ON A PROBLEM OF J. NEYMAN AND E. SCOTT[1]

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1. Introduction. Let ξ be a one-dimensional random variable distributed according to $N(a, \sigma^2)$ (that means a normal distribution with mean value a and variance σ^2) and $f(x)$ a measurable function defined on $-\infty < x < \infty$. Suppose that

$$((2\pi)^{\frac{1}{2}}\sigma)^{-1} \int_{-\infty}^{+\infty} f(x) e^{-(x-a)^2/2\sigma^2}\, dx = \omega(a, \sigma^2)$$

exists for all real numbers a and all $\sigma^2 > 0$ as a Lebesgue integral. Let λ be a fixed positive number and η a one-dimensional random variable with distribution $N(a, \sigma^2\lambda^2)$. Let ζ be a random variable such that z/σ^2 has a Pearson-Helmert distribution with ν degrees of freedom. Further, we suppose that η and ζ are independent. The question is whether or not there are unbiased estimates $H(\eta, \zeta)$ for $\omega(a, \sigma^2)$ where $-\infty < a < \infty$, and where $\sigma^2 > 0$. In a paper of Neyman and Scott [1] it is proved that there is always an unbiased estimate $H(\eta, \zeta)$ for $\omega(a, \sigma^2)$ for the class of entire functions $f(x)$ which satisfy the conditions

$$(1) \qquad \frac{1}{n} (|f^{(2n)}(0)|)^{1/n} = o(1), \qquad \frac{1}{n} (|f^{(2n+1)}(0)|^{1/n} = o(1)$$

and which take real values on the real line. Condition (1) can be expressed in the following way: $f(x)$ is an entire function of order 2 and type zero or of any smaller order. Let us recall that an entire function $f(x)$ is of order k and type $\alpha \geqq 0$ if $|f(re^{i\varphi})| = O(\exp\{(\alpha + \epsilon)r^k\})$ for every $\epsilon > 0$ but for no $\epsilon < 0$ if $r \geqq r(\epsilon)$ and $0 \leqq \varphi < 2\pi$.

In addition, the following problem is raised in the paper just mentioned: Let $f(x)$ be a measurable function defined on the whole real line such that

(i) $\int_{-\infty}^{+\infty} |f(x)| e^{-\epsilon x^2}\, dx$ converges for all $\epsilon > 0$.

Is there always an unbiased estimate $H(\eta, \zeta)$ for $\omega(a, \sigma^2)$ if $f(x)$ satisfies Condition (i)? In this paper it will be shown that there is for each $f(x)$ satisfying Condition (i) an unbiased estimate $H(\eta, \zeta)$ for $\omega(a, \sigma^2)$ if λ is any positive number in the interval $0 < \lambda \leqq 1$. However, if λ is allowed to be >1, then the unbiased estimate $H(\eta, \zeta)$ need not exist. Also, it will be shown that the theorem of Neyman and Scott mentioned above is, in a certain sense, best.

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2. Problem and solution. Let us now consider for any $\lambda > 0$ and any real number $\nu \geqq 1$ the integral equation

$$\int_{-\infty}^{+\infty} f(x)e^{-(x-a)^2/2\sigma^2}\,dx \tag{2}$$
$$= (2\lambda\sigma^2\Gamma(\nu/2))^{-1}\int_{-\infty}^{+\infty}\int_0^{\infty} H(y, z)e^{-(y-a)^2/2\sigma^2\lambda^2}e^{-z/2\sigma^2}(z/2\sigma^2)^{(\nu/2)-1}\,dy\,dz,$$

where $-\infty < a < \infty$, $\sigma^2 > 0$ and $f(x)$ is any measurable function satisfying (i). We will prove the following theorem.

THEOREM 1. *If $f(x)$ is a measurable function defined on the whole real line and satisfying (i), then for all positive numbers λ in the interval $0 < \lambda \leqq 1$ there is a solution $H(y, z)$ of (2) which satisfies the following "natural" condition*

(ii) *$\int_{-\infty}^{+\infty}\int_0^{\infty} |H(y, z)|\, z^{(\nu/2)-1}e^{-\epsilon y^2}e^{-\eta z}\,dy\,dz$ converges for all $\epsilon > 0$, $\eta > 0$.*

Moreover, there is only one solution of this kind (up to sets of measure zero, of course). For $0 < \lambda < 1$ and $\nu > 1$ this solution is given by

$$H(y, z) = \Gamma\left(\frac{\nu}{2}\right)[(\pi)^{\frac{1}{2}}(1-\lambda^2)^{\frac{1}{2}}\Gamma((\nu-1)/2)]^{-1}z^{-(\nu/2)+1}$$
$$\cdot\int_{-\infty}^{+\infty} f(x)[I(z-(x-y)^2/(1-\lambda^2))]^{(\nu-3)/2}\,dx$$
$$-\infty < y < \infty,\quad 0 < z < \infty,$$

where

$$(I(x))^{\alpha} = \begin{cases} x^{\alpha} & x > 0 \\ 0 & x \leqq 0. \end{cases}$$

for every real number α. For $\nu = 1$ the solution is given by

$$H(y, z) = \tfrac{1}{2}\{f[y-(z(1-\lambda^2))^{\frac{1}{2}}] + f[y+(z(1-\lambda^2))^{\frac{1}{2}}]\}.$$

For $\lambda = 1$ the solution is given by

$$H(y, z) = f(y), \qquad -\infty < y < \infty,\quad 0 \leqq z < \infty.$$

For the proof we observe first that the assertion concerning $\lambda = 1$ is trivial. For $\lambda < 1$, notice that $f(x)$ is locally integrable by (i). Therefore, it is obvious that

$$J(y, z, \nu) = \int_{-\infty}^{+\infty} f(x)\left(I\left(z - \frac{(x-y)^2}{1-\lambda^2}\right)\right)^{(\nu-3)/2}dx$$

is defined for all y, all nonnegative z and for $\nu \geqq 3$. Moreover, it is easy to show that $J(y, z, \nu)$ defines, for all y, each real $\nu > 1$ and almost all nonnegative z, a locally integrable function of z, which after multiplication by $\exp\{-\eta z\}$ for any $\eta > 0$ is absolutely integrable for $0 < z < \infty$. To show this, consider the identity

$$
(3)\quad \begin{aligned} &\Gamma\left(\frac{\nu-1}{2}\right)(2\sigma^2)^{(\nu-1)/2}\int_{-\infty}^{+\infty} f(y+t)e^{-t^2/2\sigma^2(1-\lambda^2)}\,dt \\ &\qquad = \int_0^{\infty} u^{(\nu-3)/2}e^{-u/2\sigma^2}\,du \int_{-\infty}^{+\infty} f(y+t)e^{-t^2/2\sigma^2(1-\lambda^2)}\,dt. \end{aligned}
$$

The iterated integral on the right side of (3) is absolutely convergent by (i) for each positive $\lambda < 1$, each $\sigma^2 > 0$, each $\nu > 1$ and all real y. But easy transformation of the integration variables shows that this iterated integral is equal to

$$
(1-\lambda^2)^{\frac{1}{2}}2^{-1}\int_0^{\infty} u^{(\nu-3)/2}\int_u^{\infty} e^{-z/2\sigma^2}(f(X)+f(Y))(z-u)^{-\frac{1}{2}}\,dz\,du
$$

where $X = y + [(z-u)(1-\lambda^2)]^{\frac{1}{2}}$ and where $Y = y - [(z-u)(1-\lambda^2)]^{\frac{1}{2}}$. The absolute convergence of this iterated integral justifies changing the order of integration and after an easy calculation we get for this integral

$$
\int_0^{\infty} e^{-z/2\sigma^2}\int_{-\infty}^{+\infty} f(x)\left(I\left(z-\frac{(x-y)^2}{1-\lambda^2}\right)\right)^{(\nu-3)/2}\,dx\,dz.
$$

Another application of Fubini's theorem and the fact that $\exp\{-z/2\sigma^2\}$ is positive and bounded for all nonnegative z and every $\sigma^2 > 0$ show that $J(y, z, \nu)$ is a locally integrable function of z for all y, each $\nu > 1$, and almost all nonnegative z. This function is, after multiplication by $\exp\{-\eta z\}$, for any $\eta > 0$ absolutely integrable for all nonnegative z. Now we use the following simple lemma.

LEMMA. *If $f(x)$ satisfies* (i) *then for any λ in $0 < \lambda < 1$ there exists exactly one solution $g(y, \sigma^2)$ of the equation*

$$
(4)\quad \int_{-\infty}^{+\infty} f(x)e^{-(x-a)^2/2\sigma^2}\,dx = \frac{1}{\lambda}\int_{-\infty}^{+\infty} g(y,\sigma^2)e^{-(y-a)^2/2\sigma^2\lambda^2}\,dy, \qquad -\infty < a < \infty,
$$

for which the integral on the right side of (4) *converges absolutely for each $\sigma^2 > 0$ and all real a. This solution is given by*

$$
(5)\quad \begin{aligned} g(y,\sigma^2) = [(2\pi)^{\frac{1}{2}}\sigma(1-\lambda^2)^{\frac{1}{2}}]^{-1}\int_{-\infty}^{+\infty} f(x)e^{-(x-y)^2/2\sigma^2(1-\lambda^2)}\,dx, \\ -\infty < y < \infty. \end{aligned}
$$

The uniqueness for a solution of (4) with the asserted property is well known. For the proof that (5) satisfies (4) it is enough to notice that

$$
\begin{aligned} &[(2\pi)^{\frac{1}{2}}\sigma((1-\lambda^2)^{\frac{1}{2}})^{-1}]\int_{-\infty}^{+\infty} f(x)\int_{-\infty}^{+\infty} e^{-(y-x)^2/2\sigma^2(1-\lambda^2)}e^{-(y-a)^2/2\sigma^2\lambda^2}\,dy\,dx \\ &\qquad = \lambda\int_{-\infty}^{+\infty} f(x)e^{-(x-a)^2/2\sigma^2}\,dx \end{aligned}
$$

and that the iterated integral on the left side of this identity converges absolutely by (i).

Now we proceed with the proof of Theorem 1. Using (3) and the above lemma it follows (again by an application of Fubini's theorem) that $J(y, z, \nu)$ is a locally integrable function for almost all y, $-\infty < y < \infty$, and almost all z, $0 < z < \infty$, and that $H(y, z)$ satisfies the integral equation (2). The lemma just mentioned and the remarks about the absolute integrability of $\exp\{-\eta z\} J(y, z, \nu)$ for all $\eta > 0$ show that $H(y, z)$ satisfies condition (ii). The asserted representation of $H(y, z)$ for the case $\nu = 1$ is now obvious. Moreover, it is well known that there is only one solution $H(y, z)$ of (1) satisfying (ii).

3. A negative result. Now we will prove that in a certain sense the result obtained by Neyman and Scott concerning the existence of unbiased estimates $H(\eta, \zeta)$ for $\omega(a, \sigma^2)$ cannot be improved.

THEOREM 2. *Let $f(x)$ be an entire function with real values on the real line and satisfying Condition* (i). *There is for each real $\alpha > 0$ an entire function of order* 2 *and type α such that* (2) *does not have a solution $H(y, z)$ which satisfies* (ii) *for any $\lambda > 1$. This means that there is no unbiased estimate for $\omega(a, \sigma^2)$ according to the usual definition of unbiased estimates.*

For the proof take $f(x) = \exp\{-\alpha x^2\}$. For a given $\alpha > 0$, the function $f(x)$ is an entire function of order 2 and type α, which takes real values on the real line. Suppose there is a solution $H(y, z)$ of (2) which satisfies Condition (ii). Then for each $\sigma^2 > 0$, $K(y, \sigma^2)$ defined by

$$K(y, \sigma^2) = (\Gamma(\nu/2))^{-1} \int_0^\infty H(y, z) e^{-z/2\sigma^2} (z/2\sigma^2)^{(\nu/2)-1}\, dz/2\sigma^2$$

exists for almost all y, is locally integrable, and

$$\int_{-\infty}^{+\infty} K(y, \sigma^2) e^{-(y-a)^2/2\sigma^2\lambda^2}\, dy$$

exists as an absolutely convergent integral for all real a, each $\sigma^2 > 0$ and any $\lambda > 0$. Hence, for each $\lambda > 1$ and each $\sigma^2 > 0$ the equation

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2} e^{-(x-a)^2/2\sigma^2}\, dx = 1/\lambda \int_{-\infty}^{+\infty} K(y, \sigma^2) e^{-(y-a)^2/2\sigma^2\lambda^2}\, dy$$

must be an identity in a for all real a. We write $a = -s$ and get

$$(6) \qquad \left[\frac{2\pi\sigma^2}{2\alpha\sigma^2 + 1}\right]^{\frac{1}{2}} e^{-\alpha s^2/(2\alpha\sigma^2+1)} e^{s^2/2\sigma^2\lambda^2} = \frac{1}{\lambda} \int_{-\infty}^{+\infty} K(y, \sigma^2) e^{-y^2/2\sigma^2\lambda^2} e^{-ys/\sigma^2\lambda^2}\, dy.$$

But it is obvious that these two expressions define for each $\sigma^2 > 0$ and each $\lambda > 1$ analytical functions of s in the whole complex plane, which are identical on the real line and so, everywhere. Take now

$$(7) \qquad \sigma^2(\lambda^2 - 1) > 1/2\alpha.$$

This is possible since $\lambda > 1$. For $s = it$ and real t the right side of (6) is a Fourier transform of an absolutely integrable function and must converge to zero for $|t| \to \infty$. But the left side of (6) goes to infinity for $|t| \to \infty$ by (7). This is a contradiction to the existence of an unbiased estimate for $\omega(a, \sigma^2)$.

REMARK 1. We notice that the same argument proves the nonexistence of more general solutions $H(y, z)$ of (2) in the case $\lambda > 1$. For instance, it is easy to show that for $f(x) = \exp\{-\alpha x^2\}$, $\alpha > 0$, there is no solution $H(y, z)$ of (2) for which the integral on the right side of (2) can be written as an iterated conditionally convergent integral of the form

$$\left[\lambda\Gamma\left(\frac{\nu}{2}\right)\right]^{-1}(2\sigma^2)^{-\nu/2}\lim_{M,N\to\infty}\int_{-M}^{N} e^{-(y-a)^2/2\sigma^2\lambda^2}\lim_{\substack{\epsilon\to 0\\ P\to\infty}}\int_{\epsilon}^{P} H(y,z)e^{-z/2\sigma^2}z^{\nu/2-1}\,dz\,dy.$$

For this we have only to consider the identity in s obtained from (6) by modifying the definition of $K(y, \sigma^2)$ and the right side of (6) in an obvious manner. But using (allowed) partial integration it is easy to show that now the integral on the right side of (6) exists also for all s of the complex plane. Further, it is well known that the Fourier transform of a conditionally integrable function is a $o(|t|)$ for $|t| \to \infty$ and this leads again to a contradiction if σ^2 satisfies (7).

REMARK 2. Using Laplace transforms it is also possible to give a more general theorem than Theorem 1 which covers also the case $\lambda > 1$. Because Theorem 2 gives a clear idea in which way such a theorem must be formulated, I do not think that it is of any value to give it here.

4. Examples. According to Theorem 1 there is a solution $H(y, z)$ of (2) for any λ in the interval $0 < \lambda \leqq 1$ if $f(x)$ is given by $\exp\{-\alpha x^2\}$, $\alpha > 0$. For $0 < \lambda < 1$ this solution is given by

$$Ce^{-\alpha y^2}z^{-(\nu/2)+1}\int_0^z e^{-\alpha u(1-\lambda)^2}(z-u)^{(\nu-3)/2}u^{-\frac{1}{2}}\cosh\left(2\alpha y(u(1-\lambda^2))^{\frac{1}{2}}\right)du,$$

where $C = \Gamma(\nu/2)[\Gamma((\nu-1)/2)\sqrt{\pi}]^{-1}$. This expression and Theorem 2 give a complete answer to a question raised in the paper of Neyman and Scott (p. 12 of [1]).

Another function $f(x)$ whose inverse is occasionally used as a normalizing transform is given by

$$f(x) = \begin{cases} (b+x)^p, & p > 0 \qquad x > -b \\ 0 & \qquad\qquad\;\; x \leqq -b. \end{cases}$$

If p is an integer it is more usual to define $f(x)$ by $(b + x)^p$ for all real x. Moreover in most practical applications $|b|$ is a very large number such that these two definitions are "almost" equivalent. It appears that there is an unbiased estimate $H(\eta, \zeta)$ in the case $0 < \lambda < 1$, but it does not seem to be appropriate for practical use in general, although it is of course possible for rational numbers p and integers $\nu = 2k + 3$, where $k \geqq 0$, to express $J(y, z, \nu)$ by elementary functions. For instance, for $y > -b$ and $z < (b + y)^2/(1 - \lambda^2)$ we get the following form of $H(y, z)$ where $p = n/m$, with $(n, m) = 1$.

$$H(y, z) = m\Gamma(\nu/2)[\sqrt{\pi}((1 - \lambda^2)^{\frac{1}{2}})\Gamma((\nu - 1)/2)]^{-1} z^{-(\nu/2)+1} \sum_{l=0}^{k} z^l \binom{k}{l} (-1)^{k-l}$$
$$\cdot (1 - \lambda^2)^{-(k-l)} \left[X^{(m+n)/m} \sum_{r=0}^{2(k-l)} (-1)^r \binom{2(k-l)}{r} \frac{(a + y)^r X^{2(k-l)-r}}{(n + m)[1 + 2(k - l) - r]} \right.$$
$$\left. - Y^{(m+n)/m} \sum_{r=0}^{k} (-1)^r \binom{2(k-l)}{r} \frac{(a + y)^r Y^{2(k-l)-r}}{(n + m)[1 + 2(k - l) - r]} \right]$$

where $X = b + y + [z(1 - \lambda^2)]^{\frac{1}{2}}$ and where $Y = b + y - [z(1 - \lambda^2)]^{\frac{1}{2}}$. If p is an integer and $f(x)$ is defined by $(b + x)^p$ for all x then $H(y, z)$ is given by this expression for all y and $z \geqq 0$ and any $\lambda > 0$ in agreement with the results of Neyman and Scott.

5. Relation to previously obtained results. The proof of Theorem 1 shows that

$$h_x(y, z) = C(\lambda, \nu) z^{-(\nu/2)+1} [I(z - (x - y)^2/(1 - \lambda^2))]^{(\nu-3)/2},$$
$$-\infty < y < \infty, \quad 0 < z < \infty$$

where

$$C(\lambda, \nu) = \Gamma\left(\frac{\nu}{2}\right) [\sqrt{\pi}(1 - \lambda^2)^{\frac{1}{2}}\Gamma((\nu - 1)/2)]^{-1}$$

is a solution of the equation

$$e^{-(x-a)^2/2\sigma^2} = (2\lambda\sigma^2\Gamma(\nu/2))^{-1} \int_{-\infty}^{+\infty} \int_0^{\infty} h_x(y, z) e^{-(y-a)^2/(2\sigma^2\lambda^2)} e^{-z/2\sigma^2} \cdot (z/2\sigma^2)^{(\nu/2)-1}\, dy\, dz$$

for every $\nu > 1$, every fixed real x and $0 < \lambda < 1$. This means $h_x(\eta, \zeta)$ is an unbiased estimate for $e^{-(x-a)^2/2\sigma^2}$. This fact is proved and used by Kolmogorov [2], who obtains the results of Theorem 1 for the following special cases: $f(x)$ is the characteristic function of a measurable set and $\lambda = (1/(\nu + 1))^{\frac{1}{2}}$, $\nu = 2, 3, 4, \cdots$. But at least for integers $\nu \geqq 2$ it is easy, using Kolmogorov's method, to extend his result to the more general assertion of Theorem 1. The method of Kolmogorov and my own method for proving Theorem 1 are of course closely related. Another paper which can be mentioned in connection with the method used here is one by Washio, Morimoto and Ikeda [3].

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