BOOK REVIEWS

Correspondence concerning reviews should be addressed to the Book Review Editor, Professor William Kruskal, Department of Statistics, University of Chicago, Chicago 37, Illinois.

E. B. DYNKIN, *Theory of Markov Processes*, (translated from the Russian by D. E. Brown). Prentice Hall, Englewood Cliffs, New Jersey and Pergamon Press, Oxford-London-Paris, 1961. $11.95, £3.2s. ix + 210 pp. E. B. DYNKIN, *Die Grundlagen der Theorie der Markoffschen Prozesse*, (translated from the Russian by Joseph Wloka), (Die Grundlehren der Mathematischen Wissenschaften, Band 108). Springer-Verlag, Berlin-Gottingen-Heidelberg, 1961. DM.29.80. 12 + 174 pp. (Stechert-Hafner, Inc., New York, $8.45).

Review by RONALD PYKE

University of Washington

The concept of a Markov process is a very common and important one to most readers of the *Annals of Mathematical Statistics*. Whether in the form of a discrete-parameter, countable-state Markov Chain such as the classical Random Walk, or in the form of a continuous-parameter Markov process such as a Birth and Death process or Brownian Motion, most readers will be quite familiar with this type of stochastic process. However, in view of the fact that Professor Dynkin's book deals with questions of a very general and abstract nature on the foundations of Markov processes, a good background in these more common types of Markov processes as well as in mathematics proper does not guarantee that a person will be able to understand or even appreciate the book.

Markov processes were named after the Russian Mathematician, A. A. Markov, who made his pioneering contributions to this subject during the first decade of this century. The books being reviewed are translations of a 1959 Russian treatise [1] written on Markov processes half a century after Markov's original work.[1]

This review is in two parts. The first part concerns the mathematical contents and contributions of Dynkin's original manuscript, while in the second part the quality of the present books' printing and translation is discussed. In summary, this reviewer finds the mathematical quality of the book quite respectable, but finds the publication details of the English translation miserable. The German translation's typography is extremely fine. (For obvious reasons, no comments are made about the quality of the translation into German.)

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[1] It has been brought to the reviewer's attention by Professor P. A. Meyer that a French translation of this book has now appeared. The exact reference to this edition, as kindly supplied to me by Professor Dynkin, is: E. B. Dynkin, *Théorie des Processus Markoviens*, Dunod, Paris, 1963, viii + 204 pp. (no. 11, Collection Universitaire de Mathématiques). Transl. by Ch. Sarthou.

The classical definition of a Markov process concerns the transition functions of the process. For example, in the case of a countable state Markov Chain $\{X_n ; n = 0, 1, 2, \cdots \}$, the standard definition would state that

$$\begin{aligned} \Pr[X_n = j \mid X_0 = i_0 , X_1 = i_1 , \cdots , X_{n-k} = i_{n-k}] \\ = \Pr[X_n = j \mid X_{n-k} = i_{n-k}] \end{aligned} \tag{1}$$

for all choices of states $j, i_0 , \cdots , i_{n-k}$ and for all $k, n \geqq 1$. In the case of a continuous-parameter, countable-state Markov process, $\{X_t:t \geqq 0\}$, the usual definition would state that

$$\Pr[X_t = j \mid X_{t_1} = i_1 , \cdots , X_{t_k} = i_k] = \Pr[X_t = j \mid X_{t_k} = i_k].$$

for all choices of time points $t_1 < t_2 < \cdots < t_k < t$ and states $i_1 , \cdots , i_k$. In the case of an arbitrary state-space Markov process, $\{X_t:t \geqq 0\}$, the usual definition would primarily concern the transition probabilities $P(s, x; t, A)$, which represent the conditional probabilities (actually a version thereof) that the process will be in the set A at time t given that at time s the process equalled x.

In the last decade, considerable research has been done on continuous-parameter Markov processes which take on values in abstract spaces. This research has been motivated in part by the applications which these processes have to non-probabilistic problems in analysis, such as potential theory and boundary value problems. For these recent researches, the classical definitions of Markov processes are unsuitable, and a more careful formulation is necessary. It was for the explicit purpose of providing such a formulation of the logical foundations of Markov processes that the present book was written.

The level and the intended audience of this book are best described simply by quoting from Professor Dynkin's preface: "This book cannot be used by the student to make his first acquaintance with the theory of Markov processes. Although we have not assumed formally any previous acquaintance with the theory of probability, in fact a reading of the book can only prove of value to someone already acquainted with an elementary exposition of the theory of Markov processes, . . . ".

In the introductory chapter (24 pages), Dynkin provides a condensed course in Measure Theory. It serves to give the reader a chance to familiarize himself with the vocabulary, definitions and theorems to be assumed in the following chapters. In the Supplementary Notes at the back of the book, sufficient references are given to enable the reader to locate the omitted proofs, as well as to find supplementary reading on the basic subjects treated in this chapter.

The author begins the main portion of the book in Chapter 2, by defining a Markov process. Although this review will not attempt to be notationally precise [that this would be impossible and undesirable is indicated by the fact that of the 142 symbols listed in the Index of Notation, 74 involve 3 or more letters, such as F_t^s or $Q_A^{\delta,\epsilon}(B)$] it is appropriate to give the definition of a Markov process which the author so carefully and skillfully investigates. [All processes considered in this book are indexed for convenience by the half-line $[0, \infty)$]. To define a Markov process, one first assumes an underlying sample space Ω, and a family of σ-algebras $\{\mathfrak{M}^s; 0 \leqq s < \infty,\}$ of subsets of Ω. Let $\zeta \leqq \infty$ be a non-negative valued function defined on Ω. [ζ will play the role of the (random) time at which the Markov process terminates.] Let $\{x_t(\omega); 0 \leqq t < \zeta(\omega), \omega \varepsilon \Omega\}$ be functions which take values in an abstract space E. With E is associated a σ-algebra $\mathfrak{B}$ of subsets of E, which includes all single-element subsets of E. For each $0 \leqq s \leqq t$ there is given a σ-algebra $\mathfrak{M}_t^s$ of subsets of the set $\Omega_t = \{\omega \varepsilon \Omega : \zeta(\omega) > t\}$. Finally, there is given for each $s \geqq 0$, and each $x \varepsilon E$, a probability measure $P_{s,x}(\cdot)$ defined on $\mathfrak{M}^s$. These given quantities, $X = (x_t, \zeta, \mathfrak{M}_t^s, \mathfrak{M}^s, P_{s,x})$ are said to define a *Markov process* if for every choice of $0 \leqq s \leqq t \leqq u$, $A \varepsilon \mathfrak{M}_t^s$, $B \varepsilon \mathfrak{B}$, and $x \varepsilon E$, it is true that

(i) $A \cap [\zeta > u] \varepsilon \mathfrak{M}_u^s \subset \mathfrak{M}^s$,

(ii) $[x_t \varepsilon B] \varepsilon \mathfrak{M}_t^s$,

(iii) the (transition) function $P(s, x; t, B) \equiv P_{s,x}([x_t \varepsilon B])$ is $\mathfrak{B}$-measurable in x,

(iv) $P(s, x; s, E - \{x\}) = 0$, and

(v) $P_{s,x}([x_u \varepsilon B] \cap A) = \int_A P(t, x_t; u, B) P_{s,x}(dy)$.

Condition (v) is the major Markov property. It essentially is equivalent to saying that the conditional probability of an event at time u given the past history of the process during the time interval $[s, t]$ depends only on the state of the process at the last time point t.

The author then defines the concept of a *Markov random function*, which embodies the idea of a Markov process with a fixed initial position. A *Markov family of random functions* is defined as a collection of these Markov random functions, which concept is then shown to be equivalent to that of the Markov process. These concepts are specialized to that of a time-homogeneous (called stationary in the text) Markov process. [Throughout the book, this very important subclass of processes is treated as a special case of the general theory, rather than being presented in a more self-contained way. Due to the great importance of these time-homogeneous processes, and the special methods of analysis involving the shift transformations which are peculiar to them, it is unfortunate that they did not receive a separate treatment.] *Equivalent* and *subordinate* processes are defined and studied. The former is a concept which equates two processes which have the same state and time spaces and the same transition function. One process is said to be subordinate to another if the sample functions of the first are in one-to-one correspondence with a subset of those of the second, and if the probability structure is essentially the same for both.

In Chapter 3 the author systematically develops the connection between subprocesses and multiplicative functionals. A subprocess of a given Markov process is defined as a randomly truncated version of the original process. A wide class of subprocesses are generated, for example, by observing the original process until the sample functions hit a specified set. If for such an example one were to define $\alpha_s^t (s < t)$ to be the conditional probability that this set will not be hit before t given the past history of the process up to time s, then one would expect to conclude from the Markovian nature of the process that $\alpha_s^u \alpha_u^t = \alpha_s^t$, $(s < u < t)$.

Functions satisfying this relationship are called multiplicative functionals. Dynkin defines and studies several classes of multiplicative functionals and gives results relating these functionals to the existence of associated subprocesses. This is the longest chapter of the book (43 pages) and represents material almost exclusively due to the author himself. Many of the proofs appear here for the first time. Since the appearance of this book, many contributions to the theory of multiplicative functionals have been made. The interested person is referred, for example, to the recent work of Meyer [4].

Chapter 4, the shortest chapter in the book (7 pages) gives the construction of a Markov process having a specified transition function. Five basic examples of transition functions are given at this point.

Throughout the history of Markov processes it has been an accepted folk-theorem that if one knows the state of the process at some known instant of time, then the future development of the process is independent of the past, *even if that instant of time is allowed to be a random variable determined by the past.* For example, one concludes from this "theorem" that if one observes 1-dimensional Brownian Motion until it is a unit distance away from its initial position for the first time, the future of the process is the same as a new and independent Brownian Motion.

One problem in the modern (post-1955) development of Markov processes has been to give a precise statement of this folk-theorem and to characterize those processes for which it is true. The property embodied in this theorem is called the Strong Markov property, and processes which possess this property are referred to as Strong Markov processes. In Chapter 5, the author rigorously defines this property and proves many of the basic consequences of the property. These results are then specialized to stationary processes. A weakened form of the Strong Markov property is given for processes whose sample functions are right continuous. Some standard sufficient conditions for a process to be Strong Markov are given in complete detail.

An important problem in the treatment of any class of stochastic processes is to obtain properties of the sample functions of the process. Chapter 6 deals with just this problem, treating in particular questions of boundedness, right-continuity, no discontinuities of the second kind, continuity, and jump-type. Conditions are also given under which an equivalent version of a process exists which possesses certain of these regularity properties.

In the Addendum, the author presents results of Choquet on capacities. These results are then used in a study of the measurability of the (random) times of first hitting a given subset of the state space.

Included in the book is a very informative set of supplementary notes which give the references, history, credits and comparative discussions for each section. (In fact, neither references nor credits are given in the main text of the book.) Also of considerable value is a notation index which gives the page on which each symbol was first defined. Most readers will make frequent references to this index.

So far, this review has been concerned only with the subject matter of the book, and is therefore applicable to the author's original Russian publication. We now turn our attention to the English publication.

The English edition of Dynkin's manuscript, when judged on the basis of its typography and the quality of the translation, is one of the worst books I have read. In view of the book's price, this low quality is inexcusable. The number of broken letters and symbols is very large. This is a great handicap to the reader of a book such as this which uses very complicated notation. (On the very first page there are 7 broken letters in mathematical expressions.) Pages 121, 188, 189 (particularly the footnote), are further examples of poor reproduction. Reading is made difficult also by the many words and symbols which are out of alignment. For example, look at lines 1, 2, 14, 21, 23, 28, 29 of page 2, or at lines 1, 16, 20, 24 of page 75. Mathematical expressions and equations (for example see Equation (4.6) on page 100) are frequently displayed in poor formats. Those expressions which come at the end of a line are often split in strange ways. (For example, on page 29, line 22, the expression $P_{s,x}(B_2)$ is split following the comma!)

The quality of the translation is annoyingly poor in several respects; for example, in the treatment of articles. Such phrases as "Let X be a random variable ...", or "Let f be a measurable mapping of ...", or "There exists a function ...", or "A necessary and sufficient condition for ...", which are very common in mathematical literature, all appear very frequently in this book, but with the article "a" replaced with the article "the". (In one instance, on line 17 of page 21, both articles are put in.) Often, as is the case in the Russian language, the articles are omitted altogether. For example, the title of Section 1, Chapter 4, is "Definition of Transition Function."

There are several inconsistencies in the translation. Random variable is occasionally translated as random magnitude. The word "restriction" is translated once as "purge" on page 47. The basic Strong Markov property is referred to throughout the book as the Strictly Markov property, while the word "strong" is used on the book's dust-jacket. The phrase "independent of the future and the s-past" is precisely defined, and yet it is occasionally translated as "not dependent on the future and the s-past." The word "extension" is translated as "prolongation" in the title of the Addendum.

It is amusing to notice that in the Preface the translator has bothered to translate the title of Feller's book [3] into Russian, where in the original Russian edition, the English title of the book had quite naturally been included.

The following minor errors were noted. On page 32, Kolmogorov-Chepmen should read Kolmogorov-Chapman. In line 11 of page 49, "... and $(t_1, \cdots)$" should read "... and $q(t_1, \cdots)$". In the fifth line from the bottom of page 103, Ω should read Ω_τ . In line 4 on page 189, "contained in" should read "containing". On page 197, "Ref. 2" should be "Ref. 21" on line 11, and "Ref. 8" in line 24 should be "Ref. 7". "Ref. 27" in line 10 of page 198 should be "Ref. 2".

It is regrettable that the name, date and publishing house of the original Russian book are not mentioned anywhere in the English translation. Even the date (July 4, 1958) is curiously deleted from the end of the author's preface.

Also, nowhere in this book nor on the dust-jacket can one find Professor Dynkin's academic address.

It is most unfortunate that the word "foundations" has been deleted from the original title of this book. As it stands, the title misleads one into believing that this is a general treatise on the theory of Markov processes, which it most certainly is not, nor was it intended to be by the author.

Many of the above comments on the quality of the English edition of Dynkin's book are quite minor when considered individually. Taken together, however, these comments should serve to indicate the overall unsatisfactory quality of this edition.

The German edition, on the other hand, is a superbly printed book. Although this reviewer can say nothing about the idiomatic clarity of this translation, he would remark that the typography and the paper are very easy on the eyes, and that only very few minor misprints were observed.

The reviewer is pleased to learn that Professor Dynkin's more extensive second book on the subject of Markov Processes, [2], has already been translated into English. This translation has been authorized by the author, and was carried out by J. Fabius, V. L. Greenberg, A. Maitra and G. D. Majone of the Statistical Laboratory, University of California, Berkeley, in consultation with Professor J. Neyman. The English edition of this book will be published shortly by Springer-Verlag of Berlin.

In summary, the reviewer would emphasize that this is a book for specialists in Markov Processes, or for students with a very good mathematical and probabilistic background who plan to work on problems in the general theory of Markov Processes. To people such as these, this book is recommended reading. However, it is suggested that anyone who requires a copy of Dynkin's treatise and whose reading knowledge of German is at all passable, will find the German edition to be more satisfactory than the English one, as well as less expensive, at $8.45. Furthermore, since Professor Dynkin's second book [2] will soon appear in English, this reviewer would reemphasize that the theme of the book being reviewed here is summarized by the word which does not appear in the title of the English edition, namely, "foundations". The second book on the other hand, presents a more extensive treatment of the mathematics of Markov Processes, and their relationships with analysis.

REFERENCES

[1] Dynkin, E. B. (1959). *Foundations of the Theory of Markov Processes*. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow [In Russian].

[2] Dynkin, E. B., *Markov Processes*, Fizmatgiz, 1963, Moscow, 860 pp. [In Russian]

[3] Feller, William (1957). *An Introduction to Probability Theory and its Applications*, 1 (2nd ed.). Wiley, New York.

[4] Meyer, P. A. (1962). Multiplicative and additive Markovian functionals. *Inst. Fourier, Grenoble*. **12** 125–130.