DISTRIBUTION OF THE MEASURE OF A RANDOM TWO-DIMENSIONAL SET

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1. Summary. This paper considers the distribution of the measure of a special random two-dimensional set. Related work, usually motivated by a search for principles for bombing operations, deals exclusively with moment problems and appears in [1], [3], [4], [5], [6], [7], [8]. A one-dimensional distribution problem appears in [2]. The random set considered is the intersection of a fixed circle with the union of N random circles. Centers of the random circles are subject to the variability imposed by the bivariate normal distribution with circular symmetry and means not necessarily coincident with the coordinates of the center of the fixed circle. The measure of interest is the ratio of the area of the intersection ("covered area") to the total area of the fixed circle. For $N = 1$, the distribution is determined and its use facilitated by the graphs in Fig. 1 and Fig. 2. A procedure for obtaining upper and lower bounds of the distribution for $N = 2$ is given. Tables I, II, III, and IV give upper and lower bounds for the percentage points of the distribution for $N = 2$ for some special illustrative situations. For N = 1 in all situations, and for $N = 2$ in many situations; the graphs and tables demonstrate that a realistic decision can be made rather easily without resorting to the usual practice of random number "Monte Carlo" devices for each ad hoc situation of interest.

2. Development of distribution. Consider a fixed circle of radius T and an aiming point at a distance R from the center of the fixed circle at which are dropped N random circles of equal radius W according to the aforementioned bivariate normal distribution specified by the parameter σ. Define $c(0 \leqq c \leqq 1)$ as the fraction coverage, that is, the ratio of the area of the intersection to the total area of the fixed circle. We are interested in finding

$$P_C = \Pr\{c \geqq C/W, T, R, \sigma, N\}$$

where P_C is the probability of getting at least C fraction coverage for specified values of the parameters: W, T, R, σ, N. In order to achieve C coverage for $N = 1$, the center of the random circle must fall on or within the circle having the same center as the fixed circle and a radius $R^* = W + aT$. The relationship between a and c is developed by integration from the geometry of the picture and is

$$c = \tfrac{1}{2}(1 + S^2) - \frac{1}{\pi}[f(\alpha) - S^2 f(\beta)]$$

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where $S = W/T$, $f(x) = \text{arc} \sin x + x\sqrt{1 - x^2}$, $2\alpha = v + (1 - S^2)/v$, $2\beta = -v/S + (1 - S^2)/vS$ and $v = S + a$. This relationship is graphically depicted in Fig. 2.

The problem, then, of finding P_C is equivalent to the probability of getting a point in an offset circle whose radius, center, and distance between center and aiming point are known. From the bivariate normal law assumption, the probability, dp, that the center of the random circle will fall within an area, dA, at a distance, D, from the aiming point is $dp = (2\pi\sigma^2)^{-1} \exp(-D^2/2\sigma^2)\, dA$. Since the center of the fixed circle is at a distance R from the aiming point, if we choose polar coordinates ρ, θ about the former, then $D^2 = R^2 + \rho^2 - 2\rho R \cos\theta$ and we may write

$$p = \frac{1}{2\pi\sigma^2} \int_0^{R^*} \int_0^{2\pi} \exp\left[-\left(\frac{R^2 + \rho^2 - 2\rho R \cos\theta}{2\sigma^2}\right)\right] \rho\, d\rho\, d\theta$$

or

$$p = \int_0^{R^*} \rho I_0(\rho R) \exp\left[-\left(\frac{R^2 + \rho^2}{2}\right)\right] d\rho$$

where R^* and R are now in σ units; $I_n(x)$ is the modified Bessel function of the first kind for order n and argument x.

It is useful in graphically depicting the distribution to have the slopes of the constant contours of probability in the RR^* plane. If we define $q(R^*, R)$ by $q(R^*, R) + p(R^*, R) = 1$, then by the theorem or implicit functions we have

$$\frac{\partial R^*}{\partial R} = -\frac{\dfrac{\partial q}{\partial R}}{\dfrac{\partial q}{\partial R^*}} = \frac{\int_{R^*}^{\infty} [\rho^2 I_1(\rho R) - \rho R I_0(\rho R)] \exp[-\tfrac{1}{2}(\rho^2 + R^2)]\, d\rho}{R^* I_0(RR^*) \exp[-\tfrac{1}{2}(R^{*2} + R^2)]}$$

since $d[I_0(z)]/dz = I_1(z)$. Integrating the numerator by parts and making use of the equality

$$\rho R I_1'(\rho R) = -I_1(\rho R) + \rho R I_0(\rho R)$$

where the prime refers to differentiation with respect to (ρR) we get after simplification, the interesting relationship $dR^*/dR = I_1(R^*R)/I_0(R^*R)$. Since for large x $I_n(x) \sim e^x/\sqrt{2\pi x}$ we get for large (R^*R) $dR^*/dR \sim 1$. Hence the family of curves in the RR^* plane defined by $P_C = \text{constant}$ (see Fig. 1) has a slope which approaches unity quite rapidly, for from [9] and [10] we see

$$\frac{I_1(5)}{I_0(5)} = .89 \qquad \frac{I_1(15)}{I_0(15)} = .97 \qquad \frac{I_1(37)}{I_0(37)} = .99.$$

Since for $R = 0$, $p = 1 - \exp(-\tfrac{1}{2}(R^{*2}))$ it becomes possible to construct the contours of equal probability because the initial point and the slopes of all points on any one contour are known. The unabridged printed listings mentioned in [11] however were used in constructing the contours.

3. Use of the graphs. Figures 1 and 2 represent the distribution for $N = 1$. As an illustration, suppose $W = 1$, $T = 1$, $R = 1$ (σ units) and $P_{.35}$ is desired. Fig. 2 shows that $a = .075$ for $C = .35$, $S = 1$. Thus $W + aT = 1.075$ and referring this to Fig. 1 we get $P_{.35} = .30$. Conversely, let $W = 3$, $T = 2$, $R = 2$, (σ units), $P_C = .50$ and C is desired. In Fig. 1, for $P_C = .50$ and $R = 2$, we get $W + aT = 2.25$, then $a = -0.375$. Then referring to Fig. 2 with $S = 1.5$, we get $C = .67$.

4. Upper and lower bounds. Let $\underline{P}(C, N)$ and $\bar{P}(C, N)$ represent lower and upper bounds for the probability of obtaining C coverage or better when N random circles are dropped for a specified set of parameters: T, W, R, σ. Then certainly

$$\underline{P}(C, N) = 1 - (1 - P_C),^N$$

$$\bar{P}(C, N) = P\{c_1 + c_2 + \cdots + c_N \geqq C\},$$

where the c_i are the coverage random variables for $N = 1$, and $P_C = P\{c_i \geqq C\}$. Now each c_i has a distribution which is a combination of a continuous distribution with density $f(c)$ at $[0, 1]$ plus discrete probabilities at $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ if $W \geqq T$, and at $\begin{bmatrix} W^2/T^2 \\ 0 \end{bmatrix}$ if $W < T$; where

$$f(c) = -\frac{d}{dc} P_c, \qquad g(c) = W + a(c) \cdot T,$$

$$P_c = \int_0^{g(c)} \rho I_0(\rho R) \exp\left[-\left(\frac{\rho^2 + R^2}{2}\right)\right] d\rho.$$

Then $f(c) = -g'(c) \cdot g(c) I_0[g(c) \cdot R] \exp -\frac{1}{2}\{[g(c)]^2 + R^2\}$ where $g'(c) = dg(c)/dc$ is never positive. A glance at Fig. 2 will demonstrate this. Thus for $N = 2$, we get

$$\bar{P}(C, 2) = P_0^2 - \int_0^C f(c_1)\, dc_1 \int_0^{C-c_1} f(c_2)\, dc_2 + 2(1 - P_0)P_C.$$

This reduces to

$$\bar{P}(C, 2) = 2P_C - P_C P_0 + \int_0^C f(c_1) P_{C-c_1}\, dc_1.$$

Thus $\bar{P}(C, 2)$ is easily determined by numerical integration since $f(c)$ and P_{C-c} can be computed for any value of c. To compute $f(c)$, it will be necessary to find $g'(c) = a'(c) \cdot T$ where the prime refers to differentiation with respect to c. To find $a'(c)$ we first determine

$$\frac{d}{dc}(\alpha) = \frac{1}{2} \frac{da}{dc}\left[1 + \frac{S^2 - 1}{v^2}\right],$$

$$\frac{d}{dc}(\beta) = \frac{1}{2} \frac{da}{dc}\left\{\frac{1}{S}\left[\frac{S^2 - 1}{v^2} - 1\right]\right\},$$

where S, α, β, are as defined previously. Then

$$\pi = \frac{da}{dc}\left\{-\sqrt{1-\alpha^2}\left[1+\frac{S^2-1}{v^2}\right]+S\sqrt{1-\beta^2}\left[\frac{S^2-1}{v^2}-1\right]\right\};$$

when $S = 1$, this reduces to

$$a'(c) = -\frac{\pi}{2}\left[\frac{1}{\sqrt{1-\left(\frac{1+a}{2}\right)^2}}\right].$$

In the numerical integration trouble results in the interval from 0 to Δc because $g'(c)$ has a discontinuity at $c = 0$. However since we are actually interested in an upper bound we can replace $\int_0^{\Delta c_1} f(c_1)P_{c-c_1}\,dc_1$ by

$$\left[-\int_0^{\Delta c_1}\exp\left\{-\frac{[g(c_1)]^2+R^2}{2}\right\}d\left\{\frac{[g(c_1)]^2}{2}\right\}\right]\cdot\left[\cdot\max_{0\text{ to }\Delta c_1} P_{c-c_1}I_0[g(c_1)\cdot R]\right]$$

or

$$\exp\left\{-\frac{[g(c_1)]^2+R^2}{2}\right\}\Bigg]_0^{\Delta c_1}\cdot\left[\max_{0\text{ to }\Delta c_1} P_{c-c_1}I_0[g(c_1)\cdot R]\right]$$

and thus avoid this difficulty.

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(Figs. 1–2 and Tables I–IV follow)

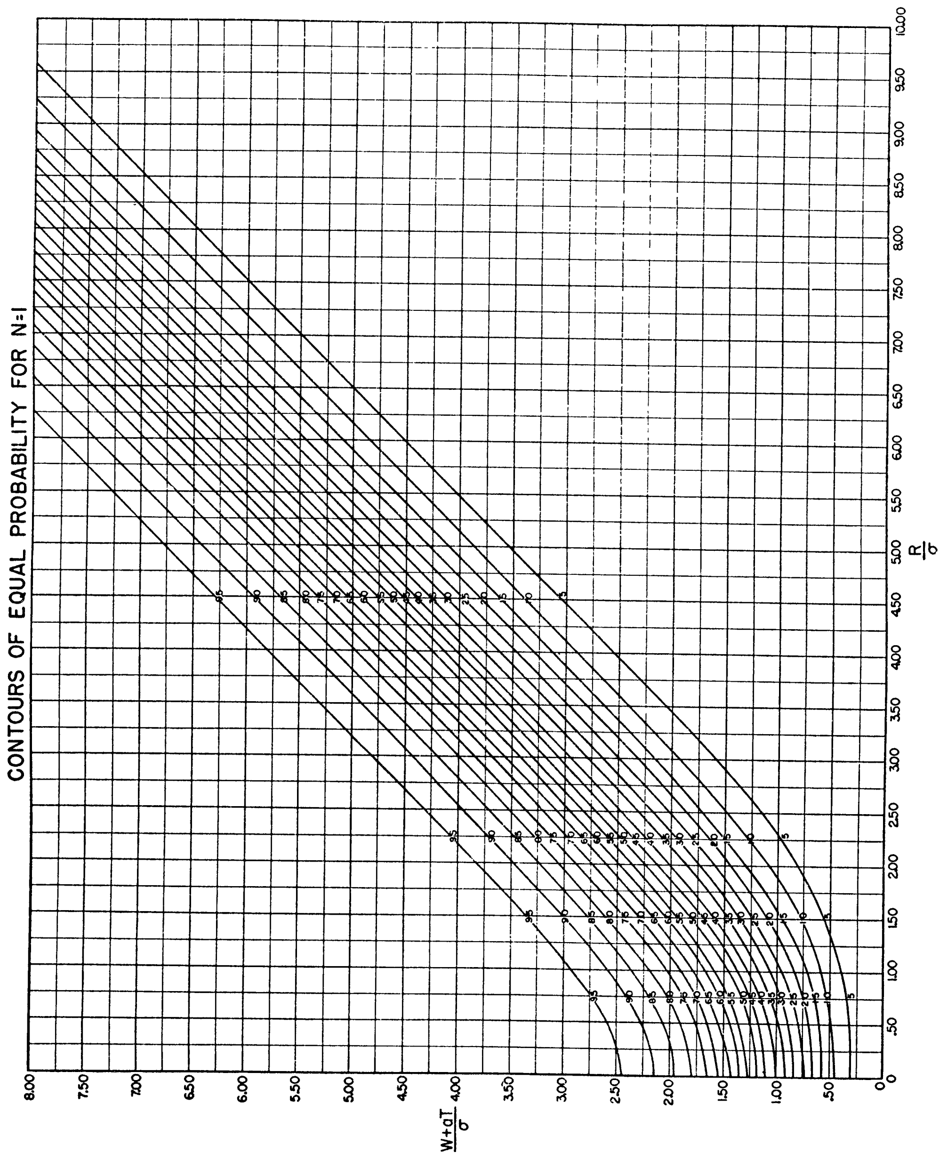
CONTOURS OF EQUAL PROBABILITY FOR N=1
$\frac{W+aT}{\sigma}$
$\frac{R}{\sigma}$

FIG. 1

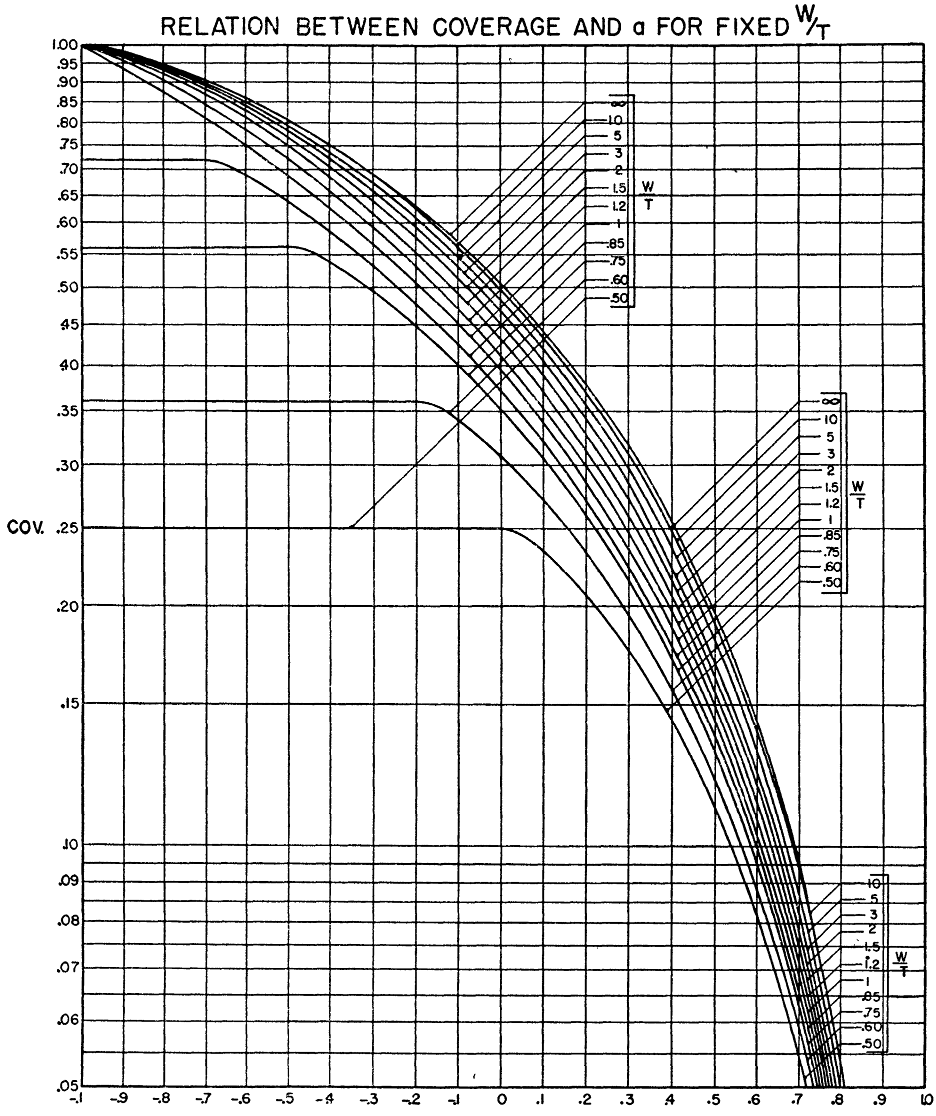
RELATION BETWEEN COVERAGE AND a FOR FIXED W/T
COV.
1.00
.95
.90
.85
.80
.75
.70
.65
.60
.55
.50
.45
.40
.35
.30
.25
.20
.15
.10
.09
.08
.07
.06
.05
-.1
-.9
-.8
-.7
-.6
-.5
-.4
-.3
-.2
-.1
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
a
∞
10
5
3
2
1.5
1.2
1
.85
.75
.60
.50
W/T

Fig. 2

TABLE I

Lower and upper bounds for probability of coverage

$W/\sigma = 1, \quad T/\sigma = 1, \quad R = 0, \quad N = 2$

C	.1	.2	.3	.4	.5	.6	.7	.8	.9
$\underline{P}_C$	.926	.853	.733	.619	.483	.329	.208	.098	.059
$\bar{P}_C$	.937	.888	.796	.726	.638	.533	.440	.342	.275

TABLE II

Lower and upper bounds for probability of coverage

$S = 1, \quad C = .5, \quad N = 2$

$\frac{T}{\sigma} = \frac{W}{\sigma}$	.5	1	1	1	2	2	2	2	3	3	3	3
$\frac{R}{\sigma}$	0	0	1	2	0	1	2	3	0	1	2	3
$\underline{P}_{.5}$	.159	.483	.339	.098	.924	.798	.445	.107	.997	.978	.815	.407
$\bar{P}_{.5}$	.185	.638	.445	.122	.993	.947	.727	.240	1.000	1.000	.990	.858

TABLE III

Lower and upper bounds for probability of coverage

$S = 2, \quad C = .5, \quad N = 2$

$\frac{W}{\sigma}$	1	2	2	2	2	3	3	3	3	3
$\frac{T}{\sigma}$	.5	1	1	1	1	1.5	1.5	1.5	1.5	1.5
$\frac{R}{\sigma}$	0	0	1	2	3	0	1	2	3	4
$\underline{P}_{.5}$	.609	.974	.916	.609	.190	.999	.996	.935	.609	.190
$\bar{P}_{.5}$	.646	.987	.963	.691	.248	1.000	.998	.983	.752	.280

TABLE IV

Lower and upper bounds for probability of coverage

$S = .5, \quad C = .2, \quad N = 2$

$\frac{W}{\sigma}$	1	1	1	1	1.5	1.5	1.5	1.5	1.5
$\frac{T}{\sigma}$	2	2	2	2	3	3	3	3	3
$\frac{R}{\sigma}$	0	1	2	3	0	1	2	3	4
$\underline{P}_{.2}$	.878	.737	.360	.069	.990	.953	.708	.278	.040
$\bar{P}_{.2}$	.975	.881	.500	.418	1.000	.996	.922	.564	.118