where $c'(\theta) = c(\theta)/k(\theta)$ and $d\nu(x) = \phi_E(x)\, d\mu(x)$. Truncation has not changed the relative density function, and the result follows from the form of (1).

Next suppose that, instead of accepting values with probability one in E and with probability zero outside E, we select according to a fixed Borel function $\phi(x)$, the chance of accepting a value x being $\phi(x)$. The new family of distributions has the same sufficient statistics for the same reason.

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ON A PROBABILITY DISTRIBUTION

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1. Introduction. The problem treated is that of generalizing the Bernouilli distribution to the case where the probability of success is not constant from trial to trial but depends on the number of previous successes. The case where the probability of an event depends on the number of trials is easily handled and is not the case treated here. Several special cases of such a distribution have been worked out at one time or another. (E.g. C. C. Craig found the solution for one such special case and thus called the author's attention to the problem.)

The solution involves the Newton divided difference expansion of powers in a form which can be utilized for computation if the number of trials is not too large. In the case where the probabilities on a single trial are small an approximation, (similar to that of the Poisson distribution to the Bernouilli distribution) can be found.

Applications can obviously be made to urn schema in which black balls are replaced, but white balls are removed. Similarly, applications can be made to the distribution of the number of plants in a given area.

2. Solution of the problem. Specifically the problem is as follows: "What is the probability that in n trials of an event it will occur x times presuming that the probability of the event on a given trial depends only on the number of previous successes?" Denote by $P(n, x)$ the probability of x successes in n trials and by p_x the probability of the event after x previous successes. As conventional denote $q_x = 1 - p_x$ and one can formulate the following equation of partial differences:

$$P(n + 1, x + 1) = p_x P(n, x) + q_{x+1} P(n, x + 1). \tag{1}$$

This equation is an obvious consequence of the statement that $x + 1$ successes in $n + 1$ trials can only occur if there are x successes in n trials and a success on the $n + 1$st or $x + 1$ successes in n trials and failure on the $n + 1$st. The boundary conditions appropriate are:

$$P(n, x) = 0 \text{ for } x < 0, \text{ or } x > n \text{ and } P(0, 0) = 1. \tag{2}$$

It is convenient and appropriate to generalize (1) while retaining the boundary conditions (2). The equation (1) will be obtained from the following equation by setting $q = 1$:

$$P(n + 1, x + 1) = (q - q_x)P(n, x) + q_{x+1}P(n, x + 1). \tag{3}$$

It will be noted for further reference at this point that:

$$P(n, 0) = q_0^n \tag{4}$$

and:

$$P(n, n) = (q - q_0)(q - q_1) \cdots (q - q_{n-1}). \tag{5}$$

This last suggests a change of variable of the form:

$$P(n, x) = F(n, x)(q - q_0)(q - q_1) \cdots (q - q_x). \tag{6}$$

Upon substituting this expression in (3) one obtains a somewhat simpler equation with the same boundary conditions as (2).

$$F(n + 1, x + 1) = F(n, x) + q_{x+1}F(n, x + 1). \tag{7}$$

Using the generating function:

$$G(x, \xi) = \sum_{n=x}^{\infty} F(n, x)\xi^n \tag{8}$$

one may obtain from (7), using the boundary conditions (2) the following ordinary linear difference equation:

$$G(x + 1, \xi) = \xi[G(x, \xi) + q_{x+1}G(x + 1, \xi)]. \tag{9}$$

From (4) it is easily seen that:

$$G(0, \xi) = 1/[1 - q_0\xi], \tag{10}$$

and hence that the solution of (9) is:

$$G(x, \xi) = \xi^x/[(1 - q_0\xi)(1 - q_1\xi) \cdots (1 - q_x\xi)]. \tag{11}$$

This may be expanded in partial fractions and the result written:

$$G(x, \xi) = \xi^x \sum_{i=0}^{x} q_i^x/[(q_i - q_0) \cdots (q_i - q_{i-1})(q_i - q_{i+1}) \cdots (q_i - q_x)(1 - q_i\xi)]. \tag{12}$$

By means of the relation in (8) one deduces readily that:

$$(13)\quad F(n, x) = \sum_{i=0}^{x} q_i^x / [(q_i - q_0) \cdots (q_i - q_{i-1})(q_i - q_{i+1}) \cdots (q_i - q_x)].$$

Jordan [1, p. 19, eq. (1)] shows this to be the xth Newton divided difference of q^n where the expansion is in terms of $(q - q_0) \cdots (q - q_x)$, for $x = 0, 1, \cdots, n$. The solution for (3) can now be written as:

$$(14)\qquad P(n, x) = (q - q_0) \cdots (q - q_{x-1}) F_n(x)$$

from which follows:

$$(15)\qquad \sum_{x=0}^{n} P(n, x) = q^n.$$

As remarked before, by setting $q = 1$ one obtains the solution of (1) subject to the boundary conditions (2).

It is clear that when all the q_i are equal that the Bernouilli distribution should come out as a special case. Since in this case the divided difference becomes the corresponding derivative divided by the appropriate factorial, one obtains:

$$(16)\qquad P(n, x) = \frac{(1 - q_0)^x}{x!} \frac{d^x q^n}{dq^x}\bigg|_{q=q_0}.$$

Upon reduction this yields the usual formula, but not in the usual way.

By choosing $p_x = \lambda_x / n$ and allowing n to increase without limit one obtains an analogue of the Poisson distribution, viz:

$$(17)\quad P(x) = (-\lambda_0) \cdots (-\lambda_x) \sum_{i=0}^{x} e^{-\lambda_i} / [(\lambda_0 - \lambda_i) \cdots (\lambda_{i-1} - \lambda_i)(\lambda_{i+1} - \lambda_i) \cdots (\lambda_x - \lambda_i)]$$

which corresponds to the expansion of $e^{-\lambda}$ about $\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_x, \cdots$ when $\lambda = 0$.

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[1] Charles Jordan, *Calculus of Finite Differences*, Chelsea Publishing Co., New York, 2nd ed., 1947.

A GRAPHICAL DETERMINATION OF SAMPLE SIZE FOR WILKS' TOLERANCE LIMITS

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1. Summary. To determine the smallest sample size for which the minimum and the maximum of a sample are the $100\beta\%$ distribution-free tolerance limits at the probability level ϵ, one has to solve the equation

$$(1)\qquad N\beta^{N-1} - (N - 1)\beta^N = 1 - \epsilon$$