where

$$a_i = 2(\sigma^2 + \lambda_i \sigma'^2)$$

The distribution of such a quadratic form has been derived by Herbert Robbins [3], and Herbert Robbins and E. J. G. Pitman [4] (Theorem 1).

For a design, the dual of which is a balanced incomplete block design, all the non-zero latent roots of D are equal and the distribution of u reduces to a chi-square distribution.

When the design is not a connected one, some of the λ's will be zero and the necessary changes can be easily made to suit that situation.

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REFERENCES

[1] MARVIN ZELEN, "Analysis of incomplete block designs," *J. Amer. Stat. Assn.*, Vol. 52 (1957), p. 204.

[2] M. H. HANSEN, W. N. HURWITZ, AND W. G. MADOW, *Sample Survey Methods and Theory*, Vol. 2, John Wiley and Sons, New York, pp. 63–65.

[3] HERBERT ROBBINS, "The distribution of a definite quadratic form," *Ann. Math. Stat.*, Vol. 19 (1948), p. 266.

[4] HERBERT ROBBINS, AND E. J. G. PITMAN, "Application of the method of mixtures to quadratic forms in normal variables," *Ann. Math. Stat.*, Vol. 20 (1949), p. 552.

A SERIES OF SYMMETRICAL GROUP DIVISIBLE INCOMPLETE BLOCK DESIGNS

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Introduction. A balanced incomplete block design (BIBD) is an arrangement of v elements in b blocks of k different elements each, so that each element occurs in r blocks and each pair of elements occurs in λ blocks. If $v = b(r = k)$, the design is said to be symmetric; it is well known that any two blocks of a symmetric BIBD have exactly λ elements in common.

It has been shown [1] that the subspaces of dimension t of $PG(m, p^n)$ form a BIBD. It is also known that the $PG(m, p^n)$ contains $PG(m, p^k)$ if k is a factor of n (see [2]). In particular, the lines of $PG(2, s^2)$ form the design $v = b = s^4 + s^2 + 1$, $r = k = s^2 + 1$, $\lambda = 1$, which therefore contains the design $v = b = s^2 + s + 1$, $r = k = s + 1$, $\lambda = 1$, (the lines of $PG(2, s)$), where $s = p^n$.

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A group divisible incomplete block design (GD design) has been defined [3] as an arrangement of v elements in b blocks of k different elements, in which the elements can be divided into m groups of n elements each, so that two elements belonging to the same group occur together in λ_1 blocks and two elements belonging to different groups occur together in λ_2 blocks. It will be shown that a series of GD designs can be obtained from the preceding series of BIBD's.

Series of symmetrical group divisible designs. Consider the BIBD (a): $v_1 = b_1 = s^4 + s^2 + 1$, $r_1 = k_1 = s^2 + 1$, $\lambda = 1$. Let set I be the set of blocks of (a) that contains the blocks of the BIBD (b): $v_2 = b_2 = s^2 + s + 1$, $r_2 = k_2 = s + 1$, $\lambda = 1$. Let set II be the remaining blocks of design (a); let set III be the set of blocks remaining in design (a) when all blocks and elements of design (b) have been deleted. Thus set I contains $b_2 = s^2 + s + 1$ blocks, set II contains $b_1 - b_2 = s^4 - s$ blocks, and set III contains $b = v = s^4 - s$ blocks and elements.

LEMMA 1. *Every element of set* III *occurs in exactly* s^2 *blocks of set* III.

PROOF. Since (b) is a symmetric BIBD, any two blocks of set I have exactly one of the $v_2 = s^2 + s + 1$ elements in common. Thus the remaining $(k_1 - k_2)$ $b_2 = s^4 - s$ elements of set I are all different, and they are also different from the v_2 elements of design (b). Since $s^4 - s = v_1 - v_2 = v$, where v is the number of elements in set III, each such element must occur in exactly one block of set I, and so must occur in exactly $r_1 - 1 = s^2$ blocks of set III.

LEMMA 2. *Every block of set* II *contains exactly one element of design* (b).

PROOF. Every pair of elements of design (b) occurs once in set I, and so cannot occur at all in set II. Therefore all elements of design (b) occur in different blocks of set II, each occurring $r_1 - r_2 = s^2 - s$ times. Thus the number of blocks of set II containing exactly one element of design (b) is $v_2(r_1 - r_2) = s^4 - s$, which is the total number of blocks of II.

Combining these lemmas, it can easily be seen that set III contains $v = s^4 - s$ elements in $b = s^4 - s$ blocks, where each block contains $k = k_1 - 1 = s^2$ elements and each element occurs $r = r_1 - 1 = s^2$ times.

Since any element of set III occurs once in set I with $k_1 - k_2 - 1$ other elements of set III, it will not occur again with any of these elements and will occur once in set III with each of the remaining elements. Thus set III is a regular GD design with parameters,

$$v = b = s^4 - s, \qquad r = k = s^2, \qquad \lambda_1 = 0,$$
$$\lambda_2 = 1, \qquad m = s^2 + s + 1, \qquad n = s^2 - s.$$

Example. The lines of PG(2, 4) form a BIBD that contains the design formed by the lines of PG(2, 2). These designs have parameters

(a): $v = b = 21$, $r = k = 5$, $\lambda = 1$, and

(*b*): $v = b = 7$, $r = k = 3$, $\lambda = 1$ as shown below.

D	*N*	*P*	*Q*	*L*	*O*	*H*	*A*	*A*	*B*	*B*	*C*	*C*	*F*	*F*	*K*	*K*	*J*	*J*	*G*	*G*
E	*R*	*U*	*T*	*M*	*S*	*I*	*N*	*R*	*H*	*I*	*H*	*I*	*D*	*E*	*D*	*E*	*D*	*E*	*D*	*E*
A	*B*	*F*	*C*	*J*	*K*	*G*	*O*	*S*	*L*	*M*	*M*	*L*	*L*	*M*	*I*	*H*	*H*	*I*	*M*	*K*
B	*F*	*C*	*J*	*K*	*G*	*A*	*P*	*T*	*P*	*Q*	*N*	*O*	*Q*	*O*	*N*	*Q*	*O*	*P*	*P*	*N*
C	*J*	*K*	*G*	*A*	*B*	*F*	*Q*	*U*	*T*	*U*	*S*	*R*	*S*	*T*	*T*	*R*	*U*	*S*	*R*	*U*

The lower left hand group of blocks constitutes the design (b), and the lower right hand group of blocks is the GD design with parameters $v = b = 14$, $r = k = 4$, $\lambda_1 = 0$, $\lambda_2 = 1$, $m = 7$, $n = 2$. The groups are (D, E), (N, R), (P, U), (Q, T), (L, M), $O, S)$, and (H, I). Thus, for example, D occurs zero times with E in the GD design and once with N, R, P, U, Q, T, L, M, O, S and I.

A design with these parameters was obtained in [4] using the method of differences and is listed as number R24 in [5]. For $s = 3$ the resulting design has parameters,

$$v = b = 78, r = k = 9, \lambda_1 = 0, \lambda_2 = 1, m = 13, n = 6.$$

REFERENCES

[1] R. C. Bose, "On the construction of balanced incomplete block designs," *Ann. Eugenics* Vol. 10 (1939), pp. 352–399.

[2] R. D. Carmichael, *Introduction to the Theory of Groups of Finite Order*, Dover Publications, 1956, pp. 334–336.

[3] R. C. Bose, and W. S. Connor, "Combinatorial properties of group divisible incomplete block designs," *Ann. Math. Stat.*, Vol. 23 (1952), pp. 367–383.

[4] R. C. Bose, S. S. Shrikhande, and K. N. Bhattacharya, "On the construction of group divisible incomplete block designs," *Ann. Math. Stat.*, Vol. 24 (1953), pp. 167–195.

[5] "Tables of partially balanced designs with two associate classes," Institute of Statistics, University of North Carolina, reprint series No. 50, p. 184 and p. 192.

ALTERNATIVE PROOF OF A THEOREM OF BIRNBAUM AND PYKE

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Let $U_1, U_2, \cdots, U_n$ be an ordered sample of a random variable (r.v.) X having a uniform distribution (0, 1). If i^* is the value of $i = 1, 2, \cdots, n$ at which $i/n - U_i$ is maximized and $U^* = U_{i^*}$, then U^* is a r.v. with values (0, 1). The probability that the sample cannot be ordered or that i^* is not uniquely defined is zero, and hence these possibilities are neglected. Theorem 3 [1] states that U^* has a uniform distribution (0, 1). Another proof of this fact was given in [2].

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