subtract the sum of the corresponding terms in the other columns multiplied by $s^{-\frac{1}{2}}$, repeat for the rows, (iv) Divide the last row and the last column by $s^{-\frac{1}{2}}$. The determinant then becomes M/s. $| C - itI |$ where $\| C \|$ is the matrix $\| c_{km} \|$, I the unit matrix and

$$\begin{aligned} c_{km} = c_{mk} = A_{km} &- (\sqrt{s}+1)^{-1}(A_{ks} + A_{ms}) + (\sqrt{s}+1)^{-2}A_{ss} \\ &- M^{-1}[M_k - (\sqrt{s}+1)^{-1}M_s][M_m - (\sqrt{s}+1)^{-1}M_s]. \end{aligned} \tag{21}$$

It should be noted that henceforward k and m run through integral values from 1 to $s-1$ only unless the contrary is specifically stated.

Thus it follows that

$$\varphi_{u_1,u_2}(t_1, t_2) = (As/M)^{\frac{1}{2}n} | C - it_1 I |^{-\frac{1}{2}(n-1)} \cdot | C - it_2 I |^{-\frac{1}{2}}.$$

Putting $C = | c_{km} |$ and noting that $\varphi_{u_1, u_2}(t_1, t_2) = 1$ when $t_1 = t_2 = 0$ clearly $As/M = C$ and the characteristic function factors into the form $\varphi_{u_1}(t_1) \cdot \varphi_{u_2}(t_2)$, where

$$\varphi_{u_1}(t_1) = C^{\frac{1}{2}(n-1)} | C - it_1 I |^{-\frac{1}{2}(n-1)} \tag{22}$$

$$\varphi_{u_2}(t_2) = C^{\frac{1}{2}} | C - it_2 I |^{-\frac{1}{2}}. \tag{23}$$

This demonstrates that u_1 and u_2 are stochastically independent and that the correlations introduced by allowing for the variations in fertility within the blocks does not affect the independence already demonstrated [16, 25].

$\| C \|$ being a square positive definite matrix of rank $s - 1$, its characteristic equation $| C - \lambda I | = 0$ must have $s - 1$ real positive roots. It follows that $| C - itI |$ must factor into $s - 1$ factors of the type α-it where α is a real positive constant. Some or all of these factors may be equal and various combinations of factors of different multiplicity are possible depending on the value of s. Only two cases will be considered here: (a) when all the roots of the characteristic equation of $\| C \|$ are equal, and (b) when all the roots of the characteristic equation are unequal.

Suppose that all the roots of the characteristic equation are equal, say to α, then $| C - itI | = (\alpha - it)^{s-1}$ and $C = \alpha^{s-1}$ giving

$$\varphi_{u_1}(t_1) = \alpha^{\frac{1}{2}(n-1)(s-1)}(\alpha - it_1)^{-\frac{1}{2}(n-1)(s-1)}, \tag{24}$$

$$\varphi_{u_2}(t_2) = \alpha^{\frac{1}{2}(s-1)}(\alpha - it_2)^{-\frac{1}{2}(s-1)} \tag{25}$$

It is seen at once that u_1 and u_2 are distributed as in (11), $f_1 = (n - 1)(s - 1)$ and $f_2 = s - 1$, and thus z or v follow the usual distribution laws.

Clearly when the variations of fertility within each block are neglected and the hypothesis tested is that $X_{jl(k)}$, or $X_{j \cdot (k)}$, is independent of k, the roots of the characteristic equation are all equal. Then there is no correlation, $\sigma_k = \sigma_\epsilon$, $A_{kk} = (2\sigma^2)^{-1} = c_{kk} = \alpha$, $A_{km} = 0 = c_{km} (k \neq m)$ and the usual results are obvious.

On the other hand when allowing for the variations of fertility within a block while testing the hypothesis that $X_{jl(k)}$ is independent of k, the variances and covariances are all equal, i.e. $\sigma_k^2 = \sigma_\eta^2 + \sigma_\epsilon^2 = \sigma^2$, $\rho_{kk} = 1$ and $\rho_{km} = \rho = -\sigma_\eta^2/\{(s - 1)(\sigma_\eta^2 + \sigma_\epsilon^2)\}$, $k \neq m$. This gives

$$\Delta_{km} = [\{1 + (s - 1)\rho\}\delta_{km} - \rho](1 - \rho)^{s-2}, \qquad \Delta = \{1 + (s - 1)\rho\}(1 - \rho)^{s-1},$$

$$A_{km} = \{1 + (s - 1)\delta_{km} - \rho\}/2\sigma^2\Delta, \qquad A = 1/(2\sigma^2)^s\Delta,$$

$$c_{km} = \delta_{km}\{2\sigma^2(1 - \rho)\}^{-1}, \qquad C = \{2\sigma^2(1 - \rho)\}^{-s+1},$$

where, as usual, $\delta_{km} = \begin{bmatrix} 1 & k = m \\ 0 & k \neq m \end{bmatrix}$ From this it follows the roots of the characteristic equation are all equal, $\alpha = \{2\sigma^2(1 - \rho)\}^{-1}$ in (24) and (25). Thus in this case also, the z-test or its equivalent gives exact control of errors of the first kind. There is, however, this difference that $u_1/(n - 1)(s - 1)$ and $u_2/(s - 1)$

are to be considered not as estimates of σ^2 but as estimates of $\sigma^2(1 - \rho) = (s-1)^{-1}\{s\sigma_\eta^2 + (s-1)\sigma_\epsilon^2\}$.

When $s = 2$, even though the variances differ, since there is only one root of the characteristic equation $\alpha = (\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)^{-1}$ the characteristic functions are of the form (24) and (25). Consequently, in this case, $s = 2$, when only two varieties are tested for the hypothesis that their average "true yields" are the same on each block then, even though the varieties may react in different ways to the fertility levels within the blocks, granting normality, the usual z-distribution applies. This, of course, includes the case when even though ρ may be zero the variances differ. $u_1/(n-1)$ and u_2 are to be considered as estimates of $\frac{1}{2}(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)$.

Proceeding next to the case in which all the roots of the characteristic equation $| C - \lambda I | = 0$ are unequal, the roots are, say, $\alpha_1 < \alpha_2 < \cdots < \alpha_{s-1}$ where, of course, all these quantities are real and positive. This case will arise in testing the hypothesis that the yield for each variety is the same for every block, that

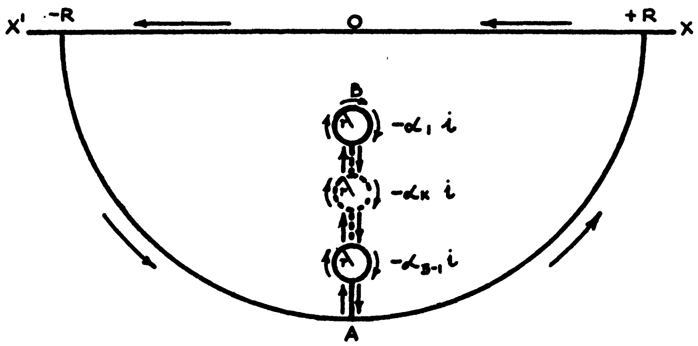

FIG. 1

$X_{j\cdot(k)} = X_{j\cdot(m)}$, while allowance is made for the different responses of the varieties to the differences in fertility within the blocks. The mathematical formulation would be the same even if there were no correlations but the variances were different for the different varieties. Then we have [26, 27, 28] for both u_1 and u_2 from (22) and (23)

$$p(u) = C^m(2\pi)^{-1}\int_{-\infty}^{\infty} e^{-iut}[\prod_k (\alpha_k - it)]^{-m}\,dt \tag{26}$$

with $m = \frac{1}{2}(n-1)$ for u_1 and $m = \frac{1}{2}$ for u_2.

Replace t by the complex variable z and integrate round the contour shown in Fig. 1. This contour consists of: (i) The real axis from $+R$ to $-R$, where $R > \alpha_{s-1}$, (ii) The quadrant $| z | = R$, $\pi \leq \arg z \leq 3\pi/2$, (iii) The imaginary axis from $A[-iR]$ to $B[-(\alpha_1 - r)i]$, cutting out the singularities by small semicircles of radius r, as shown, (iv) The imaginary axis from B to A, as in (iii), (v) The quadrant $| z | = R$, $3\pi/2 \leq \arg z \leq 2\pi$. Within this contour $f(z)$ is analytic and hence the contour integral zero. It may also be readily seen that the integrals over the two quadrants tend to zero as R increases, and by examining the changes in the amplitudes of $(\alpha_k - iz)^{-m}$, $k = 1, 2, \cdots, s-1$, as the contour circles the points $-i\alpha_1, -i\alpha_2 \cdots$, it will be seen that the integrals over the straight lines between $(-i\alpha_2, -i\alpha_3)$, $(-i\alpha_4, -i\alpha_5)$, $\cdots$ cancel whether m be half an odd or half an even integer. Then

$$p(u) = C^m(2\pi)^{-1} \sum_r \int_{D_r} e^{-iuz} [\prod_k (\alpha_k - iz)]^{-m} dz. \tag{26a}$$

The contours D_r are those shown in Fig. 2 and consist of "dumb-bells" encircling the points $(-i\alpha_r, -i\alpha_{r+1})$, $r = 1, 3, 5, \cdots$, in the negative direction. If s is even, the last integral consists of only one half the "dumb-bell" extending to

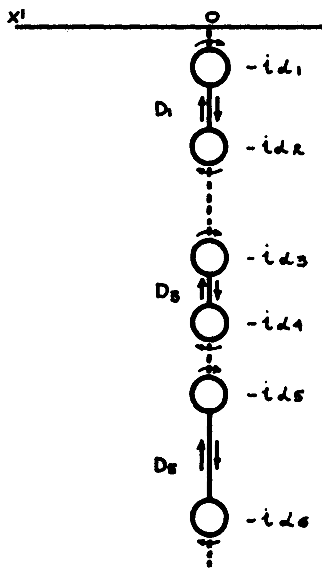

FIG. 2

$-i\infty$. It may also be noted that if n be odd and so m an integer, the other straight line integrals, those of the "dumb-bell" contours in question, also cancel and leave only the contributions of the small circles about, what are now, the poles.

Now we put $iz = w$ and $\Phi_r(w) = \prod_{k=1}^{s-1} (\alpha_k - w)^{-m}$ *omitting the terms containing* α_r *and* α_{r+1}, it follows that $\Phi_r(w)$ is analytic in and on a circle of centre $\frac{1}{2}(\alpha_r + \alpha_{r+1})$ which contains the points α_r and α_{r+1} but not α_{r-1} or α_{r+2}. Thus the function $\Phi_r(w)$ may in the interior of this circle be expanded as a uniformly convergent series in terms of $\frac{1}{2}(\alpha_r + \alpha_{r+1}) - w$ giving, $r = 1, 3, \cdots$,

$$\Phi_r(w) = \sum_{p=0}^{\infty} a_{rp}\{\tfrac{1}{2}(\alpha_r + \alpha_{r+1}) - w\}^p. \tag{27}$$

Since termwise integration is then permissible it is necessary to consider only integrals of the form

$$J_{rp} = i \int_{D_r'} \frac{\{\frac{1}{2}(\alpha_r + \alpha_{r+1}) - w\}^p e^{-uw}\, dw}{\{(\alpha_r - w)(\alpha_{r+1} - w)\}^m},$$

where D_r' is a contour similar to D_r but circling instead in a positive direction the points α_r and α_{r+1} on the real axis. We then have

$$\text{(26b)} \qquad p(u) = C^m (2\pi)^{-1} \sum_r \sum_{p=0}^{\infty} a_{rp} J_{rp}.$$

Now if

$$J_r = i \int_{D_r'} \frac{e^{-uw}\, dw}{\{(\alpha_r - w)(\alpha_{r+1} - w)\}^m}$$

it is clear that J_{rp} is obtained by applying the operator $\left\{\frac{1}{2}(\alpha_r + \alpha_{r+1}) + \frac{\partial}{\partial u}\right\}$ p times to J_r. Now putting $w - \frac{1}{2}(\alpha_r + \alpha_{r+1}) = (\alpha_{r+1} - \alpha_r)t$, it follows that

$$J_r = \frac{i e^{-\frac{1}{2}u(\alpha_r + \alpha_{r+1})}}{e^{2m\pi i}\{\frac{1}{2}(\alpha_{r+1} - \alpha_r)\}^{2m-1}} \int_{\infty}^{(1+,-1+)} \frac{e^{-\frac{1}{2}ut(\alpha_{r+1} - \alpha_r)}}{(t^2 - 1)^m}\, dt$$

and this gives [29, p. 171],

$$J_r = \frac{\pi \Gamma(\frac{1}{2}) u^{m-\frac{1}{2}} e^{-\frac{1}{2}u(\alpha_r + \alpha_{r+1})} I_{m-\frac{1}{2}}\{\frac{1}{2}u(\alpha_{r+1} - \alpha_r)\}}{2^{m-\frac{3}{2}} \Gamma(m) \{\frac{1}{2}(\alpha_{r+1} - \alpha_r)\}^{m-\frac{1}{2}}}.$$

$I_\mu(z)$ is the Bessel Function of purely imaginary argument defined, $-\pi < \arg z \leq \frac{1}{2}\pi$, by

$$\text{(28)} \qquad I_\mu(z) = \sum_{r=0}^{\infty} \frac{(\frac{1}{2}z)^{\mu+2r}}{r!\,\Gamma(\mu + r + 1)}.$$

Hence it may be found that

$$J_{rp} = \frac{2\pi\Gamma(\frac{1}{2})}{\Gamma(m)(\alpha_{r+1} - \alpha_r)^{m-\frac{1}{2}}} e^{-\frac{1}{2}u(\alpha_r + \alpha_{r+1})} \frac{\partial^p}{\partial u^p} [u^{m-\frac{1}{2}} I_{m-\frac{1}{2}}\{\frac{1}{2}u(\alpha_{r+1} - \alpha_r)\}]$$

and this gives

$$\text{(29)} \quad p(u) = \frac{C^m \Gamma(\frac{1}{2})}{\Gamma(m)} \sum_r \frac{e^{-\frac{1}{2}u(\alpha_r + \alpha_{r+1})}}{(\alpha_{r+1} - \alpha_r)^{m-\frac{1}{2}}} \sum_{p=0}^{\infty} a_{rp} \frac{\partial^p}{\partial u^p} [u^{m-\frac{1}{2}} I_{m-\frac{1}{2}}\{\frac{1}{2}u(\alpha_{r+1} - \alpha_r)\}]$$

where a_{rp} is defined by (27) with $m = \frac{1}{2}(n - 1)$ for u_1 and $m = \frac{1}{2}$ for u_2.

In the case $s = 3$ there is only one "dumb-bell" contour and $\Phi_1(w) = 1$, so that we get, for $u_1, u_2 \geq 0$.

$$\text{(30)} \quad p(u_1) = \frac{(\alpha_1 \alpha_2)^{\frac{1}{2}(n-1)} \Gamma(\frac{1}{2})}{(\alpha_2 - \alpha_1)^{\frac{1}{2}(n-2)} \Gamma\{\frac{1}{2}(n - 1)\}} e^{-\frac{1}{2}u_1(\alpha_1 + \alpha_2)} u_1^{\frac{1}{2}(n-2)} I_{\frac{1}{2}(n-2)}\{\frac{1}{2}u_1(\alpha_2 - \alpha_1)\}$$

$$\text{(31)} \quad p(u_2) = (\alpha_1 \alpha_2)^{\frac{1}{2}} e^{-\frac{1}{2}u_2(\alpha_1 + \alpha_2)} I_0\{\frac{1}{2}u_2(\alpha_2 - \alpha_1)\}.$$

It may be noted that if the series in (28) is substituted in (30) and (31) these distributions may be considered as the sum of an infinite number of χ^2-distributions all with the same σ^2 but with different degrees of freedom. It may also be noted that with $\alpha_1 = \alpha_2$ all the terms, except the first, vanish and thus a single χ^2-distribution is left.

When s is even the last contour is one from $+\infty$ circling α_{s-1} negatively. Using Hankel's integral for the Gamma-Function, putting $w - \alpha_{s-1} = \zeta/u$

$$I = i \int_{\infty}^{(\alpha_{s-1}+)} e^{-uw}(\alpha_{s-1} - w)^{-m} \, dw = ie^{-u\alpha_{s-1}} u^{m-1} \int_{\infty}^{(0+)} (-\zeta)^{-m} e^{-\zeta} \, d\zeta$$

$$= \frac{2\pi}{\Gamma(m)} e^{-u\alpha_{s-1}} u^{m-1}.$$

Denoting by D differentiation with respect to u under the sign of integration and by D^{-1} the corresponding integration from zero to u,

$$D^{-p}I = i \int_{\infty}^{(\alpha_{s-1}+)} (-w)^{-p} e^{-uw}(\alpha_{s-1} - w)^{-m} \, dw.$$

Then we can write

$$\prod_{k=1}^{s-2} (\alpha_k - w)^{-m} = (-w)^{-m(s-2)} \prod_{k=1}^{s-2} (1 - \alpha_k/w)^{-m}$$

$$= \sum_{p=0}^{\infty} a_{s-1,p}(-w)^{-p-m(s-2)}$$

the expansion being justifiable since $| \alpha_k/w | < 1$. Since $(s - 2)m$ is an integer the additional term to be added to (29) to give $p(u)$ is, therefore,

$$C^m/\Gamma(m) \cdot \sum_{p=0}^{\infty} a_{s-1,p} D^{-p-m(s-2)} \{u^{m-1} e^{-u\alpha_{s-1}}\}. \tag{32}$$

7. **Distribution of $v = u_2/u_1$.** Though the distributions of u_1 and u_2 have been given in a rather complicated form for any value of s when the roots of $| C - \lambda I | = 0$ are all unequal, the distribution of v is given only for $s = 3$. In this case, since u_1 and u_2 are independent, from (30) and (31)

$$p(u_1, u_2) = \frac{(\alpha_1\alpha_2)^{\frac{1}{2}n} \Gamma(\frac{1}{2})}{(\alpha_2 - \alpha_1)^{\frac{1}{2}(n-2)} \Gamma\{\frac{1}{2}(n-1)\}} u_1^{\frac{1}{2}(n-2)} e^{-\frac{1}{2}(u_1+u_2)(\alpha_1+\alpha_2)}$$

$$I_0\{\tfrac{1}{2}u_2(\alpha_2 - \alpha_1)\} I_{\frac{1}{2}(n-2)}\{\tfrac{1}{2}u_1(\alpha_2 - \alpha_1)\}$$

Now making the transformation $u_1 = u$, $u_2 = uv$ with $\dfrac{\partial(u_1, u_2)}{\partial(u, v)} = u$ and putting the exponential term

$$e^{-\frac{1}{2}u(1+v)(\alpha_1+\alpha_2)} = \frac{\{u(1 + v)(\alpha_1 + \alpha_2)\}^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} K_{\frac{1}{2}}\{\tfrac{1}{2}u(1 + v)(\alpha_1 + \alpha_2)\},$$

then on integrating with respect to u over the whole range of variation, from zero to infinity, we get

$$p(v) = \frac{(\alpha_1\alpha_2)^{\frac{1}{2}n}\{(1+v)(\alpha_1+\alpha_2)\}^{\frac{1}{2}}}{(\alpha_2-\alpha_1)^{\frac{1}{2}(n-2)}\Gamma\{\frac{1}{2}(n-1)\}} \int_0^\infty u^{\frac{1}{2}(n-1)} I_0\{\tfrac{1}{2}uv(\alpha_2-\alpha_1)\}$$
$$I_{\frac{1}{2}(n-2)}\{\tfrac{1}{2}u(\alpha_2-\alpha_1)\}K_{\frac{1}{2}}\{\tfrac{1}{2}u(1+v)(\alpha_1+\alpha_2)\}\,du,$$

$K_{\frac{1}{2}}(z)$ being the modified Bessel Function of the second kind.

This integral is a particular case of one investigated by Bailey [30, 31] and it gives

$$(33)\quad p(v) = \frac{(n-1)(\alpha_1\alpha_2)^{\frac{1}{2}n}}{\{\frac{1}{2}(1+v)(\alpha_1+\alpha_2\}^n} F_4\{\tfrac{1}{2}(n+1), \tfrac{1}{2}n; 1, \tfrac{1}{2}n; \beta^2v^2/(1+v)^2, \beta^2/(1+v)^2\}$$

where $\beta = (\alpha_2 - \alpha_1)/(\alpha_2 + \alpha_1)$ and F_4 is Appell's fourth hypergeometric function of two variables [32].

On performing a similar integration when $s > 3$, $p(v)$ may be obtained as a rather complicated series of terms similar to (33).

8. **Approximate Moments of the Distribution of** v. As the distribution of v is complicated even in the simplest case of $s = 3$ it appears advisable to examine its moments even though only approximately. Writing $S_r = \sum_{k=1}^{s-1} \alpha_k^{-r}$ and putting

$$\prod_{k=1}^{s-1} (1 - t/\alpha_k)^{-m} = \exp\left\{-m \sum_{k=1}^{s-1} \log (1 - t/\alpha_k)\right\}$$
$$= \exp\left\{\sum_{r=1}^{\infty} mS_r t^r/r!\right\},$$

k_r being the r-th semi-invariant of u, we get

$$k_r = S_r m\cdot(r-1)!$$

Thence the first four moments of u about its mean are

$$\bar{u} = mS_1, \qquad \mu_2 = mS_2$$
$$\mu_3 = 2mS_3, \qquad \mu_4 = 3m\{2S_4 + mS_2^2\},$$

m being $\frac{1}{2}(n-1)$ in the case of u_1 and $\frac{1}{2}$ in the case of u_2.

Now to get approximate moments for v we write $\xi = (u_1 - \bar{u}_1)/\bar{u}_1$ and $\eta = (u_2 - \bar{u}_2)/\bar{u}_2$ and, expanding in terms of ξ and η, obtain

$$v = \bar{u}_2/\bar{u}_1\cdot\{1 + \eta - \xi - \xi\eta + \xi^2 + \xi^2\eta - \cdots\}.$$

This gives, M_r being the r-th moment of v about the origin and writing $T_r = S_r/S_1^r = \sum_{k=1}^{s-1} \alpha_k^{-r} \Big/ \left(\sum_{k=1}^{s-1} \alpha_k^{-1}\right)^r$,

$$M_1 = (n-1)^{-1}[1 + 2T_2(n-1)^{-1} - 8T_3(n-1)^{-2}$$
$$+ 12\{4T_4 + (n-1)T_2^2\}(n-1)^{-3}\cdots],$$

$$M_2 = (n-1)^{-2}(1+2T_2)[1+6T_2(n-1)^{-1} - 32T_3(n-1)^{-2} + 60\{4T_4 + (n-1)T_2^2\}(n-1)^{-3} \cdots],$$

$$M_3 = (n-1)^{-3}(1+6T_2+8T_3)[1+12T_2(n-1)^{-1} - 80T_3(n-1)^{-2} + 180\{4T_4 + (n-1)T_2^2\}(n-1)^{-3} \cdots],$$

$$M_4 = (n-1)^{-4}\{1+12T_2+32T_3+12(4T_4+T_2^2)\}[1+20T_2(n-1)^{-1} - 160T_3(n-1)^{-2} + 420\{4T_4 + (n-1)T_2^2\}(n-1)^{-3} \cdots].$$

The moments around the mean may readily be found if needed. If the α's are all equal

$$M_r' = \frac{\Gamma(\frac{1}{2}f_1 - r)\Gamma(\frac{1}{2}f_2 + r)}{\Gamma(\frac{1}{2}f_1)\Gamma(\frac{1}{2}f_2)}.$$

from the known distribution of the ratio of two χ^2's with $f_2 = (s-1)$ and $f_1 = (n-1)(s-1)$ degrees of freedom respectively. Then developing M_r' as a series in terms of f_1^{-1} and f_2^{-1}

$$M_1' = (f_2/f_1)(1 + 2f_1^{-1} + 4f_1^{-2} + \cdots),$$

$$M_2' = (f_2/f_1)^2(1 + 2f_2^{-1})(1 + 6f_1^{-1} + 28f_1^{-2} + \cdots),$$

$$M_3' = (f_2/f_1)^3(1 + 6f_2^{-1} + 8f_2^{-2})(1 + 12f_1^{-1} + 100f_1^{-2} + \cdots),$$

$$M_4' = (f_2/f_1)^4(1 + 12f_2^{-1} + 44f_2^{-2} + 48f_2^{-3})(1 + 20f_1^{-1} + 260f_1^{-2} + \cdots).$$

It is then easily seen that the difference between these moments and those of v when the α's are unequal is due to the deviation of T_r from $(s-1)^{r-1}$, the value it would have if the α's were all equal.

9. **Numerical Illustrations.** The distribution of v has been obtained in workable form only when $s = 3$ and, consequently, it is only that case that is considered here. In equation (33) the variable terms in the Appell function all contain $\beta = (\alpha_2 - \alpha_1)/(\alpha_2 + \alpha_1)$ and it is this fraction or, perhaps better, its square which measures, in a sense, the deviation of the distribution of v from the usual form. There are, therefore, two stages in this examination. It will first be investigated how β changes with the correlations and variances; and then the changes in the "levels of significance" due to differences in β will be examined.

Using (9), (20) and (21) it will be found that the equation $| C - \lambda I | = 0$ for $s = 3$ becomes $4p\lambda^2 - 4q\lambda + 3 = 0$ with

$$p = \sigma_2^2\sigma_3^2\Delta_{11} + \sigma_3^2\sigma_1^2\Delta_{22} + \sigma_1^2\sigma_2^2\Delta_{33} + 2\sigma_1\sigma_2\sigma_3(\sigma_1\Delta_{23} + \sigma_2\Delta_{31} + \sigma_3\Delta_{12}),$$

$$q = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_2\sigma_3\rho_{23} - \sigma_3\sigma_1\rho_{31} - \sigma_1\sigma_2\rho_{12},$$

where, of course, $\Delta = | \rho_{km} |$ and Δ_{km} is the cofactor of ρ_{km} in Δ. This equation may readily be solved giving α_1 and α_2.

Taking first the case of zero correlation and putting $\sigma_1^2 = k_1\sigma^2$, $\sigma_2^2 = k_2\sigma^2$ and

$\sigma_3^2 = k_3\sigma^2$ it will be found that while α_1 and α_2 depend on the k's and on σ^2 the fraction $\beta = (\alpha_2 - \alpha_1)/(\alpha_2 + \alpha_1)$ depends only on the k's. For different values of the k's the following table shows the values of β^2.

TABLE 1

Values of β^2 of different values of $\sigma_1^2 : \sigma_2^2 : \sigma_3^2 = k_1 : k_2 : k_3$. No correlation.

k_1	k_2	k_3	β^2
1	1	1	0
1.0	1.1	1.2	0.003
1.0	1.5	2.0	0.037
1	2	3	0.083
1	4	4	0.111
1	9	9	0.177
1	25	25	0.221
1	1	4	0.250
1	4	9	0.250
1	9	16	0.250
1	1	9	0.529
1	1	25	0.790
1	N	N	$(N-1)^2/(2N+1)^2$
1	1	N	$(N-1)^2/(N+2)^2$

It is clear that to get a considerable value of β^2, one standard deviation must be at least three times the other two. It also seems to produce a considerably larger value of β^2 to have one large k and two small k's than to have two large ones and one small, with the same order of magnitude of the ratio large to small. Furthermore, when the ratios of the σ^2 are $1{:}1{:}N$ the limit of β^2 as N increases is 1, while if the ratios are $1 : N : N$ the limit is 0.25.

Examining now the definition of ρ_{km}, omitting the j's in equation (7), we find that ρ_{km} can be written in the form

$$\rho_{km} = -r_{km}(s-1)^{-1}[(1 + \sigma_\epsilon^2/\sigma_{\eta_k}^2)(1 + \sigma_\epsilon^2/\sigma_{\eta_m}^2)]^{-\frac{1}{2}},$$

where

$$r_{km} = \frac{s^{-1}\sum_{l=1}^{s} u_{jl(k)}\, u_{jl(m)}}{\sigma_{\eta_k}\sigma_{\eta_m}}$$

and r_{km} is itself a coefficient of correlation, i.e., the correlation between the true yields. The second part of ρ_{km} depends on s and on the relative magnitudes of the soil error and of the technical error. Its maximum value, in the case of $s = 3$, is 0.5 which occurs when the technical error is small with respect to the soil error. If both types of error have the same variance then the second term is 0.25. There appears to be no data available which enables us to assign values to

r_{km}, so the method adopted is to choose some values of r_{km} which appear likely to affect seriously the value of β^2 and then to take the second factor equal 0.5. If the values of ρ_{km} are all equal and the variances are also equal the normal theory has been shown to apply, and hence these values are taken to differ considerably. Table 2 shows the effect on β^2 of taking different correlations with various values of $\sigma_1^2 : \sigma_2^2 : \sigma_3^2$.

It is clear from the table that if there exist correlations of the order of magnitude of those assumed, they can cause the distribution of v to deviate considerably from that which arises on the usual theory. For instance, if the variances are equal the value of β^2 may be 0.444 a value it would attain if, with no correlations, one variance was seven times the other two. Taking the cases in which $\sigma_1^2 : \sigma_2^2 : \sigma_3^2 = 1 : 4 : 9$ or $1 : 9 : 16$ the value of β^2 with no correlations is, in either case, 0.250 while with the correlations it may be as low as 0.008 or as high as 0.869.

TABLE 2

Values of β^2 for different values of the correlations and of $\sigma_1^2 : \sigma_2^2 : \sigma_3^2$.

ρ_{12}	ρ_{13}	ρ_{23}	$\sigma_1^2:\sigma_2^2:\sigma_3^2$				
			1:1:1	1:4:9	1:9:16	1:25:25	1:1:25
0	0	0	0	0.250	0.250	0.221	0.790
−0.4	−0.1	0.2	0.099	0.083	0.075	0.059	0.721
0.2	−0.1	−0.4		0.523	0.549	0.543	0.851
−0.25	−0.25	0.2	0.074	0.132	0.104	0.056	0.766
0.2	−0.25	−0.25		0.423	0.429	0.402	0.843
0.4	0.2	−0.4	0.265	0.706	0.698	0.658	0.909
−0.4	0.2	0.4		0.028	0.019	0.020	0.690
0.4	0.4	−0.4	0.379	0.793	0.765	0.709	0.937
−0.4	0.4	0.4		0.016	0.016	0.038	0.673
0.4	0.4	−0.5	0.444	0.869	0.845	0.793	0.954
−0.5	0.4	0.4		0.008	0.010	0.042	0.654
0.4	0.1	−0.3	0.189	0.606	0.596	0.551	0.890
−0.3	0.1	0.4		0.055	0.035	0.012	0.721

On the other hand when β^2 is large, in the case of zero correlation, say $\beta^2 = 0.790$ when $\sigma_1^2 : \sigma_2^2 : \sigma_3^2 = 1 : 1 : 25$, the correlations, as might be expected, appear to have less effect, the values of β^2 varying from 0.654 to 0.954. We may, therefore, conclude that if such correlations exist their effect on β^2, and therefore on the distribution of v, is certainly comparable with that of fairly large differences in the variances.

We now examine

$$P\{v > v_0 \mid \beta\} = \int_{v_0}^{\infty} p(v)\, dv$$

for different values of v_0 and β. Writing $p(v)$ in full from (33) and interchanging integration and summation we get

$$P\{v > v_0 \mid \beta\} = (n-1)(1-\beta^2)^{\frac{1}{2}n} \sum_{j,k=0}^{\infty} \frac{\{\frac{1}{2}(n+1)\}_{(j+k)}(\frac{1}{2}n + k)_{(j)}}{(j!)^2 k!} \beta^{2j+2k} \cdot \int_{v_0}^{\infty} v^{2j}(1+v)^{-n-2j-2k}\,dv.$$

Changing the variable to $x = (1+v)^{-1}$ the integral part becomes

$$\int_0^{x_0} x^{2k+n-2}(1+x)^{2j}\,dx = \frac{\Gamma(2j+1)\Gamma(2k+n-1)}{\Gamma(2j+2k+n)} I_{x_0}(2k+n-1, 2j+1),$$

in the notation usually employed [15]. Substitution gives

$$P\{v > v_0 \mid \beta\} = (1-\beta^2)^{\frac{1}{2}n} \sum_{j,k=0}^{\infty} \frac{(2j)!\{\frac{1}{2}(n-1)\}_{(k)}}{2^{2j}(j!)^2 k!} \beta^{2j+2k} I_{x_0}(2k+n-1,\ 2j+1).$$

Two sets of values of this expression were obtained, one for $n = 3$, and the other for $n = 6$, while β^2 was given the values 0.1, 0.2, 0.3, 0.4, and 0.5. The values of x_0 were chosen so as to cover the 1, 5 and 10 per cent significance levels. Table 3 shows these results.

TABLE 3

$P\{v > v_0/\beta\}$ *for certain values of* v_0 *and* β

(a) $n = 3$

x_0	v_0	Values of β^2					
		0.0	0.1	0.2	0.3	0.4	0.5
0.05	19	0.0025	0.003	0.003	0.004	0.004	0.005
0.10	9	0.010	0.011	0.013	0.014	0.016	0.018
0.15	$5\frac{2}{3}$	0.0225	0.025	0.027	0.030	0.034	0.037
0.20	4	0.040	0.043	0.048	0.051	0.056	0.061
0.30	$2\frac{1}{3}$	0.090	0.095	0.100	0.106	0.112	0.117
0.40	$1\frac{1}{2}$	0.160	0.165	0.170	0.175	0.181	0.187

(b) $n = 6$

x_0	v_0	Values of β^2					
		0.0	0.1	0.2	0.3	0.4	0.5
0.3	$2\frac{1}{3}$	0.002	0.003	0.004	0.005	0.006	0.007
0.4	$1\frac{1}{2}$	0.010	0.012	0.015	0.017	0.020	0.022
0.5	1	0.031	0.035	0.039	0.043	0.047	0.050
0.6	$\frac{2}{3}$	0.078	0.082	0.087	0.092	0.098	0.106
0.7	$\frac{3}{7}$	0.168	0.171	0.173	0.176	0.181	

The 1, 5, and 10 per cent levels of significance for x_0 were obtained in both cases by graphical interpolation and the corresponding values of v_0 then calculated. Table 4 shows clearly the changes in these significance levels. It must be remembered that values of β^2 considerably in excess of 0.5 may easily arise.

TABLE 4

Changes in the levels of significance $\beta^2 = 0$ and 0.5, $n = 3$ and 6.

		1 per cent.		5 per cent.		10 per cent.	
		x_0	v_0	x_0	v_0	x_0	v_0
$n = 3$	$\beta^2 = 0$	0.10	9.00	0.22	3.47	0.32	2.16
	$\beta^2 = 0.5$	0.07	13.0	0.18	4.6	0.27	2.7
$n = 6$	$\beta^2 = 0$	0.40	1.51	0.55	0.82	0.63	0.58
	$\beta^2 = 0.5$	0.32	2.1	0.50	1.0	0.59	0.7

This work shows quite clearly that the effect of any correlation between the yields, such as that introduced by variations of fertility within a block, or of any difference between the yield variance of different varieties tends to cause a significant deviation to be recognised when, in fact, none exists. When the number of varieties tested is three, the variation in the levels of significance may be quite large.

10. **Conclusion.** The mathematical model appropriate to Randomized Block Experiments is examined and it is suggested that the use of the z-test, as ordinarily applied, is theoretically justifiable only when the variations in fertility within each block are negligible.

Correlations between the yields of the varieties, due to randomization in a limited set, are introduced when the differences in fertility within each block are allowed for.

It is suggested that, as a first approximation, a multinormal population may be used for the yields from a given block, the variances and correlations being assumed equal from block to block, though the means, of course, differ.

The simultaneous distribution of the usual sums of squares is found in this case, and these sums of squares are shown to be independently distributed as the sums of squares of $s - 1$ and $(n - 1)(s - 1)$ quantities from another multinormal population.

It is shown that the usual distribution results apply when the variances and correlations of all the varieties are equal as well, of course, as when the variances are equal and the correlations zero. It is also shown that the same is true when the number of varieties is two, though the variances may differ.

The distributions of the above sums of squares are obtained for all values of s, the number of varieties, and the distribution of their ratio for $s = 3$. The method of obtaining the distribution of the ratio for $s > 3$ is also indicated.

The relative importance of the deviations from the usual distribution produced by differences in the variances and differences in the correlations is examined when $s = 3$, and it is found when the variances are all equal that the latter can produce deviations comparable to one variance being seven times the other two.

That the presence of the correlations or of non-equality of the variances causes a tendency for a significant difference to be found when none exists is clearly shown.

In conclusion, I must express my gratitude to Prof. J. Neyman, now of the University of California, for suggesting this problem to me, to Dr. R. C. Geary and to Prof. S. S. Wilks for valuable suggestions.

REFERENCES

[1] R. A. FISHER, *Jour. Minst. of Agric.*, Vol. 33 (1926) pp. 503–513.
[2] R. A. FISHER, *Statistical Methods for Research Workers*, Oliver & Boyd (6th Ed. 1936).
[3] R. A. FISHER, *The Design of Experiments*, Oliver & Boyd (1935).
[4] J. NEYMAN, *Suppl. Jour. Roy. Stat. Soc.*, Vol. 2 (1935) pp. 107–180.
[5] R. A. FISHER, *Rothamsted Conferences* XIII (1931).
[6] H. CRAMÉR, *Math. Zeitschrift*, Vol. 4 (1936) pp. 405–414.
[7] H. CRAMÉR, *Random Variables and Probability Distributions*, Camb. Univ. Press (1937).
[8] R. A. FISHER, *Proc. Intern. Math. Congr., Toronto*, Vol. 2, (1924) pp. 805-813.
[9] R. A. FISHER, *Econometrica*, Vol. 3 (1935) pp. 353–365.
[10] J. O. IRWIN, *Jour. Roy. Stat. Soc.*, Vol. 94 (1931) pp. 284-300.
[11] J. NEYMAN AND E. S. PEARSON, *Phil Trans. Roy. Soc. A.*, Vol. 231 (1933) pp. 289–337.
[12] J. NEYMAN AND E. S. PEARSON, *Proc. Camb. Phil. Soc.*, Vol. 29 (1933) 492–510.
[13] R. A. FISHER AND F. YATES, *Statistical Tables, for Biological, Agricultural, and Medical Research*, Oliver & Boyd (1938).
[14] G. W. SNEDECOR, *Calculation and Interpretation of Analysis of Variance and Covariance*, Collegiate Press, Inc., Ames, Iowa, (1934).
[15] K. PEARSON, *Tables of the Incomplete Beta Function*, Biometrika Office, London, (1934).
[16] J. O. IRWIN, *Suppl. Jour. Roy. Stat. Soc.*, Vol. 1 (1934) pp. 236-251.
[17] J. NEYMAN AND E. S. PEARSON, *Stat. Research Mem.* Vol. 1 (1936) pp. 1-37.
[18] J. NEYMAN, *Bull. de la Soc. Math. de France*, Vol. 63 (1935) pp. 246–266.
[19] J. NEYMAN, *Skandinavisk Aktuarietidskrift*, Vol. 20 (1931) pp. 149–199.
[20] E. S. PEARSON, *Biometrika*, Vol. 23 (1931) pp. 114–133.
[21] T. EDEN AND F. YATES, *Journ. Agric. Sci.*, Vol. 23 (1933) pp. 6–17.
[22] B. L. WELCH, *Biometrika*, Vol. 29 (1937) pp. 21–52.
[23] J. WISHART AND M. S. BARTLETT, *Proc. Camb. Phil. Soc.*, Vol. 29 (1933), pp. 260–270.
[24] A. C. AITKEN, *Quart. Journ. Maths.*, Vol. 2 (1931) pp. 130–135.
[25] W. G. COCHRAN, *Proc. Camb. Phil. Soc.*, Vol. 30 (1934) pp. 178–191.
[26] P. LEVY, *Calcul des Probabilités*, Gauthier-Villars (1925).
[27] J. V. USPENSKY, *Introduction to Mathematical Probability*, McGraw-Hill (1937).
[28] S. S. WILKS, *Statistical Inference*, Edwards Brothers Inc. (1937).
[29] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, Camb. Univ. Press (1922).
[30] W. N. BAILEY, *Proc. Lond. Math. Soc.*, Vol. 40 (1935) pp. 37–48.
[31] W. N. BAILEY, *Journ. Lond. Math. Soc.*, Vol. 11 (1936) pp. 16-20.
[32] W. N. BAILEY, *Generalized Hypergeometric Series*, Camb. Univ. Press (1937).

UNIVERSITY COLLEGE,
CORK, IRELAND.