A GENERALIZATION OF GROUP DIVISIBLE DESIGNS

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1. Summary and Introduction. Roy [8] extended the idea of Group Divisible designs of Bose and Connor [1] to m-associate classes, calling such designs Hierarchical Group Divisible designs with m-associate classes. Subsequently, no literature is found in this direction. The purpose of this paper is to study these designs systematically. A compact definition of the design, under the name Group Divisible m-associate (GD m-associate) design is given in Section 2. In the same section the parameters of the design are obtained in a slightly different form than that of Roy. The uniqueness of the association scheme from the parameters is shown in Section 3. The designs are divided into $(m + 1)$ classes in Section 4. Some interesting combinatorial properties are obtained in Section 5. The necessary conditions for the existence of a class of these designs are obtained in Section 7. Finally, some numerical illustrations of these designs are given in the Appendix.

2. Definition and Parameters of a Group Divisible m-associate Design.

Definition 2.1. A Group Divisible m-associate design may be defined as follows:

(i) The experimental material is divided into b blocks of k units each, different treatments being applied to the units in the same block.

(ii) There are $v = N_1N_2 \cdots N_m$ treatments denoted by

$$v_{i_1 i_2 \cdots i_m}(i_1 = 1, 2, \cdots, N_1\,;\, i_2 = 1, 2, \cdots, N_2\,; \cdots ;\, i_m = 1, 2, \cdots, N_m).$$

Each treatment occurs once in each of the r blocks.

(iii) There can be established a relation of association between any two treatments satisfying the following requirements:

(a) Two treatments having only the first $(m - j)$ suffixes of $v_{i_1 i_2 \cdots i_m}$ the same are the jth associates $(j = 1, 2, \cdots, m)$.

(b) Each treatment has exactly n_j, jth associates.

(c) Given any two treatments which are ith associates, the number of treatments common to the jth associates of the first and the kth associates of the second is p^i_{jk} and is independent of the pair of treatments with which we start. Also, $p^i_{jk} = p^i_{kj}(i, j, k = 1, 2, \cdots, m)$.

(iv) Two treatments which are jth associates occur together in λ_j blocks.

The numbers $b, r, k, N_1, N_2, \cdots, N_m, \lambda_1, \lambda_2, \cdots, \lambda_m$ are known as the parameters of the GD m-associate design. We can easily see that

$$(2.1) \qquad n_i = N_mN_{m-1} \cdots N_{m-i+2}(N_{m-i+1} - 1), \qquad i = 1, 2, \cdots, m;$$

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and

$$(2.2)\qquad (p^i_{jk}) = \left[\begin{array}{c|c|c} 0_{(i-1)\times(i-1)} & x_{i-1} & 0_{(i-1)\times(m-i)} \\ \hline x'_{i-1} & \multicolumn{2}{c}{} \\ \cline{1-1} 0_{(m-i)\times(i-1)} & \multicolumn{2}{c}{D_{(m-i+1)\times(m-i+1)}} \end{array}\right], \qquad i = 1, 2, \cdots, m,$$

where $0_{i\times i'}$ is a null matrix of the order $i \times i'$; x_{i-1} is the $(i-1)$th order column vector with elements $n_1, n_2, \cdots, n_{i-1}$; x'_{i-1} is the transpose of x_{i-1}; and

$$D_{(m-i+1)\times(m-i+1)}$$

is the diagonal matrix with elements $N_m N_{m-1} \cdots N_{m-i+2}(N_{m-i+1} - 2)$, n_{i+1}, $n_{i+2}, \cdots, n_m$. The parameters satisfy the relations

$$N_1 N_2 \cdots N_m r = bk; \qquad \sum_{\alpha=1}^{m} n_\alpha = N_1 N_2 \cdots N_m - 1;$$

$$(2.3)\qquad \sum_{\alpha=1}^{m} n_\alpha \lambda_\alpha = r(k-1);$$

$$n_i p^i_{jk} = n_j p^j_{ik} = n_k p^k_{ij}; \qquad \sum_{k=1}^{m} p^i_{jk} = n_j - \delta_{ij}, \quad i, j, k = 1, 2, \cdots, m$$

where δ_{ij} is the Kronecker delta taking the value 1 or 0 according as $i = j$ or $i \neq j$. Since the parameters satisfy the above relations, it can be seen that a GD m-associate design is a special case of Partially Balanced Incomplete Block Designs defined by Bose and Nair [2].

3. Uniqueness of the Association Scheme. This section shows that the relations (2.1) and (2.2) imply the association scheme iii(a). In this section, we call a group of treatments which are first associates a first-associate group; a group of first-associate groups a second-associate group, etc. Let θ be any treatment. Let $\theta_1^{(i)}, \theta_2^{(i)}, \cdots, \theta_{n_i}^{(i)}$ be its ith associates ($i = 1, 2, \cdots, m$). Consider the treatments θ and $\theta_1^{(1)}$. Since $n_1 = N_m - 1$ and $p^1_{11} = N_m - 2$, the first associates of $\theta_1^{(1)}$ except θ are the same as the first associates of θ except $\theta_1^{(1)}$. Also, as

$$p^1_{1i} = 0 (i = 2, 3, \cdots, m),$$

we can divide the treatments into first-associate groups such that treatments in different first-associate groups are 2nd, 3rd, $\cdots$, or mth associates. It can be seen that each first-associate group contains N_m treatments. Thus the v treatments are divided into $N_1 N_2 \cdots N_{m-1}$ first-associate groups of N_m treatments each.

Now, consider the treatments θ and $\theta_1^{(2)}$. Since

$$n_i = p^2_{ii} = N_m N_{m-1} \cdots N_{m-i+2}(N_{m-i+1} - 1),$$

it is obvious that the ith associates of θ and $\theta_1^{(2)}$ are the same ($i = 3, 4, \cdots, m$). Also, as $p^2_{11} = 0$ and $p^2_{22} = N_m(N_{m-1} - 2)$, the $N_1 N_2 \cdots N_{m-1}$ first-associate groups of the above paragraph can be subdivided into $N_1 N_2 \cdots N_{m-2}$ second-associate groups of N_{m-1} first-associate groups of N_m treatments each such that (i) treatments in different second-associate groups are 3rd, 4th, $\cdots$, or mth associates, and (ii) treatments in different first-associate groups of a second-associate group are the second associates.

Again, consider the treatments θ and $\theta_1^{(3)}$. Since

$$n_i = p_{ii}^3 = N_m N_{m-1} \cdots N_{m-i+2}(N_{m-i+1} - 1),$$

it can be seen that the ith associates of θ and $\theta_1^{(3)}$ are the same ($i = 4, 5, \cdots, m$). Also, as $p_{11}^3 = 0 = p_{12}^3 = p_{22}^3$ and $p_{33}^3 = N_m N_{m-1}(N_{m-2} - 2)$, the $N_1 N_2 \cdots N_{m-2}$ second-associate groups can be further grouped into $N_1 N_2 \cdots N_{m-3}$ third-associate groups each containing N_{m-2} second-associate groups. These second-associate groups contain N_{m-1} first-associate groups each containing N_m treatments. Treatments in different third-associate groups are 4th, 5th, $\cdots$, or mth associates. Treatments in different second-associate groups of a third-associate group are the third associates and treatments in different first-associate groups of a second-associate group are the second-associates.

By similar reasoning, we finally obtain N_1, $(m - 1)$-associate groups of N_2, $(m - 2)$-associate groups, $\cdots$, of N_{m-1} first-associate groups of N_m treatments. The above grouping will be such that (i) treatments in different $(m - 1)$-associate groups are the mth associates, and (ii) treatments in different i-associate groups of an $(i + 1)$-associate group are the $(i + 1)$th associates

$$(i = 1, 2, \cdots, m - 2).$$

We can easily see that the above grouping of the treatments is the same as the association scheme iii(a). Hence the parameters (2.1) and (2.2) define the association scheme iii(a) uniquely and we have the following:

THEOREM 3.1. *The relations* (2.1) *and* (2.2) *for a Group Divisible m-associate design uniquely define the association scheme* iii(a).

4. Characterization of Group Divisible m-associate Designs. Let $n_{ij} = 1$, if the ith treatment occurs in the jth block; and $n_{ij} = 0$, otherwise. Then the $v \times b$ matrix $N = (n_{ij})$ is known as the incidence matrix of the GD m-associate design. From the definition of GD m-associate design, we can see that

$$\sum_{j=1}^{b} n_{ij}^2 = r, \qquad i = 1, 2, \cdots, v; \quad \text{and} \quad \sum_{j=1}^{b} n_{ij} n_{i'j} = \lambda_1, \lambda_2, \cdots, \text{or } \lambda_m$$

according as i and i' are 1st, 2nd, $\cdots$, or mth associates, $i \neq i'$; $i, i' = 1, 2, \cdots, v$. Now, by suitably marking the treatments, we have

$$NN' = \begin{bmatrix} B_m & A_m & \cdots & A_m \\ A_m & B_m & \cdots & A_m \\ \vdots & \vdots & & \vdots \\ A_m & A_m & \cdots & B_m \end{bmatrix}, \tag{4.1}$$

where, at any stage,

$$
(4.2)\qquad
\begin{aligned}
B_i &= \begin{bmatrix} B_{i-1} & A_{i-1} & \cdots & A_{i-1} \\ A_{i-1} & B_{i-1} & \cdots & A_{i-1} \\ \vdots & \vdots & & \vdots \\ A_{i-1} & A_{i-1} & \cdots & B_{i-1} \end{bmatrix}, \qquad & i = 2, 3, \cdots, m; \\
A_i &= \lambda_i E_{N_{m-i+2}N_{m-i+3} \cdots N_m}, & i = 2, 3, \cdots, m; \\
B_1 &= r,\ A_1 = \lambda_1,
\end{aligned}
$$

where $E_{N_{m-i+2}N_{m-i+3} \cdots N_m}$, is an $N_{m-i+2}N_{m-i+3} \cdots N_m$th order square matrix with positive unit elements everywhere. The orders of NN' and B_i are $N_1N_2 \cdots N_m$ and $N_{m-i+2}N_{m-i+3} \cdots N_m$ respectively $(i = 2, 3, \cdots, m)$. The matrices A_1 and B_1 are of unit order. Det (NN') can be evaluated in the usual manner and we get

$$
(4.3)\qquad |NN'| = rkP_1^{N_1-1}P_2^{N_1(N_2-1)} \cdots P_m^{N_1N_2\cdots N_{m-1}(N_m-1)},
$$

where

$$
(4.4)\qquad
\begin{aligned}
P_i = (r - \lambda_{m-i+1}) + (\lambda_1 - \lambda_{m-i+1})n_1 + \cdots + \\
(\lambda_{m-i} - \lambda_{m-i+1})n_{m-i}, \qquad i = 1, 2, \cdots, m.
\end{aligned}
$$

By replacing r by $(r - z)$ in det (NN') we can easily see that rk and P_i's $(i = 1, 2, \cdots, m)$ are the distinct characteristic roots of NN'. We know from the result of Connor and Clatworthy [4] that the characteristic roots of NN' cannot be negative for an existing design. Thus we have the following theorem:

THEOREM 4.1. *A necessary condition for the existence of a Group Divisible m-associate design is that* $P_i \geqq 0$ $(i = 1, 2, \cdots, m)$.

The designs with the following parameters violate the above necessary condition and hence are impossible. The reason of impossibility is shown in brackets against the parameters.

1. $v = 90 = b, r = 9 = k, N_1 = 3, N_2 = 15, N_3 = 2,$
$\lambda_1 = 12, \lambda_2 = 0, \lambda_3 = 1$ $(P_1, P_3 < 0)$.
2. $v = 12, b = 15, r = 5, k = 4, N_1 = 2 = N_2, N_3 = 3,$
$\lambda_1 = 0, \lambda_2 = 3, \lambda_3 = 1$ $(P_2 < 0)$.
3. $v = 8, b = 4, r = 3, k = 6, N_1 = 2 = N_2 = N_3,$
$\lambda_1 = 3, \lambda_2 = 0, \lambda_3 = 3$ $(P_1 < 0)$.
4. $v = 16 = b, r = 5 = k, N_1 = 2 = N_2, N_3 = 4,$
$\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 2$ $(P_1 < 0)$.
5. $v = 16, b = 24, r = 6, k = 4, N_1 = 2 = N_2 = N_3 = N_4,$
$\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 0, \lambda_4 = 2$ $(P_1 < 0)$.
6. $v = 32, b = 64, r = 10, k = 5, N_1 = 2 = N_2 = N_3 = N_4 = N_5,$
$\lambda_1 = 4, \lambda_2 = 0, \lambda_3 = 1, \lambda_4 = 0, \lambda_5 = 2$
$(P_1 < 0)$.

We can classify the existing designs mainly into $(m + 1)$ classes as follows:
(1) Singular GD m-associate designs characterized by $P_m = 0$;
(2) P_m — regular GD m-associate designs characterized by $P_m > 0, P_{m-1} = 0$;
.
.
.
(i) $P_m, P_{m-1}, \cdots, P_{m-i+2}$ — regular GD m-associate designs characterized by $P_m > 0$, $P_{m-1} > 0$, $\cdots$, $P_{m-i+2} > 0$, $P_{m-i+1} = 0$;
.
.
.
(m) $P_m, P_{m-1}, \cdots, P_2$ — regular GD m-associate designs characterized by $P_m > 0$, $P_{m-1} > 0$, $\cdots$, $P_2 > 0$, $P_1 = 0$; and
$(m + 1)$ Regular GD m-associate designs characterized by $P_i > 0$ $(i = 1, 2, \cdots m)$.

Excepting the last two classes, the other classes can be further divided; but, since this will be cumbersome, we do not do so.

5. Some Combinatorial Properties of Group Divisible m-associate Designs. If $P_i = 0 = P_{i+1} (i = 1, 2, \cdots, m - 1)$, we have $\lambda_{m-i+1} = \lambda_{m-i+2}$. Thus if $P_1 = 0 = P_2 = \cdots = P_m$, then $r = \lambda_1 = \cdots = \lambda_m$ and the GD m-associate design reduces to an ordinary randomised block design. Hence, we have

THEOREM 5.1. *If, in a Group Divisible m-associate design,* $P_1 = 0 = P_2 = \cdots = P_m$, *then the design reduces to a randomized block design.*

Let j consecutive λ's $(j = 2, 3, \cdots, m - 1)$ of the GD m-associate design be equal. In this case we can see from the association scheme that the design reduces to a GD $(m - j + 1)$-associate design. The above result can be written in the form of the following theorem.

THEOREM 5.2. *If, in a Group Divisible m-associate design j consecutive* λ's $(j = 2, 3, \cdots, m - 1)$ *are equal, then the design reduces to a Group Divisible* $(m - j + 1)$*-associate design.*

We now prove another important theorem.

THEOREM 5.3. *In a* $P_m, P_{m-1}, \cdots, P_2$ — *regular Group Divisible m-associate design k, is divisible by* N_1. *Further, every block contains* k/N_1 *treatments of the form* $v_{ii_2\cdots i_m}$ $(i_2 = 1, 2, \cdots, N_2; i_3 = 1, 2, \cdots, N_3; \cdots; i_m = 1, 2, \cdots, N_m)$ *for any i* $(i = 1, 2, \cdots, N_1)$.

PROOF. For any i $(i = 1, 2, \cdots, N_1)$, let e_j^i treatments of the form $v_{ii_2\cdots i_m}$ $(i_2 = 1, 2, \cdots, N_2; i_3 = 1, 2, \cdots, N_3; \cdots; i_m = 1, 2, \cdots, N_m)$ occur in the jth block $(j = 1, 2, \cdots, b)$. Then, we have

$$\sum_{j=1}^{b} e_j^i = N_2 N_3 \cdots N_m r,$$
$$(5.1) \qquad \sum_{j=1}^{b} e_j^i(e_j^i - 1) = N_2 N_3 \cdots N_m(n_1\lambda_1 + n_2\lambda_2 + \cdots + n_{m-1}\lambda_{m-1}),$$

since each of the treatments occur in r blocks and every pair of treatments of the form $v_{ii_2\cdots i_m}$ $(i_2 = 1, 2, \cdots, N_2; i_3 = 1, 2, \cdots, N_3; \cdots; i_m = 1, 2, \cdots, N_m)$

occurs in $\lambda_1, \lambda_2, \cdots$, or λ_{m-1} blocks. Using the property of $P_m, P_{m-1}, \cdots, P_2$ — regular GD m-associate design and (5.1), we get

$$\sum_{j=1}^{b} (e_j^i)^2 = N_2^2 N_3^2 \cdots N_m^2 \lambda_m . \tag{5.2}$$

Let $\overline{e^i} = b^{-1} \sum_{j=1}^{b} e_j^i = k/N_1$. Then,

$$\sum_{j=1}^{b} (e_j^i - \overline{e^i})^2 = N_2^2 N_3^2 \cdots N_m^2 \lambda_m - bk^2/N_1^2 = 0. \tag{5.3}$$

Therefore, $e_1^i = e_2^i = \cdots = e_b^i = \overline{e^i} = k/N_1$. Since $e_j^i(i = 1, 2, \cdots, N_1; j = 1, 2, \cdots, b)$ must be integral, k is divisible by N_1. Further $e_j^i = k/N_1$ $(i = 1, 2, \cdots, N_1; j = 1, 2, \cdots, b)$. This completes the proof of the theorem.

The following P_3, P_2-regular GD 3-associate designs have a non-integral value for k/N_1 and hence are non-existing:

1. $v = 12, b = 16, r = 4, k = 3, N_1 = 2, N_2 = 3, N_3 = 2,$ $\lambda_1 = 2, \lambda_2 = 0, \lambda_3 = 1.$
2. $v = 12, b = 16, r = 4, k = 3, N_1 = 2 = N_2, N_3 = 3,$ $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 1.$
3. $v = 12, b = 9, r = 3, k = 4, N_1 = 3, N_2 = N_3 = 2,$ $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 1.$
4. $v = 20, b = 32, r = 8, k = 5, N_1 = 2, N_2 = 5, N_3 = 2,$ $\lambda_1 = 4, \lambda_2 = 1, \lambda_3 = 2.$

A GD m-associate design is said to be symmetrical if $b = v$ and in consequence $r = k$. Shrikhande [9] and Chowla and Ryser [5] have obtained conditions necessary for the existence of symmetrical balanced incomplete block designs. Bose and Connor have obtained necessary conditions for the existence of symmetrical regular GD designs. We shall extend their results to symmetrical regular GD m-associate designs. With this in view, we give a brief resume of the important properties of the Legendre symbol, the Hilbert norm residue symbol and the Hasse-Minkowski invariant.

6. Some known results about the Legendre symbol, the Hilbert norm residue symbol and the Hasse-Minkowski invariant. The Legendre symbol is defined as

$$(a/p) = \begin{cases} +1, \text{ if } a \text{ is quadratic residue of } p; \\ -1, \text{ if } a \text{ is a non quadratic residue of } p. \end{cases} \tag{6.1}$$

A slight generalization of the Legendre symbol, is the Hilbert norm residue symbol $(a, b)_p$. If a and b are any non zero rational numbers, we define $(a, b)_p$ to have the value $+1$ or -1 according as the congruence

$$ax^2 + by^2 \equiv 1 \pmod{p^r}, \tag{6.2}$$

has or has not for every value of r, rational solutions x_r and y_r. Here p is any prime including the conventional prime $p_\infty = \infty$.

Many properties of $(a, b)_p$ are given by Bruck and Ryser [3], Jones [6] and

Pall [7]. For further use, we reproduce the properties of $(a, b)_p$ taken from the above references, in the form of the following theorems.

THEOREM 6.1. *If* m and m' *are integers not divisible by the odd prime* p, *then*

$$(m, m')_p = +1, \tag{6.3}$$

$$(m, p)_p = (m/p). \tag{6.4}$$

Moreover, if $m \equiv m' \not\equiv 0 \pmod{p}$, *then*

$$(m, p)_p = (m', p)_p. \tag{6.5}$$

THEOREM 6.2. *For arbitrary non-zero integers* m, m', n, n', *and for every prime* p,

$$(-m, m)_p = +1, \tag{6.6}$$

$$(m, n)_p = (n, m)_p, \tag{6.7}$$

$$(mm', n)_p = (m, n)_p(m', n)_p, \tag{6.8}$$

$$(m, nn')_p = (m, n)_p(m, n')_p, \tag{6.9}$$

$$(mm', m - m')_p = (m, -m')_p, \tag{6.10}$$

$$\prod_{j=1}^{m} (j, j+1)_p = ((m+1)!, -1)_p, \tag{6.11}$$

and

$$(as^2, b)_p = (a, b)_p. \tag{6.12}$$

Now, let $A = (a_{ij})$ be any $n \times n$ symmetric matrix with rational elements. The matrix B is said to be rationally congruent to A, written $A \sim B$, provided there exists a non-singular matrix C with rational elements, such that $A = CBC'$, where C' is the transpose of C. If D_i $(i = 1, 2, \cdots, n)$ denotes the leading principal minor determinant of order i in the matrix A, then if none of the D_i vanishes, the quantity

$$C_p = C_p(A) = (-1, -D_n)_p \prod_{i=1}^{n-1} (D_i, -D_{i+1})_p, \tag{6.13}$$

is invariant for all matrices rationally congruent to A. $C_p(A)$ defined above is known as the Hasse-Minkowski invariant.

The following lemmas regarding C_p will be useful.

LEMMA 6.1. *If* d *is a rational number and* $\Delta_m = dI_m$, *where* I_m *is the identity matrix of order* m, *then*

$$C_p(\Delta_m) = (-1, -1)_p(d, -1)_p^{m(m+1)/2}. \tag{6.14}$$

LEMMA 6.2. *If* A *and* B *are symmetric matrices with rational elements and* $U = A \dotplus B$, *is the direct sum of* A *and* B, *then*

$$C_p(U) = (-1, -1)_p C_p(A) C_p(B)(|A|, |B|)_p. \tag{6.15}$$

7. Necessary conditions for the existence of Symmetrical Regular Group Divisible m-associate Designs. Since the design is a symmetric one, $\det(NN')$ is a perfect square (cf. Connor and Clatworthy, and Shrikhande). Thus

$$P_1^{N_1-1}P_2^{N_1(N_2-1)} \cdots P_m^{N_1N_2\cdots N_{m-1}(N_m-1)}$$

is a perfect square. This result can be written in the form of the following theorem.

THEOREM 7.1. *A necessary condition for the existence of a regular symmetrical Group Divisible m-associate design is that* $P_1^{N_1-1}P_2^{N_1(N_2-1)} \cdots P_m^{N_1N_2\cdots N_{m-1}(N_m-1)}$ *is a perfect square.*

The designs with the following parameters do not satisfy the above theorem and hence are impossible.

1. $v = 24 = b, r = 6 = k, N_1 = 4, N_2 = 2, N_3 = 3,$
$\lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 1.$
2. $v = 32 = b, r = 7 = k, N_1 = 4, N_2 = 2, N_3 = 4,$
$\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 1.$
3. $v = 30 = b, r = 7 = k, N_1 = 5, N_2 = 3, N_3 = 2,$
$\lambda_1 = 2, \lambda_2 = 4, \lambda_3 = 1.$
4. $v = 30 = b, r = 12 = k, N_1 = 3, N_2 = 5, N_3 = 2,$
$\lambda_1 = 4, \lambda_2 = 6, \lambda_3 = 4.$
5. $v = 54 = b, r = 11 = k, N_1 = 3 = N_2 = N_3, N_4 = 2,$
$\lambda_1 = 6, \lambda_2 = 5, \lambda_3 = 4, \lambda_4 = 1.$

Let

$$Q_i = B_i - A_i, \qquad i = 2, 3, \cdots, m; \tag{7.1}$$

where B_i's and A_i's are as defined in Section 4. $\text{Det}(Q_i)i = 2, 3, \cdots, m$ can be found easily, and we have

$$\begin{aligned} | Q_i | = {} & \{(r - \lambda_i) + (\lambda_1 - \lambda_i)n_1 + \cdots + (\lambda_{i-1} - \lambda_i)n_{i-1}\} \\ & \{(r - \lambda_{i-1}) + (\lambda_1 - \lambda_{i-1})n_1 + \cdots + (\lambda_{i-2} - \lambda_{i-1})n_{i-2}\}^{N_{m-i+2}-1} \\ & \cdot \\ & \cdot \\ & \cdot \\ & \{r - \lambda_1\}^{N_{m-i+2}N_{m-i+3}\cdots N_{m-1}(N_m-1)}. \end{aligned} \tag{7.2}$$

Now, let us calculate the Hasse-Minkowski invariant of (NN') for odd primes using the method of Bose and Connor. Taking the direct sum with $-\lambda_m$, NN' becomes

$$(NN')_1 = \begin{bmatrix} NN' & \\ & -\lambda_m \end{bmatrix}. \tag{7.3}$$

Therefore, from Lemma 6.2,

$$C_p(NN')_1 = C_p(NN')(\lambda_m, -1)_p. \tag{7.4}$$

But

$$(7.5) \qquad (NN')_1 \sim \begin{bmatrix} Q_m & & & & L \\ & Q_m & & & L \\ & & \ddots & & \vdots \\ & & & Q_m & L \\ L & L & \cdots & L & -\lambda_m \end{bmatrix}$$

where L is an $N_2N_3 \cdots N_m$th order column vector with $-\lambda_m$ everywhere. Hence

$$(7.6) \quad C_p(NN')_1 = \{C_p(Q_m)\}^{N_1}(|Q_m|, -1)_p^{N_1(N_1+1)/2}(\lambda_m, -|Q_m|^{N_1})_p.$$

Equating (7.4) and (7.6), we get

$$(7.7) \quad C_p(NN') = \{C_p(Q_m)\}^{N_1}(|Q_m|, -1)_p^{N_1(N_1+1)/2}(\lambda_m, |Q_m|)_p^{N_1}.$$

$C_p(Q_i)i = 2, 3, \cdots, m$ can be calculated in a similar way as above and we get

$$(7.8) \quad \begin{aligned} C_p(Q_2) = {} & (r - \lambda_1, -1)_p^{N_m(N_m-1)/2}(\lambda_1 - \lambda_2, r - \lambda_1)_p^{N_m} \\ & (|Q_2|, r - \lambda_1)_p^{N_m}(|Q_2|, \lambda_1 - \lambda_2)_p, \end{aligned}$$

$$(7.9) \quad \begin{aligned} C_p(Q_i) = {} & \{C_p(Q_{i-1})\}^{N_{m-i+2}}(|Q_{i-1}|, -1)_p^{N_{m-i+2}(N_{m-i+2}+1)/2} \\ & (\lambda_{i-1} - \lambda_i, |Q_i|)_p(\lambda_{i-1} - \lambda_i, |Q_{i-1}|)^{N_{m-i+2}} \\ & (|Q_i|, |Q_{i-1}|)^{N_{m-i+2}}, \qquad i = 3, 4, \cdots, m. \end{aligned}$$

Equation (7.9) is a recurrence relation. This equation with the help of (7.2) and (7.8) finally gives $C_p(Q_m)$. Substituting this value of $C_p(Q_m)$ in (7.7), $C_p(NN')$ can be calculated. Now, since $I_v = N^{-1}(NN')(N')^{-1}$, $I_v \sim NN'$. Therefore,

$$(7.10) \qquad C_p(NN') = C_p(I_v) = (-1, -1)_p = +1.$$

Thus we have the following theorem

THEOREM 7.2. *A necessary condition for the existence of a symmetrical regular Group Divisible m-associate design is that $C_p(NN') = +1$, for odd primes p where $C_p(NN')$ is calculated from* (7.2), (7.8), (7.9) *and* (7.7).

When there are only three associate classes the above calculations can be simplified and the corollary follows:

COROLLARY 7.2.1. *A necessary condition for the existence of a regular symmetrical Group Divisible* 3*-associate design is that*

$$(7.11) \quad \begin{aligned} & (P_3, -1)_p^{\{N_1N_2N_3(N_1+N_2+N_3+3)-N_1N_2(N_1+N_2)\}/2}(\lambda_3, P_1)_p \\ & \cdot (P_1, -1)_p^{N_1(N_1+1)/2}(P_2, -1)^{N_1\{N_2(N_2+1)+(N_2-1)(N_1+1)\}/2} \\ & \cdot (\lambda_2 - \lambda_3, P_1P_2)_p^{N_1}(\lambda_1 - \lambda_2, P_2P_3)_p^{N_1N_2}(P_2, P_3)_p^{N_1N_2} \\ & \cdot (P_1, P_2)_p^{N_1N_2}(P_1, P_3)_p^{N_1N_2(N_3-1)} = +1, \qquad \text{for all odd primes } p. \end{aligned}$$

ILLUSTRATION 7.2.1. Consider the GD 3-associate design with the parameters

$$v = 27 = b, r = 7 = k, N_1 = 3 = N_2 = N_3, \lambda_1 = 6, \lambda_2 = 2, \lambda_3 = 1.$$

The left hand side of (7.11) is

$$(22, 13)_p = (13, 2)_p(13, 11)_p = -1, \qquad \text{when } p = 11.$$

Thus the corollary 7.2.1 is not satisfied and the design is impossible.

ILLUSTRATION 7.2.2. For the GD 3-associate design with the parameters

$$v = 48 = b, r = 10 = k, N_1 = 6, N_2 = 4, N_3 = 2,$$
$$\lambda_1 = 4, \lambda_2 = 1, \lambda_3 = 2.$$

the left hand side of (7.11) is

$$(12, -1)_p = (3, -1)_p = -1, \text{ for } p = 3.$$

The Corollary 7.2.1 is not satisfied and the design is impossible.

By applying the Corollary 7.2.1, it can be easily verified that the following designs are non-existing:

1. $v = 24 = b, r = 9 = k, N_1 = 2 = N_2, N_3 = 6, \lambda_1 = 6, \lambda_2 = 1, \lambda_3 = 3.$
2. $v = 24 = b, r = 10 = k, N_1 = 2 = N_2, N_3 = 6, \lambda_1 = 6, \lambda_2 = 2, \lambda_3 = 4.$
3. $v = 24 = b, r = 10 = k, N_1 = 6, N_2 = 2 = N_3, \lambda_1 = 6, \lambda_2 = 2, \lambda_3 = 4.$
4. $v = 40 = b, r = 13 = k, N_1 = 10, N_2 = 2 = N_3, \lambda_1 = 10, \lambda_2 = 1, \lambda_3 = 4.$

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APPENDIX

Here we give some numerical constructions in the useful range r, $k \leqq 10$. For convenience we denote the treatment v_{ijk} by (ijk) in the following examples.

1. $v = 8 = b$, $r = 3 = k$, $N_1 = 2 = N_2 = N_3$, $\lambda_1 = 2$, $\lambda_2 = 0$, $\lambda_3 = 1$. Taking the treatments as

(111)	(112)	(211)	(212)
(121)	(122)	(221)	(222)

the plan of the design is

	[(111)	(112)	(211)]
	[(112)	(111)	(212)]
	[(121)	(122)	(221)]
	[(122)	(121)	(222)]
	[(211)	(212)	(121)]
	[(212)	(211)	(122)]
	[(221)	(222)	(111)]
	[(222)	(221)	(112)]
Reps.	I	II	III

2. $v = 8 = b$, $r = 4 = k$, $N_1 = 2 = N_2 = N_3$, $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 2$. Taking the treatments as in the above example, the plan of the design is

	[(111)	(112)	(211)	(221)]
	[(112)	(111)	(212)	(222)]
	[(121)	(122)	(221)	(212)]
	[(122)	(121)	(222)	(211)]
	[(222)	(221)	(111)	(121)]
	[(221)	(222)	(112)	(122)]
	[(211)	(212)	(122)	(111)]
	[(212)	(211)	(121)	(112)]
Reps.	I	II	III	IV

3. $v = 8$, $b = 24$, $r = 9$, $k = 3$, $N_1 = 2 = N_2 = N_3$, $\lambda_1 = 4$, $\lambda_2 = 1$, $\lambda_3 = 3$. Taking the treatments as in Example 1, the plan of the design is

			Reps.
[(111)	(112)	(211)]	
[(112)	(111)	(212)]	
[(121)	(122)	(221)]	
[(122)	(121)	(222)]	I, II, III
[(211)	(212)	(121)]	
[(212)	(211)	(122)]	
[(221)	(222)	(111)]	
[(222)	(221)	(112)]	

	[(111)	(112)	(221)]	
	[(112)	(111)	(222)]	
	[(121)	(122)	(211)]	
	[(122)	(121)	(212)]	IV, V, VI
	[(211)	(212)	(111)]	
	[(212)	(211)	(112)]	
	[(221)	(222)	(121)]	
	[(222)	(221)	(122)]	
	[(111)	(121)	(211)]	
	[(122)	(111)	(221)]	
	[(121)	(112)	(222)]	
	[(112)	(122)	(212)]	VII, VIII, IX
	[(211)	(221)	(112)]	
	[(222)	(211)	(122)]	
	[(221)	(212)	(121)]	
	[(212)	(222)	(111)]	
Reps.	I, II, III	IV, V, VI	VII, VIII, IX	

4. $v = 8 = b$, $r = 5 = k$, $N_1 = 2 = N_2 = N_3$, $\lambda_1 = 4$, $\lambda_2 = 2$, $\lambda_3 = 3$. Taking the treatments as in Example 1, the plan of the design is

	[(111)	(112)	(211)	(221)	(222)]
	[(112)	(111)	(212)	(222)	(221)]
	[(121)	(122)	(222)	(211)	(212)]
	[(122)	(121)	(221)	(212)	(211)]
	[(211)	(212)	(121)	(111)	(112)]
	[(212)	(211)	(122)	(112)	(111)]
	[(221)	(222)	(111)	(121)	(122)]
	[(222)	(221)	(112)	(122)	(121)]
Reps.	I	II	III	IV	V

5. $v = 8 = b$, $r = 6 = k$, $N_1 = 2 = N_2 = N_3$, $\lambda_1 = 4$, $\lambda_2 = 5$, $\lambda_3 = 4$. Taking the treatments as in Example 1, the plan of the design is

	[(111)	(112)	(121)	(122)	(211)	(221)]
	[(112)	(121)	(122)	(111)	(222)	(211)]
	[(121)	(122)	(111)	(112)	(221)	(212)]
	[(122)	(111)	(112)	(121)	(212)	(222)]
	[(211)	(212)	(221)	(222)	(111)	(121)]
	[(212)	(221)	(222)	(211)	(122)	(111)]
	[(221)	(222)	(211)	(212)	(112)	(122)]
	[(222)	(211)	(212)	(221)	(121)	(112)]
Reps.	I	II	III	IV	V	VI

6. $v = 12 = b$, $r = 4 = k$, $N_1 = 2 = N_2$, $N_3 = 3$, $\lambda_1 = 3$, $\lambda_2 = 0$, $\lambda_3 = 1$.

Taking the treatments as

(111)	(112)	(113)	(211)	(212)	(213)
(121)	(122)	(123)	(221)	(222)	(223)

the plan of the design is

	[(111)	(112)	(113)	(211)]
	[(112)	(113)	(111)	(212)]
	[(113)	(111)	(112)	(213)]
	[(121)	(122)	(123)	(221)]
	[(122)	(123)	(121)	(222)]
	[(123)	(121)	(122)	(223)]
	[(211)	(212)	(213)	(121)]
	[(212)	(213)	(211)	(122)]
	[(213)	(211)	(212)	(123)]
	[(221)	(222)	(223)	(111)]
	[(222)	(223)	(221)	(112)]
	[(223)	(221)	(222)	(113)]
Reps.	I	II	III	IV

7. $v = 16 = b$, $r = 4 = k$, $N_1 = 2 = N_2 = N_3 = N_4$, $\lambda_1 = 0$, $\lambda_2 = 2$, $\lambda_3 = 0$, $\lambda_4 = 1$. Taking the treatments as

(1111)	(1112)	(1211)	(1212)
(1121)	(1122)	(1221)	(1222)
(2111)	(2112)	(2211)	(2212)
(2121)	(2122)	(2221)	(2222)

the plan of the design is

	[(1111)	(1121)	(2111)	(2121)]
	[(1121)	(1111)	(2112)	(2122)]
	[(2211)	(2221)	(1111)	(1122)]
	[(2212)	(2222)	(1122)	(1111)]
	[(2221)	(2211)	(1112)	(1121)]
	[(2222)	(2212)	(1121)	(1112)]
	[(1112)	(1122)	(2121)	(2111)]
	[(1122)	(1112)	(2122)	(2112)]
	[(2111)	(2122)	(1211)	(1222)]
	[(2112)	(2121)	(1222)	(1211)]
	[(1211)	(1221)	(2211)	(2222)]
	[(1221)	(1211)	(2212)	(2221)]
	[(1222)	(1212)	(2222)	(2211)]
	[(1212)	(1222)	(2221)	(2212)]
	[(2122)	(2111)	(1221)	(1212)]
	[(2121)	(2112)	(1212)	(1221)]
Reps.	I	II	III	IV

8. $v = 16$, $b = 32$, $r = 8$, $k = 4$, $N_1 = 2 = N_2 = N_3 = N_4$, $\lambda_1 = 4$, $\lambda_2 = 2$, $\lambda_3 = 0$, $\lambda_4 = 2$. Taking the treatments as in the above example, the plan of the design is

					Reps.
	[(1111)	(1112)	(2111)	(2121)]	
	[(1112)	(1111)	(2112)	(2122)]	
	[(2111)	(2122)	(1121)	(1122)]	
	[(2112)	(2121)	(1122)	(1121)]	I, II
	[(1211)	(1212)	(2211)	(2221)]	
	[(1212)	(1211)	(2212)	(2222)]	
	[(2211)	(2222)	(1221)	(1222)]	
	[(2212)	(2221)	(1222)	(1221)]	
	[(2221)	(2211)	(1111)	(1112)]	
	[(2222)	(2212)	(1112)	(1111)]	
	[(1121)	(1122)	(2222)	(2211)]	
	[(1122)	(1121)	(2221)	(2212)]	III, IV
	[(2121)	(2111)	(1211)	(1212)]	
	[(2122)	(2112)	(1212)	(1211)]	
	[(1221)	(1222)	(2122)	(2111)]	
	[(1222)	(1221)	(2121)	(2112)]	
	[(2111)	(2112)	(1111)	(1121)]	
	[(2112)	(2111)	(1112)	(1122)]	
	[(1111)	(1122)	(2121)	(2122)]	
	[(1112)	(1121)	(2122)	(2121)]	V, VI
	[(2211)	(2212)	(1211)	(1221)]	
	[(2212)	(2211)	(1212)	(1222)]	
	[(1211)	(1222)	(2221)	(2222)]	
	[(1212)	(1221)	(2222)	(2221)]	
	[(1121)	(1111)	(2211)	(2212)]	
	[(1122)	(1112)	(2212)	(2211)]	
	[(2221)	(2222)	(1122)	(1111)]	
	[(2222)	(2221)	(1121)	(1112)]	VII, VIII
	[(1221)	(1211)	(2111)	(2112)]	
	[(1222)	(1212)	(2112)	(2111)]	
	[(2121)	(2122)	(1222)	(1211)]	
	[(2122)	(2121)	(1221)	(1212)]	
Reps.	I, II	III, IV	V, VI	VII, VIII	

9. $v = 18$, $b = 42$, $r = 7$, $k = 3$, $N_1 = 3$, $N_2 = 3$, $N_3 = 2$, $\lambda_1 = 2$, $\lambda_2 = 0$,

$\lambda_3 = 1$. Taking the treatments as

(111)	(112)	(211)	(212)	(311)	(312)
(121)	(122)	(221)	(222)	(321)	(322)
(131)	(132)	(231)	(232)	(331)	(332)

the plan of the design is

[(111), (112), (211)]	[(111), (112), (212)]
[(121), (122), (221)]	[(121), (122), (222)]
[(131), (132), (231)]	[(131), (132), (232)]
[(211), (212), (311)]	[(211), (212), (312)]
[(221), (222), (321)]	[(221), (222), (322)]
[(231), (232), (331)]	[(231), (232), (332)]
[(311), (312), (111)]	[(311), (312), (112)]
[(321), (322), (121)]	[(321), (322), (122)]
[(331), (332), (131)]	[(331), (332), (132)]
[(111), (221), (331)]	[(111), (222), (332)]
[(111), (231), (321)]	[(111), (232), (322)]
[(112), (221), (332)]	[(112), (222), (331)]
[(112), (231), (322)]	[(112), (232), (321)]
[(121), (211), (331)]	[(121), (212), (332)]
[(121), (231), (311)]	[(121), (232), (312)]
[(122), (211), (332)]	[(122), (232), (311)]
[(122), (231), (312)]	[(122), (212), (331)]
[(131), (211), (321)]	[(131), (212), (322)]
[(131), (221), (311)]	[(131), (222), (312)]
[(132), (211), (322)]	[(132), (212), (321)]
[(132), (221), (312)]	[(131), (222), (311)]

10. $v = 24, b = 16, r = 4, k = 6, N_1 = 3, N_2 = 4, N_3 = 2, \lambda_1 = 4, \lambda_2 = 0, \lambda_3 = 1$. Taking the treatments as

(111)	(112)	(211)	(212)	(311)	(312)
(121)	(122)	(221)	(222)	(321)	(322)
(131)	(132)	(231)	(232)	(331)	(332)
(141)	(142)	(241)	(242)	(341)	(342)

the plan of the design is

[(111), (112), (211), (212), (311), (312)]
[(111), (112), (221), (222), (321), (322)]
[(111), (112), (231), (232), (331), (332)]
[(111), (112), (241), (242), (341), (342)]
[(121), (122), (211), (212), (321), (322)]
[(121), (122), (221), (222), (331), (332)]
[(121), (122), (231), (232), (341), (342)]
[(121), (122), (241), (242), (311), (312)]

[(131), (132), (211), (212), (331), (332)]
[(131), (132), (221), (222), (341), (342)]
[(131), (132), (231), (232), (311), (312)]
[(131), (132), (241), (242), (321), (322)]
[(141), (142), (211), (212), (341), (342)]
[(141), (142), (221), (222), (311), (312)]
[(141), (142), (231), (232), (321), (322)]
[(141), (142), (241), (242), (331), (332)]