INCOMPLETE SUFFICIENT STATISTICS AND SIMILAR TESTS[1]

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0. Summary. For a family of exponential densities a method is given, called "D method," for constructing a class of similar tests in the case that the minimal sufficient statistic is boundedly incomplete. This method also provides a proof of a criterion for bounded incompleteness. Under certain conditions the criterion states that a sufficient statistic for a family of exponential densities is boundedly incomplete if the number of components of the statistic is larger than the number of parameters specifying the distribution. Applications are indicated in the Behrens-Fisher problem, and in the problem of testing the ratio of mean to standard deviation in a normal population. In the latter problem it is shown that the D method generates the whole class of similar tests. Some unsolved problems concerning the existence of an optimal similar test are indicated.

1. Introduction. Lehmann and Scheffé [8], [9] have introduced the concept of completeness of a family of measures and have shown the usefulness of this notion both for unbiased estimation and for the construction of similar regions. The latter were introduced by Neyman and Pearson [11] as a means to cope with tests of composite hypotheses. If the hypothesis is composite only because of nuisance parameters, then the requirement of similarity of the test is often a convenient means of restricting the class of tests to be considered. If the hypothesis is composite both of nuisance parameters and because the parameter tested is not completely specified by the hypothesis, then similarity is often required if the test is to be unbiased. For instance, let θ be a real parameter, τ a possibly vector valued nuisance parameter, and let the hypothesis be $H:\theta \leqq \theta_0$, the alternative $\bar{H}:\theta > \theta_0$, for some specified θ_0. Suppose we want the test to be unbiased, then the power function of the test has to be $\leqq \alpha$ for $\theta \leqq \theta_0$ and $\geqq \alpha$ for $\theta > \theta_0$, where α is the level of significance. If, in addition, the power function is continuous, which is usually the case, then we have automatically that its value on the surface $\theta = \theta_0$ equals α, identically in τ. Search for an optimum unbiased test reduces then to the simpler problem of search for an optimum similar test of the hypothesis $H_1:\theta = \theta_0$ against $\bar{H}:\theta > \theta_0$.

In the presence of a sufficient statistic there exists a special class of easily constructible similar regions [10], termed *similar regions of Neyman structure* by Lehmann and Scheffé [8]. They proved that every similar region is of Neyman structure if and only if the family of distributions of the sufficient statistic, as specified by the hypothesis, is boundedly complete [8]. Unfortunately, there

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are important problems in which the latter condition is not fulfilled, in which case the class of all similar regions is larger than the class of similar regions of Neyman structure. An example is the Behrens-Fisher problem (see, for example, [13], in which also references to earlier work can be found). In this problem the similar regions of Neyman structure are of no use, since for any such region the power function is identically constant.

All remarks in the previous paragraph are equally valid if instead of similar rejection regions we consider randomized similar tests. It is clear from the discussion that in each problem of testing a composite hypothesis by means of a similar test it is important to know whether or not the problem admits a boundedly complete sufficient statistic. If not, one would like to have a method of constructing all similar tests. It is the purpose of this paper to provide partial answers to these problems. In section 3 a method termed the "D method," will be given for the construction of a large class of similar tests in the case of a family of exponential densities. In section 5 the D method will be used to derive a criterion for bounded incompleteness in the case of a family of exponential densities. Two examples of the D method are given in section 4; the first example is the Behrens-Fisher problem, the second example is the problem of testing the ratio of mean to standard deviation in a normal population. For the latter problem it is proved in section 6 that every similar test can be constructed by the D method, provided this method is given sufficiently wide scope. Some remarks on the problem of finding an optimal similar test are made in section 7. A preliminary account of the results of sections 3 and 5 appeared in [16].

2. Similar tests and boundedly incomplete sufficient statistics. Let $\mathfrak{X}$ be a space of points x, $\mathfrak{A}$ a σ-field of subsets of $\mathfrak{X}$ (with $\mathfrak{X} \varepsilon \mathfrak{A}$), and $\mathcal{P} = \{P_\theta, \theta \varepsilon \Omega\}$ a family of probability measures on $(\mathfrak{X}, \mathfrak{A})$. Expectation with respect to P_θ will be denoted by E_θ. If $\omega \subset \Omega$ and T is a sufficient statistic for $\mathcal{P}_\omega = \{P_\theta, \theta \varepsilon \omega\}$, we shall also say that T is a sufficient statistic for ω. The range of T is denoted by $\mathfrak{I}$, and is understood to be a Borel subset of a Euclidean space. Let $\mathfrak{B}$ be the σ-field of Borel subsets of $\mathfrak{I}$. We recall the following definitions: A sufficient statistic for ω is called *minimal* if the sufficient sub σ-field which it induces in $\mathfrak{X}$ is "essentially" contained in every sufficient sub σ-field for ω (see Bahadur [2] for a precise definition).[3] A sufficient statistic T for ω is called *complete* for ω if, for every $\mathfrak{B}$-measurable numerical function g

$$(1) \qquad E_\theta g(T) = 0 \quad \text{for all} \quad \theta \varepsilon \omega \Rightarrow g \equiv 0 \quad \text{a.e.} \quad (\mathcal{P}_\omega).$$

If the implication (1) holds for every *bounded* $\mathfrak{B}$-measurable numerical function, then T is called *boundedly complete* for ω. The following implications are true [8].

$$(2) \qquad \text{Completeness} \Rightarrow \text{bounded completeness} \Rightarrow \text{minimality.}$$

[3] The term *minimal* was introduced by Lehmann and Scheffé [8], whereas Bahadur [2] describes the same concept with the term *necessary and sufficient* statistic.

Suppose a composite hypothesis H specifies $\theta \in \omega \subset \Omega$. We shall consider randomized tests for H with test functions ϕ, where, for each $x \in \mathfrak{X}$, $0 \leqq \phi(x) \leqq 1$, ϕ measurable, and H is rejected with probability $\phi(x)$ if x is observed. Among all tests we restrict ourselves to similar tests, defined by the condition that $E_\theta \phi$ is independent of θ if $\theta \in \omega$. If T is a sufficient statistic for ω, $\mathcal{A}_0 \subset \mathcal{A}$ its sufficient sub σ-field and ϕ any test, we can consider the $\mathcal{A}_0$-measurable function $E(\phi \mid \mathcal{A}_0)$. If α is a number, $0 < \alpha < 1$, and if ϕ is such that $E(\phi \mid \mathcal{A}_0) \equiv \alpha$, then clearly $E_\theta \phi = \alpha$ for all $\theta \in \omega$, so that ϕ is similar. Such a ϕ is called a *test of Neyman structure* [8]. If T is a boundedly complete sufficient statistic for ω, then every similar test has Neyman structure [8]. On the other hand, if a sufficient statistic T is not boundedly complete for ω, then there exist similar tests which do not have Neyman structure. This follows from the fact that the bounded incompleteness implies the existence of a $\mathcal{B}$-measurable numerical function g on $\mathfrak{J}$, bounded below by $-\alpha$, above by $1 - \alpha$, different from 0 on a set of positive probability (with respect to $\mathcal{P}_\omega$), with $E_\theta g(T) = 0$ for all $\theta \in \omega$. With f on $\mathfrak{X}$ defined by $f(x) = g(T(x))$, we have that $\phi = f + \alpha$ is similar of size α, but $E(\phi \mid \mathcal{A}_0) - \alpha = f \not\equiv 0$ on a set of positive probability, so ϕ is not a test of Neyman structure. Conversely, for any similar test ϕ we can form the function $f = E(\phi \mid \mathcal{A}_0) - \alpha$ and define g on $\mathfrak{J}$ by $g(T(x)) = f(x)$, so that

$$E_\theta g(T) = 0$$

for all $\theta \in \omega$. It follows that all similar tests can be found by constructing all bounded numerical functions g on $\mathfrak{J}$ whose expectations vanish for all $\theta \in \omega$.

3. The D method for constructing similar tests in the case of a family of regular exponential densities. In this section the restriction of θ to ω will be understood. Let the distribution of T, induced by P_θ, have a density with respect to m-dimensional Lebesgue measure, and let this density p_θ be of the form

$$p_\theta(t) = c(\theta) \exp\left[-\sum_{i=1}^{m} s_i(\theta) t_i\right] h(t) \tag{3}$$

in which $t = (t_1, \cdots, t_m)$, and $s_1, \cdots, s_m$ are real valued functions on ω. We shall assume that the function h is of such a nature that it is possible to find a closed m-dimensional cube C on which h is bounded away from 0. With this restriction on h, the family (3) will be called a family of *regular exponential densities*. Exponential densities which arise in statistics are always regular.

If ω is an m-dimensional subset of an m-dimensional Euclidean space, then, under mild conditions, T with density (3) is complete for ω [9]. In that case every similar test has Neyman structure. From the point of view of the present paper the interesting case arises when ω is a subset of an $m - 1$ dimensional Euclidean space. In that case θ has at most $m - 1$ components, so that the m functions s_i are functions of at most $m - 1$ parameters. Eliminating those parameters will result in a functional relation between the s_i. Suppose that this relation can be put in the form

$$P(s_1, \cdots, s_m) = 0 \tag{4}$$

in which P is a polynomial of positive degree in at least one of the s_i. It should be kept in mind that (4) holds identically in θ.

As discussed in section 2, a similar test of non-Neyman structure can be constructed by constructing a bounded function g on $\mathfrak{Z}$, $g \not\equiv 0$ on a set of positive probability, such that

$$\int g(t)p_\theta(t)\,dt \underset{\theta}{\equiv} 0 \tag{5}$$

Using (3), remembering that h is bounded away from 0 on some m-dimensional cube C, it suffices to construct a bounded function F which is $\not\equiv 0$ on a subset of C of positive Lebesgue measure, vanishes outside C, and satisfies

$$\int F(t)\exp\left[-\sum_i^m s_i(\theta)t_i\right]dt \underset{\theta}{\equiv} 0 \tag{6}$$

The function g in (5) can then be taken as F/h. The left hand side of (6) is the m-dimensional Laplace transform of F, denoted by $\mathfrak{L}(F)$:

$$\int F(t_1\cdots,t_m)\exp\left[-\sum_i^m s_i\,t_i\right]dt = \mathfrak{L}(F)(s_1\cdots,s_m). \tag{7}$$

The problem is to construct F in such a way that $\mathfrak{L}(F) = 0$ for all values of $s(\theta)$, $\theta\,\varepsilon\,\omega$. This can be done with help of (4). Let P be of degree d and let G be a function on $\mathfrak{Z}$ possessing all partial derivatives of dth order in the interior of C, vanishing outside C, and having all partial derivatives of $d - 1$st order continuous on the boundary of C. An example of such a function is the following. Let C be given by $a_i \leqq t_i \leqq a_i + l$ $(i = 1, \cdots, m)$, then on C we can take $G(t) = \prod_{i=1}^m (t_i - a_i)^d(a_i + l - t_i)^d$. Now denote by D the differential operator

$$D = P\left(\frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial t_m}\right). \tag{8}$$

We then have

$$\mathfrak{L}(DG)(s) = P(s)\mathfrak{L}(G)(s) \tag{9}$$

in which $s = (s_1, \cdots, s_m)$. Since the right hand side of (9) is $\equiv 0$ by (4), we may take F in (7) to be $F = DG$. The final result is therefore

$$g(t) = (DG(t))/h(t) \tag{10}$$

for suitably chosen G, and

$$\phi(t) = \alpha + (DG(t))/h(t) \tag{11}$$

is a size α similar test of non-Neyman structure.

Even for one m-dimensional cube C the number of choices for G is large. In addition there will usually be a large number of m-dimensional cubes on each of which h is bounded away from 0, and finally one may consider regions other than cubes for which the construction of functions G is possible. Thus, there will be a large class of functions g satisfying (5) which can be generated by the differential operator method, called the *D method* henceforth. Whether this method, in general, will give all those functions g, is still an open question. In one particular case the question has been answered in the affirmative, provided the definition of D method is taken sufficiently wide (see section 6).

Suppose that with the help of the D method a similar test $\phi(T)$ is constructed, and that it is desired to consider similar tests which do not necessarily depend on T only. Let ψ be a test function defined on the sample space $\mathfrak{X}$. If ψ is chosen to satisfy $E(\psi \mid t) = \phi(t)$, then ψ is also similar. In particular, it will usually be possible to construct in this way a similar rejection region w, in which case ψ is the indicator of w (this construction fails is $\mathfrak{X}$ is a subspace of a Euclidean space with same dimension as $\mathfrak{I}$) A similar region w is constructed by demanding

$$P(w \mid t) = \phi(t). \tag{12}$$

In other words, on each surface $T = t$ in the sample space a region is selected which has conditional probability $\phi(t)$. This generalizes the construction of a similar region of Neyman structure [10]. Equation (12) will be used in section 4, example 2.

4. Examples of the D method. EXAMPLE 1 (Behrens-Fisher problem). Let $X_1, \cdots, X_{n_1}$ be n_1 independent observations on a normal variable with mean μ_1, variance σ_1^2, and $Y_1, \cdots, Y_{n_2}$, n_2 independent observations on a normal variable with mean μ_2, variance σ_2^2. The X's and Y's are independent, and all parameters are unknown. Under the hypothesis tested, which is $\mu_1 = \mu_2$, the joint distribution of the X's and Y's has an exponential density with exponential factor

$$\exp\left[-\frac{1}{2\sigma_1^2}\sum_1^{n_1} x_i^2 + \frac{\mu}{\sigma_1^2}\sum_1^{n_1} x_i - \frac{1}{2\sigma_2^2}\sum_1^{n_2} y_j^2 + \frac{\mu}{\sigma_2^2}\sum_1^{n_2} y_j\right]$$

in which μ is the common value of μ_1 and μ_2. We may take

$$T_1(x) = \sum_1^{n_1} x_i^2, \qquad T_2(x) = \sum_1^{n_1} x_i, \qquad T_3(x) = \sum_1^{n_2} y_j^2, \qquad T_4(x) = \sum_1^{n_2} y_j,$$

$$s_1(\theta) = \frac{1}{2\sigma_1^2}, \qquad s_2(\theta) = \frac{-\mu}{\sigma_1^2}, \qquad s_3(\theta) = \frac{1}{2\sigma_2^2}, \qquad s_4(\theta) = \frac{-\mu}{\sigma_2^2}.$$

The s_i are linearly independent, from which it can be shown that

$$T = (T_1, T_2, T_3, T_4)$$

is a minimal sufficient statistic for ω. T has a regular exponential density of form (3), with

$$h(t) = (n_1t_1 - t_2^2)^{(n_1-3)/2}(n_2t_3 - t_4^2)^{(n_2-3)/2} \tag{13}$$

if $n_1t_1 \geqq t_2^2$, $n_2t_3 \geqq t_4^2$, and $h(t) = 0$ otherwise. By eliminating μ, σ_1, σ_2 from the four s_i we obtain $s_1 s_4 - s_2 s_3 = 0$ as a realization of (4). The differential operator D in (8) is then

$$D = \frac{\partial}{\partial t_1}\frac{\partial}{\partial t_4} - \frac{\partial}{\partial t_2}\frac{\partial}{\partial t_3} \tag{14}$$

and for suitably chosen G the test

$$\phi(t) = \alpha + h^{-1}(t)\left(\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_4} - \frac{\partial}{\partial t_2}\frac{\partial}{\partial t_3}\right) G(t) \tag{15}$$

is similar and of size α, where $h(t)$ is given by (13). Whether this method can be used to show the existence of an invariant similar region, such as the one proposed by Welch [1], [15], has not yet been investigated.

It should perhaps be mentioned here that the approach to the Behrens-Fisher problem by Wald [14] is essentially different, since Wald does not require the test to be similar.

Example 2. (Standardized mean of a normal population). Suppose we make $n + 1$ independent observations on a normal variable and consider hypotheses concerning the ratio of mean to standard deviation. By an orthogonal transformation this problem can be brought in the following form: Let $X_0, \cdots, X_n$ be independent and normal, with common, unknown variance σ^2. X_0 has unknown mean μ, $X_1, \cdots, X_n$ have mean 0. Denote $\mu/\sigma = r$, then for some given r_0 the hypothesis tested is $r = r_0$. For the time being the alternative to be considered is immaterial. For later reference, however, suppose that the alternative is $r > r_0$. We then have

$$\Omega = \{(r, \sigma): r \geqq r_0, \sigma > 0\},$$

$$\omega = \{(r, \sigma): r = r_0, \sigma > 0\}.$$

Under the hypothesis the joint distribution of the X_i has the form given by (3), with exponential factor

$$\exp\left[-\frac{1}{2\sigma^2}\sum_0^n x_i^2 + \frac{r_0}{\sigma}x_0\right]$$

so that we may take

$$T_1(x) = \sum_0^n x_i^2, \qquad T_2(x) = x_0, \qquad s_1(\sigma) = \frac{1}{2\sigma^2}, \qquad s_2(\sigma) = -\frac{r_0}{\sigma}.$$

$T = (T_1, T_2)$ is minimal sufficient, since s_1 and s_2 are linearly independent. Elimination of σ from s_1 and s_2 gives $s_2^2 - 2r_0^2 s_1 = 0$, so that we can take

$$P(s_1, s_2) = s_2^2 - 2r_0^2 s_1 \tag{16}$$

and

$$D = \frac{\partial^2}{\partial t_2^2} - 2r_0^2\frac{\partial}{\partial t_1} \tag{17}$$

The function h in (3) is found to be

$$h(t_1, t_2) = (t_1 - t_2^2)^{(n/2)-1} \tag{18}$$

if $t_1 \geqq t_2^2$, and $h = 0$ otherwise. For suitably chosen $G(t_1, t_2)$ the test function

$$\phi(t_1, t_2) = \alpha + (t_1 - t_2^2)^{1-n/2}\left(\frac{\partial^2}{\partial t_2^2} - 2r_0^2 \frac{\partial}{\partial t_1}\right) G(t_1, t_2) \tag{19}$$

is similar and of size α.

Equation (19) can be used to demonstrate the existence of similar tests which are not invariant. In the present problem an invariant test is a function of $T_2/\sqrt{T_1}$ only. Choose for G in (19) the following function:

$$G(t_1, t_2) = c(t_1 - t_2^2)^{(n/2)+1} e^{-t_1} \tag{20}$$

if $t_1 \geqq t_2^2$ and $G = 0$ otherwise, with $c > 0$ chosen so small that ϕ is bounded between 0 and 1. It is easily checked that after substitution into (19) the resulting test function is not invariant. This example can also be used to show the existence of similar rejection regions which are not equivalent to a cone in the sample space (we shall call two tests *equivalent* if they have the same power function, and by a *cone* is meant a union of rays through the origin). If w is any rejection region, ϕ the corresponding test function, given by (12), then w and ϕ are equivalent since T is sufficient, not only for ω, but also for Ω. If w_1 is a cone in the sample space, then the corresponding ϕ_1 is invariant. Let w_2 be any rejection region equivalent to w_1, ϕ_2 the corresponding test function; then ϕ_1 and ϕ_2 are equivalent. Now T is not only sufficient for Ω, it is also complete for Ω. Since ϕ_1 and ϕ_2 have the same power functions, it follows then that $\phi_1 = \phi_2$ a.e. and thus ϕ_2 is also invariant. The existence of a noninvariant similar test ϕ implies then the existence of a similar region which is not equivalent to any cone in the sample space.[4]

5. A criterion for bounded incompleteness in the case of regular exponential densities. Let the family of distributions be given by (3), with $\theta \varepsilon \omega$. By (2), if T is not minimal sufficient for ω, then T cannot be boundedly complete. This happens, for instance, if the s_i are linearly dependent on ω because the exponent $-\Sigma s_i t_i$ in (3) can then be written as a linear combination of fewer than m of the t_i. The incompleteness in this case also follows from the applicability of the D method of section 3, because of the existence of a polynomial P, linear in this case, for which (4) holds. On the other hand, if the m functions s_i are linearly independent on ω, then T is minimal sufficient for ω. Even if this is the case, T may still be boundedly incomplete. Theorem 2 below tells when this will happen. Its proof uses the D method of section 3. The conditions of Theorem 2 are designed to guarantee the existence of the polynomial P on the left

[4] This seems to contradict a statement by Patnaik [12] to the effect that in the problem under consideration every similar region is equivalent to a cone in the sample space. However, Patnaik's proof is unconvincing, and the non-invariant ϕ exhibited above provides a counter example.

hand side of (4), such that P is not the zero polynomial. This is made possible by the following theorem, due to A. Seidenberg (private communication). The proof is given in Appendix 1.

THEOREM 1. (Seidenberg). *Let for each i, $i = 1, \cdots, m$, $P_i(s_i; \theta_1, \cdots, \theta_k)$ be a polynomial in s_i and the $\theta_j (j = 1, \cdots, k)$, with coefficients in some field K, where $k < m$ and P_i is of positive degree in s_i. Let $A_i(\theta)$ be the leading coefficient of P_i as a polynomial in s_i. Then there is a polynomial $P(s_1, \cdots, s_m)$ with coefficients in K, which is not the zero polynomial, and a power product $B(\theta)$ of the $A_i(\theta)$, such that $B(\theta)P(s) = 0$ whenever $P_i = 0$ for all i.*

COROLLARY. *If θ is restricted to a set Θ, and if, for each $\theta \varepsilon \Theta$ and each i,*

$$A_i(\theta) \neq 0,$$

then $P = 0$ whenever $P_i = 0$ for all i.

For, if $A_i(\theta) \neq 0$, $i = 1, \cdots, m$, then $B(\theta) \neq 0$.

In the application we want to make of the corollary, the set Θ is ω. Furthermore, we shall assume the s_i of section 3 to be algebraic functions of the θ_j, for $\theta \varepsilon \omega$. Then for each i there is a polynomial P_i in s_i and the θ_j such that $P_i(s_i; \theta_1, \cdots, \theta_k) = 0$ if $\theta \varepsilon \omega$. We shall further assume that the $A_i(\theta)$ are $\neq 0$ if $\theta \varepsilon \omega$. These conditions will be satisfied in particular if, for each i, s_i on ω is a rational function of the θ_j, with nonvanishing denominator.

THEOREM 2. *Suppose a family of regular exponential densities is given by* (3), *with $\theta \varepsilon \omega$; ω is a subset of a k-dimensional Euclidean space, with $k < m$; on ω, the m functions s_i are algebraic functions of the k parameters θ_j, so that*

$$P_i(s_i; \theta_1, \cdots, \theta_k) = 0$$

for some polynomial P_i $(i = 1, \cdots, m)$; $A_i(\theta)$, the leading coefficient of P_i as a polynomial in s_i, does not vanish anywhere on ω for any i. Then T is boundedly incomplete for ω.

The proof follows immediately from the constructibility, by the D method of section 3, of a bounded function g, $g \neq 0$ on a set of positive probability, satisfying $E_\theta g(T) = 0$ for all $\theta \varepsilon \omega$.

In both examples in section 4 the s_i are rational functions of the θ_j, with nonvanishing denominators, and in both cases $k = m - 1 < m$, so that Theorem 2 applies. This provides another proof of the well-known fact that in the Behrens-Fisher problem, as well as in the problem of testing the ratio of mean to standard deviation in a normal population, the minimal sufficient statistic is boundedly incomplete.

It would be interesting to know how much the assumptions of Theorem 2 can be relaxed. It is certainly not necessary that the s_i be algebraic functions of the θ_j, for, if $m = 2$, $k = 1$, $s_1 = \cos\theta$, $s_2 = \sin\theta$, then $s_1^2 + s_2^2 - 1 = 0$, as a realization of (4), so that the D method applies. It is not even necessary for incompleteness that there exists a polynomial P in the s_i which vanishes for all $\theta \varepsilon \omega$, as the next example will show. Take $m = 2$, $k = 1$, $s_1 = -\ln\theta$, $s_2 = -\ln(1 - \theta)$, with $0 < \theta < 1$. Instead of (4) we have a transcendental equation:

$\exp[-s_1] + \exp[-s_2] - 1 = 0$. With help of this equation one can easily construct functions F of the kind mentioned in section 3. For example, the function F whose 2-dimensional Laplace transform is

$$\mathcal{L}(F)(s_1, s_2) = \frac{1}{s_1 s_2}(e^{-s_1} + e^{-s_2} - 1)(e^{-a_1 s_1} - e^{-b_1 s_1})(e^{-a_2 s_2} - e^{-b_2 s_2})$$

is bounded between -1 and 1, vanishes outside the rectangle

$$a_1 \leqq t_1 \leqq b_1 + 1, \qquad a_2 \leqq t_2 \leqq b_2 + 1,$$

and has vanishing Laplace transform for all θ between 0 and 1. On the other hand, the fact that $k < m$ is not sufficient for bounded incompleteness, nor is the additional restriction of analyticity of the s_i sufficient. The following example is due to L. J. Savage (private communication). In (3) choose $m = 2$, $k = 1$, $s_1 = \theta \cos \theta$, $s_2 = \theta \sin \theta (\theta > 0)$, $h(t) = 1$ for t in some square, $h = 0$ otherwise. Here the s_i are analytic functions of θ, but yet it can be shown that the family of distributions is complete. Another example is due to D. L. Burkholder (private communication) and differs from Savage's example only in that $s_1 = \theta \cos (1/\theta)$, $s_2 = \theta \sin (1/\theta)$. This example is a little less regular than Savage's example, but on the other hand the completeness of the family of distributions is easier to show.

6. Completeness of the D method in the case of a hypothesis concerning the standardized mean of a normal population. In this section it will be shown that in Example 2 of section 4 all similar tests can be generated by the D method, provided the D method is defined in a sufficiently broad manner. That is, we want to show that for each similar test ϕ there exists a function G satisfying (19) and certain other conditions. In section 3 the functions G were restricted to some m-dimensional cube on which h is bounded away from 0 but it was remarked there that this restriction is not necessary. We shall not even demand that $G = 0$ whenever $h = 0$. In fact, the main thing of importance was the validity of (9), and even this we shall relax slightly in the problem under consideration.

Equation (19) can be put in the form

$$\left(\frac{\partial}{\partial t_1} - \frac{1}{2r_0^2}\frac{\partial^2}{\partial t_2^2}\right) G = \frac{\sqrt{2\pi}}{r_0}\varphi \tag{21}$$

where φ is defined by

$$\varphi(t) = -(\sqrt{8\pi} r_0)^{-1} h(t)(\phi(t) - \alpha). \tag{22}$$

Equation (21) can be considered as the heat equation in one dimension, if t_1 is interpreted as time, t_2 as position, G as temperature, and $(\sqrt{2\pi}/r_0)\varphi$ as a heat source, capable of producing both positive and negative heat, whose strength and spatial distribution varies with time. If this were an actual heat problem, its solution could be written down at once, employing the usual Green's function for the heat operator:

$$G(t_1, t_2) = \int\int \varphi(t_1', t_2')(t_1 - t_1')^{-1/2} \exp\left[-\frac{r_0^2}{2}\frac{(t_2 - t_2')^2}{t_1 - t_1'}\right] dt_1'\, dt_2' \tag{23}$$

where the integration is over the strip $0 \leqq t_1' \leqq t_1$. Since $h(t')$, and therefore, $\varphi(t')$, is zero unless $t_2'^2 \leqq t_1'$, we may integrate over $t_2'^2 \leqq t_1' \leqq t_1$. The question to be answered next is whether, and if so, in what sense, the formal solution (23) to (21), and therefore, to (19) is a representation of ϕ.

We shall at once study the power function of any similar test ϕ, since some of the results are needed in section 7. Let Ω and ω be as defined in section 4, Example 2. We shall assume $r_0 > 0$. As remarked in section 4, the statistic

$$T = (T_1, T_2)$$

is sufficient for Ω, and it suffices therefore to consider test functions ϕ which depend only on T. The power function of ϕ is $\beta(r, \sigma) = E_{r,\sigma}\phi(T_1, T_2)$. Suppose ϕ satisfies (19), then we get after substitution:

$$\begin{aligned}\beta(r, \sigma) &= \alpha + c(r, \sigma) \int\int \\ &\quad \cdot \exp\left[-\frac{1}{2\sigma^2} t_1 + \frac{r}{\sigma} t_2\right]\left(\frac{\partial^2}{\partial t_2^2} - 2r_0^2 \frac{\partial}{\partial t_1}\right) G(t_1, t_2)\, dt_1\, dt_2\end{aligned} \tag{24}$$

where the integration is over $0 \leqq t_1 < \infty$, $-\infty < t_2 < \infty$. We may effect this integration by taking the upper limits on t_1 and t_2 as A, B respectively, and then let $A \to \infty$, $B \to \infty$ in any order. With respect to the types of functions G to be considered it will not be necessary to do something similar with the lower limit on t_2. If the upper limits on t_1 and t_2 are A and B, one can integrate by parts, obtaining an integral

$$\frac{r^2 - r_0^2}{\sigma^2} \int_0^A dt_1 \int_{-\infty}^B G(t_1, t_2) \exp\left[-\frac{1}{2\sigma^2} t_1 + \frac{r}{\sigma} t_2\right] dt_2 \tag{25}$$

plus the following integrated terms:

$$-2r_0^2 \int_{-\infty}^B G(A, t_2) \exp\left[-\frac{1}{2\sigma^2} A + \frac{r}{\sigma} t_2\right] dt_2 \tag{26}$$

$$-\frac{r}{\sigma} \int_0^A G(t_1, B) \exp\left[-\frac{1}{2\sigma^2} t_1 + \frac{r}{\sigma} B\right] dt_1 \tag{27}$$

$$\int_0^A \frac{\partial G(t_1, B)}{\partial t_2} \exp\left[-\frac{1}{2\sigma^2} t_1 + \frac{r}{\sigma} B\right] dt_1 \tag{28}$$

There is also an integral involving G on the t_2-axis. For any G given by (23), $G(0, t_2) = 0$, so that the integral mentioned in the preceding sentence vanishes trivially. It is sufficient, then, to consider only functions G which vanish if $t_1 = 0$. Now if G is given by (23), with φ defined by (22) and ϕ similar of size α, then it can be shown that (26)–(28) vanish in the limit if we let first $B \to \infty$ and then $A \to \infty$. A proof is given in Appendix 2. Using (24) and (25) it follows that

$$\beta(r, \sigma) = \alpha + c(r, \sigma) \frac{r^2 - r_0^2}{\sigma^2} \lim_{A\to\infty} \lim_{B\to\infty} \int_0^A dt_1 \int_{-\infty}^B G(t_1, t_2) \cdot \exp\left[-\frac{1}{2\sigma^2} t_1 + \frac{r}{\sigma} t_2\right] dt_2 \tag{29}$$

We see from (29) that $\beta(r_0, \sigma) = \alpha$ identically in σ, as it should.

The reason we could get the power function in the form (29) is that in this problem the density of T is of the exponential form (3) on the whole of Ω. The exponent of the exponential factor is $-t_1/(2\sigma^2) + rt_2/\sigma$, so that on Ω we have $s_1 = 1/(2\sigma^2)$, $s_2 = -r/\sigma$. The polynomial (16) is now defined on the whole of Ω:

$$P(s) = s_2^2 - 2r_0^2 s_1 = \frac{r^2 - r_0^2}{\sigma^2} \tag{30}$$

(On ω, $r = r_0$, so $P = 0$ as it should). We made the integrated terms (26)–(28) vanish by taking limits in a special way. This suggests, for this problem to re-define the 2-dimensional Laplace transform as follows:

$$\mathcal{L}(F)(s_1, s_2) = \lim_{A\to\infty} \lim_{B\to\infty} \int_0^A dt_1 \int_{-\infty}^B F(t_1, t_2) \exp[-s_1 t_1 - s_2 t_2]\, dt_2 \tag{31}$$

With P and $\mathcal{L}$ defined by (30) and (31), we have proved that if ϕ is similar, and G is the corresponding function given by (23), then (9) is valid on the whole of Ω. Adding α to both sides of (9) then produces (29).

In order to characterize the whole class of similar tests, consider the class $\mathcal{C}$ of functions G defined on the right half (t_1, t_2) plane which satisfy the following conditions (with D defined by (17)):

(i) $DG(t_1, t_2) = 0$ if $t_2^2 > t_1$,

(ii) $-\alpha \leqq (t_1 - t_2^2)^{1-n/2} DG(t_1, t_2) \leqq 1 - \alpha$ if $t_2^2 \leqq t_1$,

(iii) $G = 0$ if $t_1 = 0$, and $G(t_1, t_2) \to 0$ as $t_2 \to -\infty$, for each t_1,

(iv) The integrals (26)–(28) approach 0 if we let first $B \to \infty$ and then $A \to \infty$.

For every similar size α test function ϕ there is, by (23) and (22), a unique G, satisfying the conditions (i)–(iv), so that $G \in \mathcal{C}$. Conversely, for any $G \in \mathcal{C}$ we have shown that ϕ given by (19) is similar and of size α. Thus, there is a one-to-one correspondence between the members of $\mathcal{C}$ and the similar size α test functions. The class $\mathcal{C}$ gives therefore a complete characterization of the similar tests. Unfortunately, condition (iv) is not a very easy one. There is an important subclass of $\mathcal{C}$ where (iv) is obviously fulfilled, consisting of those functions G in $\mathcal{C}$ which vanish identically for $t_2 > \sqrt{t_1}$. This is the case, for instance, with all functions G leading to an invariant test. For a proof of this fact see Appendix 3. It would be desirable if (iv) could be replaced by a simpler condition. The possibility is not excluded that conditions (i)–(iv) imply that $G(t_1, t_2) = 0$ for all $t_2 > \sqrt{t_1}$, but whether this is so is an open question.

7. Some remarks on the search for an optimum test in the problem of section 6. Consider the class $\mathcal{C}$ defined in section 6. Let ϕ_1, ϕ_2 be two similar size α tests, G_1, G_2 the corresponding functions in $\mathcal{C}$, and β_1, β_2 their power functions. It follows from (29), since $r^2 \geqq r_0^2$, that if $G_1 \geqq G_2$, then $\beta_1 \geqq \beta_2$, so that ϕ_1 is uniformly more powerful than ϕ_2. Since every similar ϕ has a representative $G \varepsilon \mathcal{C}$, if there would exist a $G_0 \varepsilon \mathcal{C}$ such that $G_0 \geqq G$ for every $G \varepsilon \mathcal{C}$, then the test function ϕ_0 corresponding to G_0 would be UMP (uniformly most powerful) among all similar tests. To decide whether or not such a dominating function G_0 exists, the following observations may be of help. The first observation is that in the problem under consideration every invariant test—that is, depending only on $T_2/\sqrt{T_1}$—is similar. Secondly, if we denote by $\mathcal{C}^*$ the subclass of $\mathcal{C}$ representing invariant tests, then in $\mathcal{C}^*$ there is a function G_0^* which dominates every $G^* \varepsilon \mathcal{C}^*$. The corresponding test function ϕ_0^* is therefore UMP among all invariant tests. ϕ_0^* is nonrandomized, with a rejection region of the form $t_2/\sqrt{t_1} >$ constant. That ϕ_0^* is UMP invariant is a known result [12], obtainable more directly by the observation that $T_2/\sqrt{T_1 - T_2^2}$ has a noncentral t-distribution with a monotonic likelihood ratio [3], [4], [7]. The third observation we want to make is that if the dominating function G_0 exists, it has to coincide with G_0^*. This follows from the following proposition: *If a* UMP *similar test based on T exists, it is necessarily invariant.* The analogous statement, with "similar" replaced by "unbiased," is well known [5], [6]. In fact, both statements are special cases of the following more general theorem, due to E. L. Lehmann (private communication): *Let $\mathcal{G}$ be a group of transformations which leaves the problem invariant, and let $\mathcal{K}$ be a class of tests which is closed under $\mathcal{G}$. If there is a unique* UMP *test in $\mathcal{K}$, it is almost invariant.* (The uniqueness is understood to be a.e.). The proof of this theorem follows the same lines as in the special case that $\mathcal{K}$ is the class of unbiased tests of fixed size. In our problem $\mathcal{K}$ is the class of similar tests of size α, based on T. $\mathcal{K}$ is clearly closed under $\mathcal{G}$. If there is a UMP test in $\mathcal{K}$, its uniqueness follows from the completeness of T for Ω. Finally, in our problem an almost invariant function can be shown to be invariant (see also [17], footnote 3, and [5]).

The conclusion drawn from the preceding discussion is that there is a dominating function $G_0 \varepsilon \mathcal{C}$ if and only if G_0^* is the dominating function. Whether or not this is so is still an open question, and consequently, it is still unknown whether a UMP similar test exists. A last remark may be added to this. As remarked in section 6 and proved in Appendix 3, the functions G^* in $\mathcal{C}^*$ have the remarkable property that they vanish for $t_2 \geqq \sqrt{t_1}$. This property holds then in particular for G_0^*. Taking into account that $G \varepsilon \mathcal{C} \Rightarrow -aG \varepsilon \mathcal{C}$ for sufficiently small $a > 0$, we conclude that if G_0^* is a dominating function in $\mathcal{C}$, then every $G \varepsilon \mathcal{C}$ must also have the property $G(t_1, t_2) = 0$ if $t_2 \geqq \sqrt{t_1}$. If this were indeed true, then condition (iv) in section 6 could be replaced by the much simpler condition $G(t_1, t_2) = 0$ if $t_2 \geqq \sqrt{t_1}$. However, as remarked in section 6, even this property has not yet been proved.

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Appendix 1. PROOF OF THEOREM 1 (Seidenberg). For the purpose of this proof we shall replace the s_i by x_i. Let $P_i = A_i(\theta)x_i^{d_i} + \cdots$. Let $d = \max\{d_i\}$. Multiplying P_i by $x_i^{d-d_i}$, we may suppose all the d_i equal. Multiplying P_1 by

$A_2 \cdot \cdots \cdot A_m$, P_2 by $A_1A_3 \cdot \cdots \cdot A_m$, etc., we may suppose all the A_i equal. Now we have $P_i = \Delta(\theta)x_i^d + \cdots$, $i = 1, \cdots, m$, where Δ is some polynomial in $\theta_1, \cdots, \theta_k$.

Suppose we have a congruence of the form

$$\Pi x_i^{r_i}\Delta^{\Sigma r_i} \equiv R(x, \theta) \bmod (P_1, \cdots, P_m)$$

(i.e. the two sides are equal whenever all P_i vanish), with R a polynomial in the x's and θ's. Let $M = \max \{\deg_\theta P_i\}$. The left hand side has degree in the θ's at most $M\Sigma r_i$. Assume this to be the case also for $R(x, \theta)$. Assume further that $\deg_{x_i} R(x, \theta) \leqq d - 1$, $i = 1, \cdots, m$. Multiplying the congruence by $x_j\Delta$, on the left we get a power product of degree $1 + \Sigma r_i$ in the x_i times $\Delta^{1+\Sigma r_i}$. On the right there possibly appears a power x_j^d: if so, we replace Δx_j^d by

$$(\Delta x_j^d - P_j) \bmod P_j .$$

In this way we get a congruence

$$\Pi x_i^{s_i}\Delta^{\Sigma s_i} \equiv R'(x, \theta) \bmod (P_1, \cdots, P_m)$$

with $\Sigma s_i = 1 + \Sigma r_i$, $\deg_{x_i} R' \leqq d - 1$ $(i = 1, \cdots, m)$, $\deg_\theta R' \leqq M\Sigma s_i$. The congruences

$$x_i^d\Delta^d \equiv \Delta^{d-1}(x_i^d - P_i) \bmod (P_1, \cdots, P_m)$$

are of the above form. Multiplying by various power products of the $x_j\Delta$, we again get congruences of the stated form. Let $s \geqq s_0 = m(d - 1) + 1$. Then any power product of the x_i of degree s must have a factor x_i^d for at least one i. Hence we can get a congruence of the desired form with any power product of the $x_j\Delta$ of degree s on the left. For any such power product there may be several congruences: choose one.

For a fixed integer $\gamma \geqq s_0$ (to be determined in a moment), we consider all the power products of the $x_i\Delta$ of degree between s_0 and γ; and all the congruences, one for each power product. We still multiply each of these by an appropriate power of Δ so that Δ^γ is the power of Δ occurring on the left. On the right, then, all polynomials are of degree $\leqq M\gamma$ in the θ's and of degree $\leqq d - 1$ in each x_i.

Let $N(p, q)$ be the number of distinct power products of degree p or less in q letters. Then $N(p, q) = (p + q)(p + q - 1) \cdots (p + 1) / q!$ We are considering, then, $N(\gamma, m) - N(s_0 - 1, m)$ congruences. The right hand sides of these congruences are linear combinations over K of power products of degree $\leqq M\gamma$ in the θ's and of degree $\leqq m(d - 1)$ in the x's; therefore in at most $N(M\gamma, k)N(m(d - 1), m)$ power products. Since

$$\deg_\gamma [N(\gamma, m) - N(s_0 - 1, m)] = m > k = \deg_\gamma N(M\gamma, k)N(m(d - 1), m),$$

we see that for sufficiently large γ,

$$N(\gamma, m) - N(s_0 - 1, m) > N(M\gamma, k)N(m(d - 1), m).$$

Let γ be taken large enough for this to be realized. Then there exist $c_{i_1,\cdots,i_m} \varepsilon K$, not all $= 0$, such that

$$\Delta^{\gamma}\Sigma c_{i_1,\cdots,i_m} x_1^{i_1} \cdot \cdots \cdot x_m^{i_m} \equiv 0 \bmod (P_1, \cdots, P_m). \qquad \text{Q.E.D.}$$

Appendix 2. It will be proved here that the integrated terms (26)–(28) vanish in the limit $B \to \infty$, then $A \to \infty$. Since ϕ is similar and of size α, the function φ defined by (22) has the property

$$\int_0^{\infty} \int_{-\sqrt{t_1}}^{\sqrt{t_1}} \varphi(t_1, t_2) \exp\left[-\frac{1}{2\sigma^2} t_1 + \frac{r_0}{\sigma} t_2\right] dt_2\, dt_1 \underset{\sigma}{\equiv} 0. \tag{32}$$

This property is crucial for showing (26) $\to 0$, but is not needed for (27) and (28).

We shall first treat (26). Since σ is an arbitrary positive number, we shall give the proof with σ replaced by $r\sigma/r_0$, which will be useful for later purposes. With this change we substitute (23) into (26) and get

$$\begin{aligned}
&\int_{-\infty}^{B} G(A, t_2) \exp\left[-\frac{r_0^2}{r^2}\frac{A}{2\sigma^2} + \frac{r_0}{\sigma} t_2\right] dt_2 \\
&\quad = \exp\left[-\frac{r_0^2}{r^2}\frac{A}{2\sigma^2}\right] \int_0^A dt_1' \int_{-\sqrt{t_1'}}^{\sqrt{t_1'}} \varphi(t_1', t_2')\, dt_2' \int_{-\infty}^{B} (A - t_1')^{-1/2} \\
&\qquad\qquad \cdot \exp\left[-\frac{r_0^2}{2}\frac{(t_2 - t_2')^2}{A - t_1'} + \frac{r_0}{\sigma} t_2\right] dt_2 \\
&\quad = \frac{\sqrt{2\pi}}{r_0} \exp\left[\left(1 - \frac{r_0^2}{r^2}\right)\frac{A}{2\sigma^2}\right] \int_0^A dt_1' \int_{-\sqrt{t_1'}}^{\sqrt{t_1'}} \varphi(t_1', t_2') \\
&\qquad \cdot \exp\left[-\frac{1}{2\sigma^2} t_1' + \frac{r_0}{\sigma} t_2'\right] dt_2' \int_{-\infty}^{B} \frac{r_0}{\sqrt{2\pi}} (A - t_1')^{-1/2} \\
&\qquad\qquad \cdot \exp\left[-\frac{r_0^2}{2}\frac{(t_2 - (A - t_1')/\sigma r_0 - t_2')^2}{A - t_1'}\right] dt_2
\end{aligned}$$

The integral over t_2 can be written

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{B'} \exp\left[-\frac{1}{2} z^2\right] dz,$$

in which

$$B' = r_0(A - t_1')^{-1/2}(B - (A - t_1')/\sigma r_0 - t_2').$$

As $B \to \infty$, $B' \to \infty$ and the integral converges monotonically increasing to 1. The integration over t_2' and t_1' can be considered as a double integral of the form $\int\int f_B(t_1', t_2')\, dt_1'\, dt_2'$, in which f_B is bounded in absolute value by

$$|\varphi(t_1', t_2')| \exp\left[-\frac{1}{2\sigma^2} t_1' + \frac{r_0}{\sigma} t_2'\right]$$

which is integrable. Applying the Lebesgue bounded (dominated) convergence theorem, we may take the limit as $B \to \infty$ under the integral. We have

$$\lim_{B\to\infty} f_B(t_1', t_2') = \int_0^A dt_1' \int_{-\sqrt{t_1'}}^{\sqrt{t_1'}} \varphi(t_1', t_2') \exp\left[-\frac{1}{2\sigma^2} t_1' + \frac{r_0}{\sigma} t_2'\right] dt_2'.$$

By (32) we may replace the integral on the right in the above equation by minus the integral with same integrand but the t_1' integration running from A to ∞. Thus we get

$$\lim_{B\to\infty} \int_{-\infty}^{B} G(A, t_2) \exp\left[-\frac{r_0^2}{r^2}\frac{A}{2\sigma^2} + \frac{r_0}{\sigma} t_2\right] dt_2 = \xi(A)$$

with

$$\xi(A) = -\frac{\sqrt{2\pi}}{r_0} \exp\left[\left(1 - \frac{r_0^2}{r^2}\right)\frac{A}{2\sigma^2}\right] \int_A^\infty dt_1' \int_{-\sqrt{t_1'}}^{\sqrt{t_1'}} \varphi(t_1', t_2') \cdot \exp\left[-\frac{1}{2\sigma^2} t_1' + \frac{r_0}{\sigma} t_2'\right] dt_2'.$$

Since ϕ is bounded, we see by (22) that $\varphi(t_1', t_2')$ is bounded in absolute value by const. $h(t_1', t_2')$, which is bounded by const. $t_1'^{(n/2)-1}$. In the integration over t_2' we have that

$$\int_{-\sqrt{t_1'}}^{\sqrt{t_1'}} \exp\left[\frac{r_0}{\sigma} t_2'\right] dt_2'$$

is bounded by $2\sqrt{t_1'} \exp\, [(r_0 / \sigma)\sqrt{t_1'}]$. Thus

$$|\,\xi(A)\,| < \text{const.} \exp\left[\left(1 - \frac{r_0^2}{r^2}\right)\frac{A}{2\sigma^2}\right] \int_A^\infty t_1'^{(n-1)/2} \exp\left[-\frac{1}{2\sigma^2} t_1' + \frac{r_0}{\sigma}\sqrt{t_1'}\right] dt_1'.$$

We make the substitutions $t_1' = \sigma^2(u + r_0)^2$, $A = \sigma^2K^2$, then A and K go to ∞ together. Put $\xi(A) = \eta(K)$, then

$$|\,\eta(K)\,| < \text{const.} \exp\left[\left(1 - \frac{r_0^2}{r^2}\right)\frac{K^2}{2}\right] \int_{K-r_0}^\infty (u + r_0)^n \exp\,[-\tfrac{1}{2}u^2]\, du.$$

In the integrand, $(u + r_0)^n$ can be bounded by const. u^n, and by partial integration one finds that

$$\int_{K-r_0}^\infty u^n \exp\,[-\tfrac{1}{2}u^2]\, du$$

is bounded by const. $K^{n-1} \exp\,[-\frac{1}{2}(K - r_0)^2]$. This leads then to

$$|\,\eta(K)\,| < \text{const.}\, K^{n-1} \exp\left[-\frac{r_0^2}{r^2}\frac{K^2}{2} + r_0 K\right]$$

which $\to 0$ as $K \to \infty$. Q.E.D.

Of the integrated terms (27) and (28) we shall only treat (28), since (27) is a little simpler and follows the same pattern. It can be shown that (23) can be differentiated partially with respect to t_2 under the integral sign, provided $t_2^2 > t_1$. Substituting the result into (28) we obtain, apart from a multiplicative constant,

$$\iiint (t_1 - t_1')^{-3/2}(B - t_2')\varphi(t_1', t_2') \cdot \exp\left[-\frac{1}{2\sigma^2} t_1 + \frac{r}{\sigma} B - \frac{r_0^2}{2}\frac{(B - t_2')^2}{t_1 - t_1'}\right] dt_2'\, dt_1'\, dt_1 \tag{33}$$

in which the integration is over the region $t_2'^2 \leqq t_1' \leqq t_1 \leqq A$. We shall show that (33) $\to 0$ as $B \to \infty$ for fixed A, after which taking the limit $A \to \infty$ yields then trivially 0. Clearly the integrand in (33) approaches 0 as $B \to \infty$. It suffices therefore to show that limit and integral may be interchanged. By the Lebesgue bounded convergence theorem it is sufficient to show that the integrand is bounded in absolute value by an integrable function independent of B (but possibly dependent on A). Let $B_0 > \sqrt{A}$ and consider only values of $B \geqq B_0$. The integrand is bounded in absolute value by $|\varphi(t_1', t_2')|f_1f_2f_3$, in which

$$f_1 = \exp\left[\frac{r}{\sigma} B - \frac{r_0^2}{4}\frac{(B - t_2')^2}{t_1 - t_1'}\right], \qquad f_2 = \exp\left[-\frac{r_0^2}{4}\frac{(B - t_2')^2}{t_1 - t_1'}\right]\left(\frac{B - t_2'}{\sqrt{t_1 - t_1'}}\right)^3,$$

and $f_3 = (B - t_2')^{-2}$. Now f_1 is bounded by

$$\exp\left[\frac{r}{\sigma} B - \frac{r_0^2}{4}\frac{(B - \sqrt{A})^2}{A}\right]$$

which is bounded by a constant; f_2 is of the form $y^3 \exp[-(r_0^2/4)y^2]$ and is therefore also bounded by a constant; f_3 is bounded by the constant $(B_0 - \sqrt{A})^{-2}$. Finally we have then that the integrand in (33) is bounded in absolute value by const. $|\varphi(t_1', t_2')|$, which is integrable over the bounded region $t_2'^2 \leqq t_1' \leqq t_1 \leqq A$. Q.E.D.

Appendix 3. We will show that if ϕ is invariant, then $G = 0$ in the region $t_2 \geqq \sqrt{t_1}$. Let $y = t_2/\sqrt{t_1}$, and $y' = t_2'/\sqrt{t_1'}$. If ϕ is invariant, it is a function of y only. Put $\phi(t_1, t_2) = \phi^*(y)$, so that by (22) and (18)

$$\varphi(t_1, t_2) = \text{const. } t_1^{(n/2)-1}(1 - y^2)^{(n/2)-1}(\phi^*(y) - \alpha).$$

After substitution into (23) and making the change of variable $\tau = t_1'/t_1$, we can write (23) as

$$G(t_1, t_2) = \text{const. } t_1^{n/2} \exp\left[-\frac{r_0^2}{2} y^2\right] \int_0^1 (1 - y'^2)^{(n/2)-1}(\phi^*(y') - \alpha)f(y, y')\, dy', \tag{34}$$

in which

$$f(y, y') = \int_0^1 \tau^{(n-1)/2}(1 - \tau)^{-1/2} \exp\left[-\frac{r_0^2}{2}\frac{y^2\tau - 2yy'\sqrt{\tau} + y'^2\tau}{1 - \tau}\right] d\tau \tag{35}$$

Throughout we restrict y and y' to $y > 1$, $y' \leqq 1$. Let the differential operator D_y be defined as

$$D_y = \frac{1}{r_0^2} \frac{\partial^2}{\partial y^2} - y \frac{\partial}{\partial y} - (n + 1) \tag{36}$$

and the operator $D_{y'}$ similarly by replacing in (36) y by y'. Then f satisfies the two equations

$$D_y f(y, y') = 0 \tag{37}$$

$$D_{y'} f(y, y') = 0. \tag{38}$$

Furthermore, it can be seen from (35) that $f \to 0$ if $y \to \infty$ for fixed y', or if $y' \to -\infty$ for fixed y. Two linearly independent solutions of the equation

$$D_y u(y) = 0 \tag{39}$$

are u_1 and u_2, with $u_2(y) = u_1(-y)$, and

$$u_1(y) = \int_0^\infty t^{(n-1)/2} \exp\left[-\tfrac{1}{2}t + r_0 \sqrt{t}\, y\right] dt \tag{40}$$

When $y \to \infty$, $u_1(y) \to \infty$ whereas $u_2(y) \to 0$, with the opposite behavior as $y \to -\infty$. It follows from (37) and (38), from the behavior of the functions u_1 and u_2, and from the behavior of f as $y \to \infty$ or $y' \to -\infty$, that f must equal

$$f(y, y') = \text{const. } u_1(y')u_2(y). \tag{41}$$

Substituting (41) into (34), it remains to be shown that the integral

$$\int_0^1 (1 - y'^2)^{(n/2)-1} (\phi^*(y') - \alpha) u_1(y')\, dy' \tag{42}$$

equals 0, with u_1 given by (40). Replacing in (40) t by t_1 and in (42) y' by $t_2/\sqrt{t_1}$, the integral (42) is nothing else but the expectation of $\phi - \alpha$ with respect to the distribution specified by $r = r_0$, $\sigma = 1$. Since ϕ is similar of size α this expectation vanishes, Q.E.D.