TABLE OF THE ASYMPTOTIC DISTRIBUTION OF THE SECOND EXTREME

By E. J. Gumbel and J. Arthur Greenwood

New York City and Manhattan Life Insurance Company

The asymptotic distributions of the extreme values taken from an initial distribution of the exponential type are now widely used, for example in flood control [6] and in problems connected with the breaking strength of material [1]. Therefore, the corresponding distribution of the penultimate (and of the second) value may also be of practical interest.

Let $F(x)$ be the initial probability; let $f(x) = F'(x)$ be the initial density (distribution). Let n be a large sample size; let the rank $m(m << n)$ be counted from the top. Finally, let the parameters u_m and α_m be defined as the solutions of

(1) $$F(u_m) = 1 - m/n; \qquad \alpha_m = nf(u_m)/m.$$

Then the asymptotic distribution $\varphi_m(x_m)$ of the mth largest value x_m is [2]

$$\varphi_m(x_m) = \frac{m^m}{\Gamma(m)} \alpha_m \exp[-my_m - me^{-y_m}],$$

where

$$y_m = \alpha_m(x_m - u_m),$$

provided that the initial distribution is of the exponential type. The asymptotic distribution ${}_m\varphi({}_mx)$ of the mth smallest value is

$${}_m\varphi({}_mx) = \varphi_m(-x_m).$$

The probability function $\Phi_m(x_m)$ is obtained from

$$\Phi_m(x_m) = \frac{m^m}{\Gamma(m)} \int_{-\infty}^{y_m} \exp[-my - me^{-y}]\, dy$$
$$= \frac{1}{\Gamma(m)} \int_{me^{-y_m}}^{\infty} z^{m-1} e^{-z}\, dz,$$

whence

(2) $$\Phi_m(x_m) = 1 - I(t_m, m - 1),$$

where

$$t_m = \sqrt{m}\, e^{-y_m}$$

and I is the incomplete Gamma function ratio of Karl Pearson [5]. In the special case $m = 2$, the probability function of the penultimate value is

(3) $$\Phi_2(x_2) = 1 - I(\sqrt{2}\, e^{-y_2}, 1).$$

The modal penultimate value is, of course, u_2, and the intervals corresponding

TABLE I

Probability $\Phi_2(y_2)$ of the penultimate value y_2

y_2	Φ_2	δ^2	y_2	Φ_2	δ^2	y_2	Φ_2	δ^2
−1.95	.00001		0.55	.67935	− 89	3.05	.99579	− 4
−1.90	.00002		0.60	.69990	91	3.10	.99618	− 4
−1.85	.00004		0.65	.71954	91	3.15	.99653	
−1.80	.00007		0.70	.73827	92	3.20	.99685	
−1.75	.00013		0.75	.75608	− 92	3.25	.99714	
−1.70	.00021	+ 5	0.80	.77297	90	3.30	.99741	
−1.65	.00034	7	0.85	.78896	89	3.35	.99765	
−1.60	.00054	10	0.90	.80406	87	3.40	.99787	
−1.55	.00084	14	0.95	.81829	85	3.45	.99807	
−1.50	.00128	+ 17	1.00	.83167	− 81	3.50	.99825	
−1.45	.00189	24	1.05	.84424	80	3.55	.99841	
−1.40	.00274	30	1.10	.85601	75	3.60	.99856	
−1.35	.00389	38	1.15	.86703	74	3.65	.99869	
−1.30	.00542	47	1.20	.87731	69	3.70	.99882	
−1.25	.00742	+ 56	1.25	.88690	− 65	3.75	.99893	
−1.20	.00998	68	1.30	.89584	64	3.80	.99903	
−1.15	.01322	77	1.35	.90414	59	3.85	.99912	
−1.10	.01723	89	1.40	.91185	55	3.90	.99920	
−1.05	.02213	100	1.45	.91901	54	3.95	.99928	
−1.00	.02803	+ 110	1.50	.92563	− 49	4.00	.99935	
−0.95	.03503	121	1.55	.93176	46	4.05	.99941	
−0.90	.04324	129	1.60	.93743	44	4.10	.99946	
−0.85	.05274	135	1.65	.94266	40	4.15	.99951	
−0.80	.06359	142	1.70	.94749	39	4.20	.99956	
−0.75	.07586	+ 145	1.75	.95193	− 34	4.25	.99960	
−0.70	.08958	147	1.80	.95603	34	4.30	.99964	
−0.65	.10477	145	1.85	.95979	29	4.35	.99967	
−0.60	.12141	142	1.90	.96326	29	4.40	.99970	
−0.55	.13947	138	1.95	.96644	27	4.45	.99973	
−0.50	.15891	+ 130	2.00	.96935	− 23	4.50	.99976	
−0.45	.17965	121	2.05	.97203	23	4.55	.99978	
−0.40	.20160	111	2.10	.97448	20	4.60	.99980	
−0.35	.22466	99	2.15	.97673	19	4.65	.99982	
−0.30	.24871	86	2.20	.97879	18	4.70	.99984	
−0.25	.27362	+ 71	2.25	.98067	− 16	4.75	.99985	
−0.20	.29924	57	2.30	.98239	14	4.80	.99987	
−0.15	.32543	44	2.35	.98397	15	4.85	.99988	
−0.10	.35206	29	2.40	.98540	12	4.90	.99989	
−0.05	.37898	11	2.45	.98671	11	4.95	.99990	
0	.40601	+ 1	2.50	.98791	− 11	5.00	.99991	
0.05	.43305	− 13	2.55	.98900	9	5.05	.99992	
0.10	.45996	25	2.60	.99000	9	5.10	.99993	
0.15	.48662	37	2.65	.99091	8	5.15	.99993	
0.20	.51291	47	2.70	.99174	8	5.20	.99994	
0.25	.53873	− 56	2.75	.99249	− 6	5.25	.99995	
0.30	.56399	65	2.80	.99318	7	5.35	.99996	
0.35	.58860	72	2.85	.99380	5	5.50	.99997	
0.40	.61249	77	2.90	.99437	5	5.65	.99998	
0.45	.63561	82	2.95	.99489	5	5.90	.99999	
0.50	.65791	− 86	3.00	.99536	− 4	6.45	1.00000	

TABLE II

Probability points

$\Phi_2(y_2)$	y_2	$\Phi_2(y_2)$	y_2
.005	−1.31239	.995	2.96138
.010	−1.19972	.990	2.59995
.025	−1.02454	.975	2.11110
.050	−0.86371	.950	1.72777
.100	−0.66519	.900	1.32461
.250	−0.29737	.750	0.73264
.500	0.17534		

to the probabilities $P_1 = 0.68269$, $P_2 = 0.95445$, and $P_3 = .99730$ are $y_2 = \pm 0.75409$, $y_2' = 1.78196$, $y_2'' = 3.27883$, respectively.

The present five-decimal table was computed by interpolation in Pearson's table. The last six lines indicate the first values of y_2 for which Φ_2 differs from the value indicated by less than $5 \cdot 10^{-6}$. The table was checked by differencing and by comparison with the short table of percentage points (Table II) which was

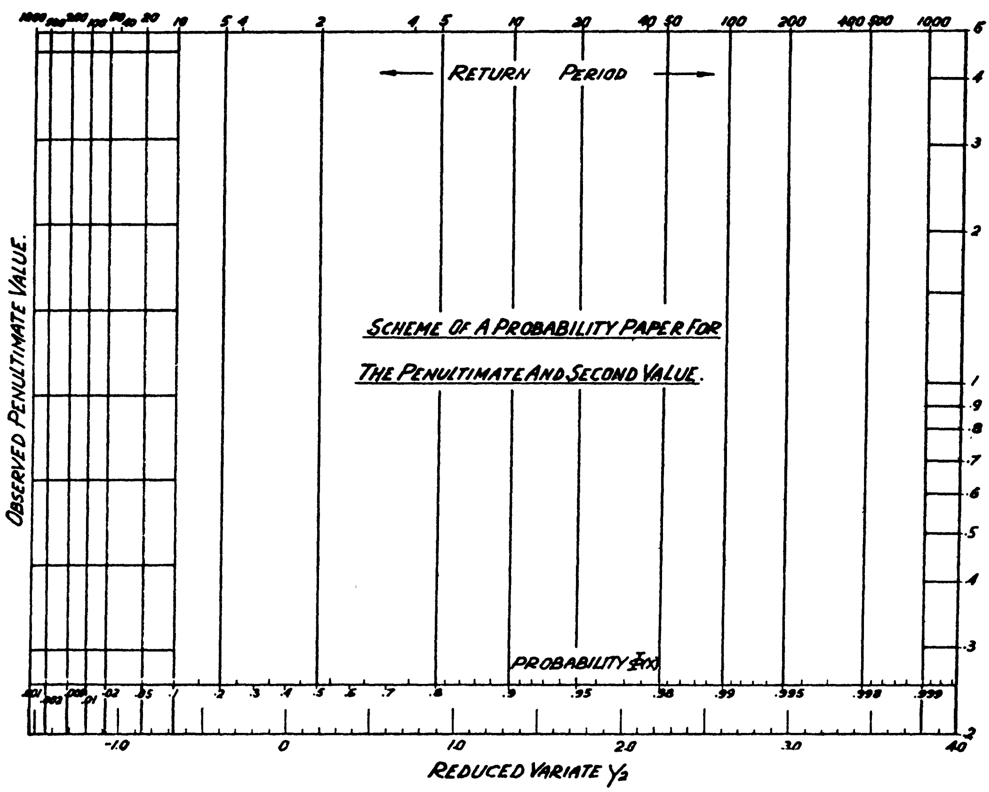

FIGURE 1

computed by noting that

$$2me^{-y_m}$$

has the χ^2 distribution with $2m$ degrees of freedom, and so transforming the percentage points given by Thompson [7] (pp. 188–189, line $y = 4$), setting $y_2 = \ln 4 - \ln \chi^2$.

More decimal places may be obtained by direct substitution in (3), by use of the relation

$$\Phi_2(x_2) = \Phi_1(z) + \varphi_1(z), \tag{4}$$

where $z = y_2 - \ln 2$ and Φ_1 and φ_1, respectively, the probability and density of the largest value, are given in a seven-decimal table originally calculated by Greenwood [4], and from the nine-decimal table of $(x + 1)e^{-x}$ by Miller and Rosebrugh [3], pp. 80–101, where

$$x = 2e^{-y_2}.$$

Table I is basic for the construction of a probability paper (Figure 1) for the penultimate value which can be used in the same way as the probability paper of the largest value [6]. If the variate is replaced by its logarithm, the paper may be applied to the penultimate value taken from a distribution of the Cauchy type. Finally, if the probability Φ_2 is replaced by $1 - \Phi_2$, the paper may serve for the second value.

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