SMALL SAMPLE DISTRIBUTIONS FOR MULTI-SAMPLE STATISTICS OF THE SMIRNOV TYPE

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1. Introduction and Summary. Let

$$X_1^{(i)}, X_2^{(i)}, \cdots, X_{n_i}^{(i)}, \qquad i = 1, 2, \cdots, c, \tag{1.1}$$

be samples of c independent random variables $X^{(i)}$ with continuous cumulative distribution functions $F^{(i)}$, and let

$$\begin{aligned} F^{*(i)}(x) &= 0 & & x < X_1^{(i)} \\ F^{*(i)}(x) &= k/n_i & & X_k^{(i)} \leqq x < X_{k+1}^{(i)}, \ 1 \leqq k < n_i \\ F^{*(i)}(x) &= 1 & & X_{n_i}^{(i)} \leqq x \end{aligned} \tag{1.2}$$

be the corresponding c empirical distribution functions. We define the statistics

$$D(n_1, n_2, \cdots, n_c) = \sup_{\substack{x,i,j \\ (i,j=1,2,\cdots,c)}} | F^{*(i)}(x) - F^{*(j)}(x) | \tag{1.3}$$

and

$$D^+(n_1, n_2, \cdots, n_c) = \sup_{\substack{x,i,j \\ (i<j;i,j=1,2,\cdots,c)}} [F^{*(i)}(x) - F^{*(j)}(x)]. \tag{1.4}$$

The well known Kolmogorov-Smirnov statistics $D(m, n)$ and $D^+(m, n)$ are special cases of (1.3) and (1.4), respectively, with $c = 2$, $n_1 = m$, $n_2 = n$.

The exact small sample distribution, under the null hypothesis

$$F^{(i)} = F^{(j)} \qquad \text{for all } i, j = 1, 2, \cdots, c, \tag{1.5}$$

of the statistics defined by (1.3) and (1.4) for any number c of samples, and for any sample sizes $n_1, n_2, \cdots, n_c$, can be obtained by solving simple difference equations which lend themselves to programming for machine computation. Using this procedure, tables of values of

$$P[D(n, n, n) \leqq r], \qquad P[D(n, n) \leqq r], \qquad P[D^+(n, n) \leqq r]$$

were computed for selected values of n between 1 and 40 and of $r = k/n$, $k = 1, 2, \cdots, n$.

Furthermore, the inequalities

$$P[D(n, n, \cdots, n) \leqq r] \geqq 1 - [c(c-1)/2]P[D(n, n) > r]$$

$$P[D(n, n, \cdots, n) \leqq r] \geqq 1 - [c(c-1)(c-2)/6]P[D(n, n, n) > r]$$

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are noted, which may be useful for values of $c \geqq 4$ for which tables are not available.

2. The general difference equations. The set of all possible values of the c-dimensional random variable $[F^{*(1)}(x), F^{*(2)}(x), \cdots, F^{*(c)}(x)]$, for fixed x, consists of the points

$$(2.1) \qquad (k_1/n_1, k_2/n_2, \cdots k_c/n_c) \qquad k_i = 1, 2, \cdots n_i; \quad i = 1, 2, \cdots, c,$$

of the c dimensional unit cube.

By the transformation $y_i = n_i x_i$ the c dimensional unit cube is transformed into the c-dimensional rectangular prism with sides $n_1, n_2, \cdots n_c$, and the points (2.1) are transformed into the points

$$(2.2) \qquad (k_1, k_2, \cdots, k_c) \qquad k_i = 1, 2, \cdots n_i; \quad i = 1, 2, \cdots, c.$$

Under the null hypothesis (1.5) the c samples may be considered as c successive drawings of $n_1, n_2, \cdots n_c$ observations from the same population, with equal probabilities of each of the $N!$ ways of drawing the ordered sample of size N, where

$$(2.4) \qquad N = n_1 + n_2 + \cdots + n_c.$$

The points (2.2) may be interpreted as being obtained in the following manner: the sample values $X_k^{(i)}$, $k = 1, 2, \cdots, n_i$; $i = 1, 2, \cdots, c$, are observed and plotted on the x-axis. It is agreed that, as one moves along the x-axis from $-\infty$ to $+\infty$, the coordinate k_i of the point (2.2) is increased by a unit whenever a value $X_k^{(i)}$ is crossed. By this procedure one obtains a path through points of the form (2.2), starting at $(0, 0, \cdots, 0)$ and ending at $(n_1, n_2, \cdots, n_c)$, and each set of the c samples determines such a path. Under the null hypothesis (1.5) all these paths are equally probable, and their number is clearly

$$(2.5) \qquad Q(n_1, n_2, \cdots, n_c) = N!/(n_1!n_2! \cdots n_c!).$$

We define, generally,

$$(2.6) \qquad \begin{aligned} &Q(k_1, k_2, \cdots, k_c) \\ &\qquad = \text{number of paths from } (0, 0, \cdots, 0) \text{ to } (k_1, k_2, \cdots, k_c) \end{aligned}$$

for any non-negative integers $k_1, k_2, \cdots, k_c$. The function Q satisfies the difference equation

$$(2.7) \qquad \begin{aligned} Q(k_1, k_2, \cdots, k_c) &= Q(k_1 - 1, k_2, \cdots, k_c) \\ &\quad + Q(k_1, k_2 - 1, \cdots, k_c) \\ &\quad \cdots \\ &\quad + Q(k_1, k_2, \cdots, k_c - 1) \end{aligned}$$

since the number of ways of getting from $(0, 0, \cdots, 0)$ to $(k_1, k_2, \cdots, k_c)$ is evidently the sum of the numbers of ways of getting to points from which

$(k_1, k_2, \cdots, k_c)$ can be reached in one step. We have for Q the initial condition

$$Q(0, 0, \cdots, 0) = 1. \tag{2.8}$$

To compute all values of Q for $0 \leqq k_i \leqq n_i$, $i = 1, 2, \cdots, c$, one may start with (2.8) and use (2.7) recursively, a procedure which can be programmed for an electronic computer.

Let now R be a given set of points of the form (2.2), and let

$$\begin{aligned} Q(k_1, k_2, \cdots, k_c; R) = {} & \text{number of paths from } (0, 0, \cdots 0) \text{ to} \\ & (k_1, k_2, \cdots, k_c) \text{ which do not pass through any points in } R. \end{aligned} \tag{2.9}$$

Again the difference equation

$$\begin{aligned} Q(k_1, k_2, \cdots, k_c; R) = {} & Q(k_1 - 1, k_2, \cdots, k_c; R) \\ & + Q(k_1, k_2 - 1, \cdots, k_c; R) \\ & \cdots \\ & + Q(k_1, k_2, \cdots, k_c - 1; R) \end{aligned} \tag{2.10}$$

is satisfied, and can be solved recursively under condition (2.8) and the additional conditions

$$Q(k_1, k_2, \cdots, k_c; R) = 0 \quad \text{for} \quad (k_1, k_2, \cdots, k_c) \text{ in } R. \tag{2.10.1}$$

This, again, is an algorithm which can be programmed for an electronic computer but the program must now, among others, contain the instruction for the computer to decide at each point $(k_1, k_2, \cdots, k_c)$ whether it belongs to R or not.

We now define

$$P_R(n_1, n_2, \cdots, n_c) = Q(n_1, n_2, \cdots, n_c; R)/Q(n_1, n_2, \cdots, n_c), \tag{2.11}$$

the probability that, under the null hypothesis (1.5), the samples determine a path from $(0, 0, \cdots, 0)$ to $(n_1, n_2, \cdots, n_c)$ which does not pass through any point of R.

If, for a given set R, we agree to reject the hypothesis (1.5) whenever the samples determine a path containing points in R, then $1 - P_R$ is the probability of an error of the first kind, i.e. of rejecting the hypothesis when it is true. The tabulation of P_R is manageable for reasonable numbers of samples c and sample sizes $n_1, n_2, \cdots, n_c$, and for R such that one can program for the computer a rule for deciding whether a point is in R or not.

For the Kolmogorov-Smirnov statistic $D(n_1, n_2)$ the sets R are usually defined by $D(n_1, n_2) > r$, which is equivalent with

$$R_r: \quad | n_2k_1 - n_1k_2 | > n_1n_2r, \tag{2.12}$$

and for the one-sided statistic $D^+(n_1, n_2)$ by $D^+(n_1, n_2) > r$, equivalent with

$$R_r': \quad n_2k_1 - n_1k_2 > n_1n_2r. \tag{2.12.1}$$

For $n_1 = n_2$, (2.12) and (2.12.1) become $|k_1 - k_2| > nr$ and $k_1 - k_2 > nr$, respectively.

Analogous multi-sample tests can be defined by using the statistics (1.3) or (1.4) and the regions of rejection

$$D(n_1, n_2, \cdots, n_c) > r \quad \text{and} \quad D^+(n_1, n_2, \cdots, n_c) > r,$$

respectively. The corresponding sets R are

$$\operatorname*{Sup}_{(i,j=1,\cdots,c)} |n_j k_i - n_i k_j| > r, \tag{2.13}$$

and

$$\operatorname*{Sup}_{(i<j)} (n_j k_i - n_i k_j) > r \tag{2.13.1}$$

respectively.

It may be noted that the computations involved in tabulating

$$P_R(n_1, n_2, \cdots, n_c)$$

would not be much more difficult to program and more time-consuming if (2.13) or (2.13.1) were replaced by more general sets R such as

$$|n_j k_i - n_i k_j| > f(k_1, k_2, \cdots, k_c)$$

for some reasonably simple function f.

The tables described in the next section were computed by using difference equations (2.7) and (2.10). It should be stated that these difference equations have been well known and used for the case $c = 2$, and that closed expressions for $P_R(n_1, n_2)$ were obtained in special cases, e.g. by Gnedenko and Korolyuk [3] and by Drion [2]. An excellent summary of the history of these methods may be found in the paper by Hodges [4]. A more recent paper by David [1] contains the derivation of the small-sample distribution and the asymptotic distribution of the statistic

$$\operatorname{Max}\{\sup_{(x)} [F^{*(2)}(x) - F^{*(1)}(x)], \quad \sup_{(x)} [F^{*(3)}(x) - F^{*(2)}(x)], \quad \sup_{(x)} [F^{*(1)}(x) - F^{*(3)}(x)]\}.$$

3. Tables. Table 1 contains the probabilities $P[D(n, n, n) \leqq r]$ for $n = 1$ (1) 20 (2) 40 and consecutive integer values nr such that the probabilities for each n range from less than .90 to more than .995.

Table 2 contains the probabilities $P[D(n, n) \leqq r)$ for $n = 1$ (1) 40 and $nr = 1$ (1) min $(n, 20)$.

Table 3 contains the probabilities $P[D^+(n, n) \leqq r]$ for $n = 1$ (1) 40 and $nr = 1$ (1) min $(n, 20)$.

All probabilities are given to six decimal places. Conservative error estimates assure an error $<5.10^{-6}$ throughout Table 1 and error $<(2.3)10^{-6}$ throughout Tables 2 and 3, but the actual errors are likely to be much smaller.

TABLE 1

$P[D(n, n, n) \leqq r]$

r \ n	1	2	3	4	5	6	7	8	9	10
1	1 000000	0 400000	0 128571	0 037402	0 010275	0 002719	0 000701	0 000177	0 000044	0 000010
2		1 000000	0 771428	0 539220	0 355929	0 226374	0 140271	0 085256	0 051053	0 030213
3			1 000000	0 926406	0 811093	0 684084	0 562086	0 453012	0 359715	0 282279
4				1 000000	0 978188	0 932164	0 868227	0 793917	0 715417	0 637148
5					1 000000	0 993829	0 977501	0 950288	0 913501	0 869301
6						1 000000	0 998303	0 992915	0 982475	0 966446
7							1 000000	0 999541	0 997847	0 994114
8								1 000000	0 999877	0 999362
9									1 000000	0 999967
10										1 000000

r \ n	11	12	13	14	15	16	17	18	19	20
1	0 000002	0 000000	0 000000							
2	0 017709	0 010297	0 005948							
3	0 219397	0 169169	0 129569							
4	0 562027	0 491832	0 427525							
5	0 819975	0 767590	0 713862							
6	0 944960	0 918575	0 888073	0 854312	0 818130	0 780302	0 741507	0 702328	0 663250	0 624670
7	0 987711	0 978261	0 965629	0 949882	0 931228	0 909969	0 886458	0 861064	0 834155	0 806081
8	0 998093	0 995691	0 991826	0 986249	0 978802	0 969415	0 958096	0 944916	0 929988	0 913459
9	0 999815	0 999399	0 998539	0 997044	0 994729	0 991436	0 987039	0 981452	0 974624	0 966539
10	0 999991	0 999947	0 999814	0 999518	0 998964	0 998049	0 996668	0 994723	0 992126	0 988805
11	1 000000	0 999997	0 999985	0 999943	0 999844	0 999646	0 999298	0 998744	0 997922	0 996773
12		1 000000	0 999999	0 999995	0 999983	0 999950	0 999881	0 999754	0 999539	0 999205
13			1 000000	0 999999	0 999998	0 999994	0 999984	0 999961	0 999915	0 999834
14				1 000000	0 999999	0 999999	0 999998	0 999995	0 999987	0 999971
15					1 000000	0 999999	0 999999	0 999999	0 999998	0 999996

r \ n	22	24	26	28	30	32	34	36	38	40
8	0 876276	0 834896	0 790312	0 744128	0 697257					
9	0 946679	0 922382	0 893835	0 862177	0 828007	0 792099	0 754883	0 717132	0 679257	0 641658
10	0 979784	0 967557	0 951678	0 932761	0 910963	0 886657	0 860253	0 832162	0 802779	0 772473
11	0 993268	0 987984	0 980202	0 970227	0 957886	0 943250	0 926465	0 907727	0 887261	0 865309
12	0 998039	0 996114	0 992702	0 988029	0 981779	0 973852	0 964215	0 952885	0 939929	0 925445
13	0 999504	0 998965	0 997584	0 995632	0 992789	0 988907	0 983879	0 977629	0 970120	0 961345
14	0 999892	0 999844	0 999284	0 998556	0 997392	0 995668	0 993276	0 990117	0 986113	0 981208
15	0 999980	0 999928	0 999811	0 999569	0 999139	0 998444	0 997404	0 995937	0 993968	0 991430
16					0 999741	0 999486	0 999073	0 998447	0 997552	0 996333

TABLE 2

$P[D(n, n) \leqq r]$

nr	n									
	1	2	3	4	5	6	7	8	9	10
1	1 000000	0 666666	0 400000	0 228571	0 126984	0 069264	0 037296	0 019891	0 010530	0 005542
2		1 000000	0 900000	0 771428	0 642857	0 525974	0 424825	0 339860	0 269888	0 213070
3			1 000000	0 971428	0 920634	0 857142	0 787878	0 717327	0 648292	0 582476
4				1 000000	0 992063	0 974025	0 946969	0 912975	0 874125	0 832178
5					1 000000	0 997835	0 991841	0 981351	0 966433	0 947552
6						1 000000	0 999417	0 997513	0 993706	0 987659
7							1 000000	0 999844	0 999259	0 997943
8								1 000000	0 999958	0 999783
8									1 000000	0 999989
10										1 000000
	11	12	13	14	15	16	17	18	19	20
1	0 002903	0 001514	0 000787	0 000408	0 000211	0 000109	0 000056	0 000028	0 000014	0 000007
2	0 167412	0 131018	0 102194	0 079484	0 061668	0 047743	0 036892	0 028460	0 021922	0 016863
3	0 520849	0 463902	0 411803	0 364515	0 321861	0 283588	0 249392	0 218952	0 191938	0 168030
4	0 788523	0 744224	0 700079	0 656679	0 614453	0 573706	0 534647	0 497409	0 462071	0 428664
5	0 925339	0 900453	0 873512	0 845065	0 815583	0 785465	0 755040	0 724581	0 694310	0 664409
6	0 979260	0 968563	0 955727	0 940970	0 924535	0 906673	0 887622	0 867606	0 846826	0 825466
7	0 995633	0 992140	0 987350	0 981217	0 973751	0 965002	0 955047	0 943981	0 931910	0 918942
8	0 999345	0 998503	0 997125	0 995100	0 992344	0 988800	0 984439	0 979252	0 973250	0 966458
9	0 999937	0 999795	0 999500	0 998979	0 998162	0 996984	0 995389	0 993331	0 990776	0 987701
10	0 999997	0 999982	0 999937	0 999836	0 999646	0 999329	0 998847	0 998160	0 997232	0 996032
11	1 000000	0 999999	0 999995	0 999981	0 999947	0 999880	0 999761	0 999570	0 999285	0 998884
12		1 000000	0 999999	0 999998	0 999994	0 999983	0 999960	0 999916	0 999843	0 999729
13			1 000000	1 000000	0 999999	0 999998	0 999994	0 999987	0 999971	0 999944
14				1 000000	1 000000	0 999999	0 999999	0 999998	0 999995	0 999990
15					1 000000	1 000000	1 000000	0 999999	0 999999	0 999998
16						1 000000	1 000000	1 000000	0 999999	0 999999
17							1 000000	1 000000	1 000000	1 000000
18								1 000000	1 000000	1 000000
19									1 000000	1 000000
20										1 000000
	21	22	23	24	25	26	27	28	29	30
1	0 000003	0 000001	0 000001	0 000000	0 000000	0 000000	0 000000	0 000000	0 000000	0 000000
2	0 012955	0 009942	0 007622	0 005838	0 004468	0 003417	0 002611	0 001993	0 001521	0 001160
3	0 146921	0 128321	0 111963	0 097599	0 085006	0 073980	0 064337	0 055914	0 048563	0 042153
4	0 397187	0 367613	0 339899	0 313982	0 289796	0 267262	0 246302	0 226833	0 208772	0 192036
5	0 635020	0 606260	0 578218	0 550963	0 524546	0 499004	0 474362	0 450633	0 427822	0 405929

TABLE 2—(*Continued*)

n_r	n									
	21	22	23	24	25	26	27	28	29	30
6	0 803687	0 781631	0 759421	0 737166	0 714957	0 692876	0 670992	0 649361	0 628035	0 607054
7	0 905183	0 890738	0 875705	0 860177	0 844239	0 827971	0 811443	0 794721	0 777865	0 760926
8	0 958911	0 950653	0 941731	0 932196	0 922101	0 911498	0 900437	0 888969	0 877140	0 864996
9	0 984094	0 979952	0 975279	0 970086	0 964388	0 958206	0 951561	0 944480	0 936988	0 929112
10	0 994532	0 992710	0 990548	0 988034	0 985162	0 981927	0 978330	0 974375	0 970069	0 965419
11	0 998343	0 997641	0 996759	0 995679	0 994385	0 992865	0 991109	0 989109	0 986859	0 984356
12	0 999561	0 999326	0 999009	0 998598	0 998079	0 997439	0 996666	0 995750	0 994681	0 993451
13	0 999899	0 999831	0 999732	0 999594	0 999409	0 999167	0 998861	0 998482	0 998020	0 997469
14	0 999980	0 999963	0 999936	0 999895	0 999837	0 999756	0 999647	0 999505	0 999325	0 999100
15	0 999996	0 999993	0 999986	0 999976	0 999960	0 999936	0 999901	0 999853	0 999790	0 999706
16	0 999999	0 999998	0 999997	0 999995	0 999991	0 999985	0 999975	0 999961	0 999940	0 999912
17	1 000000	0 999999	0 999999	0 999999	0 999998	0 999996	0 999994	0 999990	0 999984	0 999976
18	1 000000	1 000000	1 000000	0 999999	0 999999	0 999999	0 999998	0 999998	0 999996	0 999994
19	1 000000	1 000000	1 000000	1 000000	0 999999	0 999999	0 999999	0 999999	0 999999	0 999998
20	1 000000	1 000000	1 000000	1 000000	1 000000	1 000000	0 999999	0 999999	0 999999	0 999999

n_r	31	32	33	34	35	36	37	38	39	40
1	0 000000	0 000000	0 000000	0 000000	0 000000	0 000000	0 000000	0 000000	0 000000	0 000000
2	0 000884	0 000674	0 000513	0 000390	0 000297	0 000226	0 000171	0 000130	0 000099	0 000075
3	0 036570	0 031710	0 027482	0 023808	0 020615	0 017844	0 015440	0 013354	0 011546	0 009980
4	0 176546	0 162222	0 148989	0 136773	0 125505	0 115119	0 105553	0 096746	0 088644	0 081194
5	0 384946	0 364860	0 345656	0 327315	0 309815	0 293133	0 277243	0 262120	0 247737	0 234068
6	0 586454	0 566263	0 546505	0 527197	0 508355	0 489989	0 472106	0 454713	0 437810	0 421399
7	0 743954	0 726991	0 710076	0 693241	0 676518	0 659934	0 643511	0 627272	0 611234	0 595412
8	0 852579	0 839930	0 827085	0 814080	0 800946	0 787713	0 774409	0 761059	0 747686	0 734312
9	0 920879	0 912317	0 903453	0 894313	0 884922	0 875305	0 865485	0 855485	0 845325	0 835027
10	0 960438	0 955137	0 949530	0 943629	0 937451	0 931011	0 924322	0 917402	0 910264	0 902925
11	0 981599	0 978588	0 975325	0 971814	0 968060	0 964067	0 959843	0 955395	0 950731	0 945858
12	0 992054	0 990483	0 988735	0 986806	0 984695	0 982400	0 979921	0 977260	0 974418	0 971396
13	0 996821	0 996069	0 995206	0 994228	0 993128	0 991904	0 990551	0 989067	0 987450	0 985698
14	0 998825	0 998494	0 998102	0 997644	0 997113	0 996507	0 995820	0 995049	0 994189	0 993239
15	0 999600	0 999466	0 999302	0 999104	0 998868	0 998589	0 998265	0 997891	0 997464	0 996981
16	0 999875	0 999825	0 999762	0 999683	0 999586	0 999467	0 999325	0 999156	0 998958	0 998729
17	0 999964	0 999947	0 999925	0 999896	0 999859	0 999812	0 999754	0 999683	0 999598	0 999496
18	0 999990	0 999985	0 999978	0 999968	0 999955	0 999938	0 999916	0 999888	0 999854	0 999812
19	0 999997	0 999996	0 999994	0 999991	0 999987	0 999981	0 999973	0 999963	0 999950	0 999934
20	0 999999	0 999999	0 999998	0 999997	0 999996	0 999994	0 999992	0 999988	0 999984	0 999978

TABLE 3

$P[D^+(n, n) \leqq r]$

r \ n	1	2	3	4	5	6	7	8	9	10
0	500000	333333	250000	200000	166666	142857	125000	111111	100000	90909
1	1 000000	0 833333	0 700000	0 600000	0 523809	0 464285	0 416666	0 377777	0 345454	0 318181
2		1 000000	0 950000	0 885714	0 821428	0 761904	0 708333	0 660606	0 618181	0 580419
3			1 000000	0 985714	0 960317	0 928571	0 893939	0 858585	0 823776	0 790209
4				1 000000	0 996031	0 987012	0 973484	0 956487	0 937062	0 916083
5					1 000000	0 998917	0 995920	0 990675	0 983216	0 973776
6						1 000000	0 999708	0 998756	0 996853	0 993829
7							1 000000	0 999922	0 999629	0 998971
8								1 000000	0 999979	0 999891
9									1 000000	0 999994
10										1 000000

r \ n	11	12	13	14	15	16	17	18	19	20
0	83333	76923	71428	66666	62500	58823	55555	52631	50000	47619
1	0 294871	0 274725	0 257142	0 241666	0 227941	0 215686	0 204678	0 194736	0 185714	0 177489
2	0 546703	0 516483	0 489285	0 464705	0 442401	0 422084	0 403508	0 386466	0 370779	0 356295
3	0 758241	0 728021	0 699579	0 672875	0 647832	0 624354	0 602339	0 581681	0 562281	0 544042
4	0 894230	0 872010	0 849789	0 827829	0 806308	0 785345	0 765018	0 745371	0 726425	0 708187
5	0 962669	0 950226	0 936753	0 922523	0 907765	0 892672	0 877401	0 862076	0 846798	0 831646
6	0 989630	0 984281	0 977863	0 970485	0 962267	0 953336	0 943808	0 933796	0 923399	0 912705
7	0 997816	0 996070	0 993675	0 990608	0 986875	0 982500	0 977523	0 971990	0 965955	0 959470
8	0 999672	0 999251	0 998562	0 997550	0 996172	0 994400	0 992219	0 989626	0 986625	0 983229
9	0 999968	0 999897	0 999750	0 999489	0 999081	0 998492	0 997694	0 996665	0 995388	0 993850
10	0 999998	0 999991	0 999968	0 999918	0 999823	0 999664	0 999423	0 999080	0 998616	0 998016
11	1 000000	0 999999	0 999997	0 999990	0 999973	0 999940	0 999880	0 999785	0 999642	0 999442
12		1 000000	0 999999	0 999999	0 999997	0 999991	0 999980	0 999958	0 999921	0 999864
13			1 000000	1 000000	0 999999	0 999999	0 999997	0 999993	0 999985	0 999972
14				1 000000	1 000000	1 000000	0 999999	0 999999	0 999997	0 999995
15					1 000000	1 000000	1 000000	0 999999	0 999999	0 999999
16						1 000000	1 000000	1 000000	1 000000	0 999999
17							1 000000	1 000000	1 000000	1 000000
18								1 000000	1 000000	1 000000
19									1 000000	1 000000
20										1 000000

TABLE 3—(*Continued*)

nr	n: 21	22	23	24	25	26	27	28	29	30
0	45454	43478	41666	40000	38461	37037	35714	34482	33333	32258
1	0 169960	0 163043	0 156666	0 150769	0 145299	0 140211	0 135467	0 131034	0 126881	0 122983
2	0 342885	0 330434	0 318846	0 308034	0 297924	0 288451	0 279556	0 271190	0 263306	0 255865
3	0 526877	0 510702	0 495441	0 481025	0 467390	0 454479	0 442237	0 430617	0 419574	0 409069
4	0 690650	0 673801	0 657621	0 642086	0 627173	0 612856	0 599108	0 585903	0 573216	0 561022
5	0 816681	0 801950	0 787488	0 773321	0 759466	0 745936	0 732738	0 719875	0 707348	0 695154
6	0 901793	0 890731	0 879577	0 868380	0 857183	0 846022	0 834926	0 823922	0 813028	0 802262
7	0 952590	0 945365	0 937846	0 930077	0 922100	0 913953	0 905672	0 897287	0 888826	0 880316
8	0 979455	0 975326	0 970865	0 966097	0 961050	0 955747	0 950216	0 944479	0 938562	0 932486
9	0 992047	0 989976	0 987639	0 985043	0 982194	0 979103	0 975780	0 972239	0 968493	0 964555
10	0 997266	0 996355	0 995274	0 994017	0 992581	0 990963	0 989165	0 987187	0 985034	0 982709
11	0 999171	0 998820	0 998379	0 997839	0 997192	0 996432	0 995554	0 994554	0 993429	0 992178
12	0 999780	0 999663	0 999504	0 999299	0 999039	0 998719	0 998333	0 997875	0 997340	0 996725
13	0 999949	0 999915	0 999866	0 999797	0 999704	0 999583	0 999430	0 999241	0 999010	0 998734
14	0 999990	0 999981	0 999968	0 999948	0 999918	0 999878	0 999823	0 999752	0 999662	0 999550
15	0 999998	0 999996	0 999993	0 999988	0 999980	0 999968	0 999950	0 999926	0 999895	0 999853
16	0 999999	0 999999	0 999998	0 999997	0 999995	0 999992	0 999987	0 999980	0 999970	0 999956
17	1 000000	1 000000	0 999999	0 999999	0 999999	0 999998	0 999997	0 999995	0 999992	0 999988
18	1 000000	1 000000	1 000000	1 000000	0 999999	0 999999	0 999999	0 999998	0 999998	0 999997
19	1 000000	1 000000	1 000000	1 000000	1 000000	1 000000	0 999999	0 999999	0 999999	0 999999
20	1 000000	1 000000	1 000000	1 000000	1 000000	1 000000	0 999999	0 999999	0 999999	0 999999

nr	31	32	33	34	35	36	37	38	39	40
0	31250	30303	29411	28571	27777	27027	26315	25641	24999	24390
1	0 119318	0 115864	0 112605	0 109523	0 106606	0 103840	0 101214	0 098717	0 096341	0 094076
2	0 248830	0 242169	0 235854	0 229858	0 224158	0 218732	0 213562	0 208630	0 203919	0 199416
3	0 399064	0 389525	0 380422	0 371726	0 363411	0 355454	0 347832	0 340525	0 333514	0 326782
4	0 549298	0 538019	0 527164	0 516712	0 506644	0 496940	0 487582	0 478554	0 469840	0 461425
5	0 683290	0 671750	0 660528	0 649616	0 639007	0 628693	0 618666	0 608916	0 599435	0 590215
6	0 791638	0 781167	0 770856	0 760713	0 750743	0 740949	0 731333	0 721895	0 712638	0 703559
7	0 871777	0 863229	0 854689	0 846173	0 837693	0 829262	0 820888	0 812582	0 804349	0 796197
8	0 926272	0 919939	0 913505	0 906988	0 900402	0 893763	0 887081	0 880371	0 873642	0 866904
9	0 960438	0 956157	0 951724	0 947152	0 942454	0 937643	0 932729	0 927724	0 922638	0 917480
10	0 980219	0 977568	0 974764	0 971814	0 968725	0 965504	0 962160	0 958699	0 955130	0 951459
11	0 990799	0 989294	0 987662	0 985907	0 984029	0 982033	0 979921	0 977697	0 975365	0 972929
12	0 996027	0 995241	0 994367	0 993403	0 992347	0 991200	0 989960	0 988630	0 987209	0 985698
13	0 998410	0 998034	0 997603	0 997113	0 996564	0 995952	0 995275	0 994533	0 993725	0 992849
14	0 999412	0 999247	0 999051	0 998821	0 998556	0 998253	0 997910	0 997524	0 997094	0 996619
15	0 999800	0 999733	0 999651	0 999552	0 999434	0 999294	0 999132	0 998945	0 998732	0 998490
16	0 999937	0 999912	0 999881	0 999841	0 999793	0 999733	0 999662	0 999578	0 999479	0 999364
17	0 999982	0 999973	0 999962	0 999948	0 999929	0 999906	0 999877	0 999841	0 999799	0 999748
18	0 999995	0 999992	0 999989	0 999984	0 999977	0 999969	0 999958	0 999944	0 999927	0 999905
19	0 999998	0 999998	0 999997	0 999995	0 999993	0 999990	0 999986	0 999981	0 999975	0 999967
20	0 999999	0 999999	0 999999	0 999998	0 999998	0 999997	0 999996	0 999994	0 999992	0 999989

Table 2 is an extension of the table given by Massey [5]. Tables 1 and 3 appear to be new. Table 3 could also have been computed by a method due to Drion [2].

The computations were programmed for and carried out on the IBM 650 of the Research Computer Laboratory of the University of Washington. The authors wish to express their sincere appreciation to Professor D. B. Dekker for his generous help in planning and performing these computations.

4. Case of $c > 3$. With increasing number of samples c, the computations are not more complicated in structure but quickly become prohibitive in view of the increasing demand on the storage capacity of the computer and the number of additions required. Tabulations similar to those presented in the preceding section, while feasible, would hardly be worth the effort for many values of $c > 3$. Should exact tests based on the statistics $D(n_1, n_2, \cdots, n_c)$, $D^+(n_1, n_2, \cdots, n_c)$ be practically needed then, instead of computing tables, it may be preferable to prepare a program for an electronic computer which, for given sample values, would calculate the single probability needed in every specific case.

Lacking such a program, one may for $c \geqq 3$ make use of the following simple inequalities.

One clearly has, for $c \geqq 3$,

$$\begin{aligned} P[D(n_1, n_2, \cdots, n_c) \leqq r] &= P[\max_{1 \leqq i < j \leqq c} \sup_x | F^{*(i)}(x) - F^{*(j)}(x) | \leqq r] \\ &= 1 - P[\sup_x | F^{*(i)}(x) - F^{*(j)}(x) | > r \text{ for some } i < j] \\ &\geqq 1 - \sum_{1 \leqq i < j \leqq c} \sum P[D(n_i, n_j) > r] \end{aligned}$$

and, for $n_1 = n_2 = \cdots = n_c$, $c \geqq 3$,

$$P[D(n, n, \cdots, n) \leqq r] \geqq 1 - [c(c-1)/2]P[D(n, n) > r]. \tag{4.1}$$

For $c \geqq 4$, one similarly obtains

$$P[D(n, n, \cdots, n) \leqq r] \geqq 1 - [c(c-1)(c-2)/6]P[D(n, n, n) > r]. \tag{4.2}$$

These inequalities make it possible to use the statistic $D(n, n, \cdots, n)$ for testing the hypothesis (1.5) using only Table 1 or Table 2, whichever yields a greater value for the right side of (4.2) or (4.1), respectively. The test will be conservative, i.e. the probability of error of the first kind is less than that obtained from (4.2) or (4.1), but for the conventional "significance levels" and c not too large the right sides in both inequalities should be close approximations to the left side.

Similar inequalities are easily obtained for the statistic $D^+(n, n, \cdots, n)$.

It has been pointed out quite strikingly by Hodges [4] that, for $c = 2$, asymptotic expressions such as that due to Smirnov [6, 7] are inaccurate even for fairly large values of n, to an extent which makes it inadvisable to use them. It appears, therefore, rather doubtful that good approximations can be found for $c > 3$, and as long as such approximations are not available inequalities of the kind of (4.1) or (4.2) may be of practical use.

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