THIRD ORDER ROTATABLE DESIGNS IN THREE DIMENSIONS: SOME SPECIFIC DESIGNS[1]

By Norman R. Draper[2]

Mathematics Research Center, Madison, Wisconsin

0. Summary. A recent paper [3] described a method for constructing infinite classes of third order rotatable designs in three dimensions. This note gives nine specific designs selected from some of the infinite classes that have been shown to exist.

1. Preliminary remarks. Two papers by Bose and Draper [1] and Draper [2] dealt with the formation of second order rotatable designs in three [1] and in four or more [2] dimensions. It was shown there that certain point sets could be combined in such a way that infinite classes of designs, which included as special cases all previously known designs, could be formed. In further work [3] it has been established that certain pairs of the second order design classes found in [1] could be combined in such a way that infinite classes of *third order* rotatable designs for three factors were formed. Gardiner, Grandage and Hader [5] found four particular third order designs for three factors and it was shown [3] that two of their designs were the extreme cases of an infinite class. Several other infinite classes of third order rotatable designs have been tabulated in unpublished work by the author. (All are combinations of the infinite classes of second order rotatable designs given in [3].) This note presents nine specific designs, chosen from these infinite classes. The calculations by which they were obtained are illustrated elsewhere [3] and have not been repeated here.

The notation of [3] has been used below without further explanation; in particular, the symbol D_i refers to a second order rotatable design class to be found in Table II, on page 871 of [3]. The third order rotatable designs listed below consist of two separate second order rotatable designs, one each from two of the design classes in the table mentioned. The values of N, $\lambda_2 N$, $\lambda_4 N$, $\lambda_6 N$, λ_4/λ_2^2 and $\lambda_6\lambda_2/\lambda_4^2$ for each complete third order design are provided. Since it will be desirable to provide extra center points in some designs, N has been given in the form (number $+ n_0$).

It will be noticed that the designs are given in terms of a parameter θ; thus they are not scaled in the usual sense (i.e., $\lambda_2 \neq 1$). If desired, however, θ can be chosen so that $\lambda_2 = 1$ in order to make use of formulae given in [5] for the inversion of certain matrices needed in the analysis of third order rotatable designs. Formulae suitable when $\lambda_2 \neq 1$ are given in [4]; in this case, it is most convenient to set $\theta = 1$.

Received December 1, 1960; revised February 13, 1961.

[1] Sponsored by the United States Army under Contract No. DA-11-022-ORD-2059 at the Mathematics Research Center.

[2] Now at the Department of Statistics, University of Wisconsin.

2. Nine third order rotatable designs.

$$(1)\quad \begin{cases} D_1 \text{ with } a = \theta,\ c_1 = c_2 = 2^{\frac{1}{2}}\theta & (20 \text{ points}) \\ D_4 \text{ with } f = 2^{\frac{1}{2}}\theta,\ a = \theta,\ c = 2\theta & (26 \text{ points}) \end{cases}$$

$$\begin{aligned} \lambda_2 N &= 48\theta^2 & N &= 46 + n_0 \\ \lambda_4 N &= 32\theta^4 & \lambda_4/\lambda_2^2 &= N/72 \\ \lambda_6 N &= 16\theta^6 & \lambda_6\lambda_2/\lambda_4^2 &= \tfrac{3}{4} \end{aligned}$$

$$(2)\quad \begin{cases} D_2 \text{ with } a_1 = a_2 = \theta,\ c = 2\theta & (22 \text{ points}) \\ D_3 \text{ with } c_1 = c_2 = f = 2^{\frac{1}{2}}\theta & (24 \text{ points}) \end{cases}$$

$$\begin{aligned} \lambda_2 N &= 48\theta^2 & N &= 46 + n_0 \\ \lambda_4 N &= 32\theta^4 & \lambda_4/\lambda_2^2 &= N/72 \\ \lambda_6 N &= 16\theta^6 & \lambda_6\lambda_2/\lambda_4^2 &= \tfrac{3}{4} \end{aligned}$$

(Note: Designs (1) and (2) are identical if the overall third order design is considered. The split into two second order designs is, however, different in the two cases.)

$$(3)\quad \begin{cases} D_2 \text{ with } a_1 = 0.5578\theta,\ a_2 = 0.4097\theta,\ c = \theta & (22 \text{ points}) \\ D_4 \text{ with } f = 0.7201\theta,\ a = 0,\ c = 0.8564\theta & (26 \text{ points}) \end{cases}$$

$$\begin{aligned} \lambda_2 N &= 11.4472\theta^2 & N &= 48 + n_0 \\ \lambda_4 N &= 2.0756\theta^4 & \lambda_4/\lambda_2^2 &= 0.0158\, N \\ \lambda_6 N &= 0.2789\theta^6 & \lambda_6\lambda_2/\lambda_4^2 &= 0.74102 \end{aligned}$$

$$(4)\quad \begin{cases} D_2 \text{ with } a_1 = 0.5878\theta,\ a_2 = 0.2739\theta,\ c = \theta & (22 \text{ points}) \\ D_6 \text{ with } p = 0,\ q = 0.6609\theta,\ c = 0.9347\theta & (30 \text{ points}) \end{cases}$$

$$\begin{aligned} \lambda_2 N &= 14.0999\theta^2 & N &= 52 + n_0 \\ \lambda_4 N &= 2.5263\theta^4 & \lambda_4/\lambda_2^2 &= 0.0127\, N \\ \lambda_6 N &= 0.3333\theta^6 & \lambda_6\lambda_2/\lambda_4^2 &= 0.73643 \end{aligned}$$

$$(5)\quad \begin{cases} D_3 \text{ with } f = c_1 = c_2 = \theta & (24 \text{ points}) \\ D_4 \text{ with } f = 0.9357\theta,\ a = 0.8646\theta,\ c = 1.5653\theta & (26 \text{ points}) \end{cases}$$

$$\begin{aligned} \lambda_2 N &= 29.8850\theta^2 & N &= 50 + n_0 \\ \lambda_4 N &= 11.5368\theta^4 & \lambda_4/\lambda_2^2 &= 0.0129\, N \\ \lambda_6 N &= 3.3421\theta^6 & \lambda_6\lambda_2/\lambda_4^2 &= 0.75043 \end{aligned}$$

$$(6)\quad \begin{cases} D_3 \text{ with } f = c_1 = c_2 = \theta & (24 \text{ points}) \\ D_5 \text{ with } p = 1.4078\theta,\ q = 0.5112\theta,\ a = 0.7832\theta & (32 \text{ points}) \end{cases}$$

$$\begin{aligned} \lambda_2 N &= 36.9447\theta^2 & N &= 56 + n_0 \\ \lambda_4 N &= 15.8435\theta^4 & \lambda_4/\lambda_2^2 &= 0.0116\, N \\ \lambda_6 N &= 5.0949\theta^6 & \lambda_6\lambda_2/\lambda_4^2 &= 0.74986 \end{aligned}$$

$$(7)\quad \begin{cases} D_3 \text{ with } f = c_1 = c_2 = \theta & (24 \text{ points}) \\ D_5 \text{ with } p = 1.5205\theta,\ q = a = 0.5980\theta & (32 \text{ points}) \end{cases}$$

TABLE 1

Values of the parameters for the component second order rotatable designs

Third Order Design	Components	$\lambda_2 N$	$\lambda_4 N$	$\lambda_4/\lambda_2^2 N$	N
(1)	D_1 D_4	$16\ \theta^2$ $32\ \theta^2$	$8\ \theta^4$ $24\ \theta^4$	$\frac{1}{32}$ $\frac{3}{128}$	$20 + n_0$ $26 + n_0$
(2)	D_2 D_3	$24\ \theta^2$ $24\ \theta^2$	$16\ \theta^4$ $16\ \theta^4$	$\frac{1}{36}$ $\frac{1}{36}$	$22 + n_0$ $24 + n_0$
(3)	D_2 D_4	$5.8321\ \theta^2$ $5.6151\ \theta^2$	θ^4 $1.0756\ \theta^4$	0.0294 0.0341	$22 + n_0$ $26 + n_0$
(4)	D_2 $*D_6$	$5.3641\ \theta^2$ $8.7358\ \theta^2$	θ^4 $1.5263\ \theta^4$	0.0348 0.0200	$22 + n_0$ $30 + n_0$
(5)	D_3 D_4	$12\ \theta^2$ $17.8850\ \theta^2$	$4\ \theta^4$ $7.5368\ \theta^4$	$\frac{1}{36}$ 0.0236	$24 + n_0$ $26 + n_0$
(6)	D_3 D_5	$12\ \theta^2$ $24.9447\ \theta^2$	$4\ \theta^4$ $11.8436\ \theta^4$	$\frac{1}{36}$ 0.0190	$24 + n_0$ $32 + n_0$
(7)	D_3 D_5	$12\ \theta^2$ $27.0797\ \theta^2$	$4\ \theta^4$ $15.2772\ \theta^4$	$\frac{1}{36}$ 0.0208	$24 + n_0$ $32 + n_0$
(8)	D_3 D_5	$10.8284\ \theta^2$ $26.2681\ \theta^2$	$4\ \theta^4$ $14.9930\ \theta^4$	0.0341 0.0217	$24 + n_0$ $32 + n_0$
(9)	D_3 D_6	$12\ \theta^2$ $17.2230\ \theta^2$	$4\ \theta^4$ $5.9998\ \theta^4$	$\frac{1}{36}$ 0.0202	$24 + n_0$ $30 + n_0$

* Singular, center points essential; center points are also desirable in some of the other designs in which λ_4/λ_2^2 is not much greater than the singular value of 0.6, when $n_0 = 0$.

$$\lambda_2 N = 39.0797\theta^2 \qquad N = 56 + n_0$$
$$\lambda_4 N = 19.2772\theta^4 \qquad \lambda_4/\lambda_2^2 = 0.0126\, N$$
$$\lambda_6 N = 7.4641\theta^6 \qquad \lambda_6\lambda_2/\lambda_4^2 = 0.78493$$

$$(8)\quad \begin{cases} D_3 \text{ with } f = \theta,\ c_1 = 2^{\frac{1}{2}}\theta,\ c_2 = 0 & (24 \text{ points}) \\ D_5 \text{ with } p = 1.5167\theta,\ q = 0.6037\theta,\ a = 0.5042\theta & (32 \text{ points}) \end{cases}$$

$$\lambda_2 N = 37.0965\theta^2 \qquad N = 56 + n_0$$
$$\lambda_4 N = 18.9930\theta^4 \qquad \lambda_4/\lambda_2^2 = 0.0138\, N$$
$$\lambda_6 N = 7.4639\theta^6 \qquad \lambda_6\lambda_2/\lambda_4^2 = 0.76756$$

$$(9)\quad \begin{cases} D_3 \text{ with } f = c_1 = c_2 = \theta & (24 \text{ points}) \\ D_6 \text{ with } p = 0.9848\theta,\ q = 0.5748\theta,\ c = 1.4453\theta & (30 \text{ points}) \end{cases}$$

$$\lambda_2 N = 29.2230\theta^2 \qquad N = 54 + n_0$$
$$\lambda_4 N = 9.9999\theta^4 \qquad \lambda_4/\lambda_2^2 = 0.0117\, N$$
$$\lambda_6 N = 2.5405\theta^6 \qquad \lambda_6\lambda_2/\lambda_4^2 = 0.74244$$

3. Warning. The λ-values quoted above are those that apply for the complete third order rotatable design. If one second order part of the whole design is run first and a second order regression is performed, the λ-values appropriate to *it* must be used in the second order $\mathbf{X'X}$ matrix. These are provided in Table 1.

REFERENCES

[1] R. C. Bose, and Norman R. Draper, "Second order rotatable designs in three dimensions," *Ann. Math. Stat.*, Vol. 30 (1959), pp. 1097–1112.

[2] Norman R. Draper, "Second order rotatable designs in four or more dimensions," *Ann. Math. Stat.*, Vol. 31 (1960), pp. 23–33.

[3] Norman R. Draper, "Third order rotatable designs in three dimensions," *Ann. Math. Stat.*, Vol. 31 (1960), pp. 865–874.

[4] Norman R. Draper, "Some rotatable designs of third order in three factors," Technical Report No. 179, Mathematics Research Center, United States Army, Madison 6, Wisconsin, 1960.

[5] D. A. Gardiner, A. H. E. Grandage and R. J. Hader, "Third order rotatable designs for exploring response surfaces," *Ann. Math. Stat.*, Vol. 30 (1959), pp. 1082–1096