$$\tilde{H}(s, T) = e^{-(s+\alpha)T} + \sum_{k=1}^{\infty} \frac{T(T+k\epsilon)^{k-1}}{k!} e^{-\alpha T}(\alpha e^{-\alpha\epsilon})^k (e^{-s(T+k\epsilon)}). \tag{14}$$

(b) $$B(t) = 1 - e^{-\beta t}, \qquad t \geqq 0, \beta > 0.$$

To determine $P(x, T)$ from (3) integrate both sides of (3) with respect to $dB(T)$ and solve for $\int_0^\infty P(x, t)\, dB(t)$.

$$\int_0^\infty P(x, t)\, dB(t) = \frac{\alpha + \beta - \sqrt{(\alpha+\beta)^2 - 4\alpha\beta x}}{2\alpha x} \tag{15}$$

so

$$P(x, T) = \exp\left\{-\frac{T}{2}\left(\alpha - \beta + \sqrt{(\alpha+\beta)^2 - 4\alpha\beta x}\right)\right\}. \tag{16}$$

To determine $\tilde{H}(s, T)$ from (7) integrate both sides of (7) with respect to $dB(T)$ and solve for $\int_0^\infty \tilde{H}(s, t)\, dB(t)$.

$$\int_0^\infty \tilde{H}(s, t)\, dB(t) = \frac{s + \alpha + \beta - \sqrt{(s+\alpha+\beta)^2 - 4\alpha\beta}}{2\alpha} \tag{17}$$

so

$$\tilde{H}(s, T) = \exp\left\{-\frac{T}{2}\left(s + \alpha - \beta + \sqrt{(s+\alpha+\beta)^2 - 4\alpha\beta}\right)\right\}. \tag{18}$$

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DISTRIBUTION OF THE "BLOCKS ADJUSTED FOR TREATMENTS" SUM OF SQUARES IN INCOMPLETE BLOCK DESIGNS

By A. M. Kshirsagar

Bombay University

Introduction. Marvin Zelen [1] has stated that the distribution of the "blocks adjusted for treatments" sum of squares in an incomplete block design is unknown. The present paper is intended to derive this distribution.

Notation and derivation. Let there be v treatments and b blocks having $k_1, k_2, \cdots, k_b$ plots respectively and let the ith treatment be replicated r_i times; $(i = 1, 2, \cdots, v)$. Let n_{ij} (which is either zero or one) be the number of times the ith treatment occurs in the jth block, $(i = 1, 2, \cdots v; j = 1, 2, \cdots, b)$. Then

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$$\underset{v \times b}{N} = [n_{ij}]$$

is the incidence matrix of the design. Let the total yields of the blocks be denoted by $B_1, B_2, \cdots, B_b$ respectively and the total yields of the treatments by $T_1, T_2, \cdots, T_v$ respectively. Let

$$B = \begin{bmatrix} B_1 \\ B_2 \\ \cdots \\ B_b \end{bmatrix}, \qquad T = \begin{bmatrix} T_1 \\ T_2 \\ \cdots \\ T_v \end{bmatrix}.$$

We shall assume that the yield of a plot consists of a general effect μ, the effect of the block containing the plot, the effect of the treatment which is applied to the plot and the error component. Let α_j denote the effect of the jth block, and let

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_b \end{bmatrix}.$$

It is assumed that the errors are independently and normally distributed with zero mean and variance σ^2, the block effects are independently and normally distributed with zero mean and variance σ'^2 and that the block effects and error components are independently distributed of each other.

Let E_1, V_1, cov_1 denote respectively the expectation, variance-covariance matrix and covariance matrix, in the conditional distribution of the yields when the block effects are fixed; and E, V, cov will denote the same quantities, in the absolute distribution of the yields when the block effects are also normally distributed. Let

$$P = B - N' \operatorname{diag}\left(\frac{1}{r_1}, \frac{1}{r_2}, \cdots, \frac{1}{r_v}\right) T,$$

where diag stands for a diagonal matrix, the elements in the diagonal being written in the adjoining bracket. P is the vector of the block totals, adjusted for treatment effects. Let

$$D = \operatorname{diag}(k_1, k_2, \cdots, k_b) - N' \operatorname{diag}\left(\frac{1}{r_1}, \frac{1}{r_2} \cdots, \frac{1}{r_v}\right) N.$$

Then it can be readily seen that

$$E_1(P) = D\alpha, \qquad V_1(P) = \sigma^2 D.$$

Since the sum of all the elements in any row (or column) of D is zero, one of its latent roots is zero. If the design is a connected one (i.e., all the treatment contrasts and block contrasts are estimable), the rank of D is $b - 1$. Let the non-zero latent roots of D be $\lambda_1, \lambda_2, \cdots, \lambda_{b-1}$, the corresponding orthogonal normalized latent vectors (column vectors) being $m_1, m_2, \cdots, m_{b-1}$. Then $(D - \lambda_i I)m_i = 0$, $i = 1, 2, \cdots, b-1$; and hence

$$
\begin{aligned}
E_1(m_i'P) &= m_i'D\alpha = \lambda_i m_i'\alpha, && i = 1, 2, \cdots, b-1; \\
V_1(m_i'P) &= \sigma^2 m_i'Dm_i = \sigma^2\lambda_i, && i = 1, 2, \cdots, b-1; \\
\text{cov}_1\,(m_i'P, m_j'P) &= \sigma^2 m_i'Dm_j && (i \neq j) \\
&= 0 && i, j = 1, 2, \cdots, b-1.
\end{aligned}
$$

Hence the "blocks adjusted for treatments," sum of squares with $b - 1$ degrees of freedom is

$$u = \sum_{i=1}^{b-1} (m_i'P)^2/\lambda_i .$$

When the block effects α_j are not fixed but are random variables obeying a normal distribution with zero mean and variance σ'^2, the means, variances and covariances of $m_i'P$ can be found as below by using theorems 14 and 15 about expectations, variances and covariances proved in [2].

$$
\begin{aligned}
V(m_i'P) &= (EV_1 + VE_1)m_i'P \\
&= E(\sigma^2\lambda_i) + V(\lambda_i m_i'\alpha) \\
&= \sigma^2\lambda_i + \sigma'^2\lambda_i^2, && i = 1, 2, \cdots b-1; \\
\text{cov}\,(m_i'P, m_j'P) &= \text{cov}\,\{E_1(m_i'P), E_1(m_j'P)\} + E\{\text{cov}_1\,(m_i'P), (m_j'P)\} \\
&= \text{cov}\,(\lambda_i m_i'\alpha, \lambda_j m_j'\alpha) && (i \neq j) \\
&= 0, && i, j = 1, 2, \cdots b-1;
\end{aligned}
$$

and

$$
\begin{aligned}
E(m_i'P) &= E\{E_1(m_i'P)\} \\
&= E(\lambda_i m_i'\alpha) \\
&= 0, && i = 1, 2, \cdots, b-1.
\end{aligned}
$$

Therefore, if

$$x_i = \frac{m_i'P}{(\sigma^2\lambda_i + \sigma'^2\lambda_i^2)^{1/2}},$$

then $x_1, x_2, \cdots, x_{b-1}$ are standard normal independent variables and the (adjusted) block sum of squares is

$$u = \sum_{i=1}^{b-1} \frac{a_i}{2} x_i^2,$$

where

$$a_i = 2(\sigma^2 + \lambda_i\sigma'^2)$$

The distribution of such a quadratic form has been derived by Herbert Robbins [3], and Herbert Robbins and E. J. G. Pitman [4] (Theorem 1).

For a design, the dual of which is a balanced incomplete block design, all the non-zero latent roots of D are equal and the distribution of u reduces to a chi-square distribution.

When the design is not a connected one, some of the λ's will be zero and the necessary changes can be easily made to suit that situation.

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A SERIES OF SYMMETRICAL GROUP DIVISIBLE INCOMPLETE BLOCK DESIGNS

BY D. A. SPROTT

Waterloo College, Ontario, Canada

Introduction. A balanced incomplete block design (BIBD) is an arrangement of v elements in b blocks of k different elements each, so that each element occurs in r blocks and each pair of elements occurs in λ blocks. If $v = b(r = k)$, the design is said to be symmetric; it is well known that any two blocks of a symmetric BIBD have exactly λ elements in common.

It has been shown [1] that the subspaces of dimension t of PG(m, p^n) form a BIBD. It is also known that the PG(m, p^n) contains PG(m, p^k) if k is a factor of n (see [2]). In particular, the lines of PG(2, s^2) form the design $v = b = s^4 + s^2 + 1$, $r = k = s^2 + 1$, $\lambda = 1$, which therefore contains the design $v = b = s^2 + s + 1$, $r = k = s + 1$, $\lambda = 1$, (the lines of PG(2, s)), where $s = p^n$.

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