THE AUTOCORRELATION FUNCTION OF A SEQUENCE UNIFORMLY DISTRIBUTED MODULO 1

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1. Introduction. The study of sequences x_j uniformly distributed modulo 1 has been pursued mainly by number-theoreticians in the field of analytic number theory and diophantine analysis [2]. The properties of uniformly distributed sequences have recently become important in the application of Monte-Carlo methods and in the formation of random number generators and tables [3]. This paper proves a theorem which will facilitate the determination of the autocorrelation function of a uniformly distributed sequence.

The symbol $\{x\}$ denotes the *fractional part* of x, so that

$$0 \leqq \{x\} < 1. \tag{1.1}$$

The function $\{x\}$ is periodic with period 1 and assumes the value 0 only when x is integral. Let

$$\rho(x) = \tfrac{1}{2} - \{x\}, \tag{1.2}$$

then the function $\rho(x)$ is also periodic with period 1, assumes the value $\frac{1}{2}$ only when x is integral, and satisfies

$$-\tfrac{1}{2} < \rho(x) \leqq \tfrac{1}{2}. \tag{1.3}$$

Since $\rho(x)$ is a function of bounded variation, it possesses a convergent Fourier expansion which is

$$\rho(x) = \sum_{k=1}^{\infty} (\sin 2\pi kx/\pi k). \tag{1.4}$$

Let T denote the number of solutions of the inequalities

$$0 \leqq \{x_j\} \leqq \gamma, \qquad 1 \leqq j \leqq N, \qquad 0 \leqq \gamma < 1, \tag{1.5}$$

in which γ is fixed; then the sequence x_j is said to be uniformly distributed modulo 1 if and only if

$$T = \gamma N + o(N) \tag{1.6}$$

for each γ. An important criterion is that of Weyl [2] which states that the sequence x_j is uniformly distributed modulo 1 if and only if

$$\sum_{j=1}^{N} e^{i2\pi hx_j} = o(N), \qquad \text{for each integral } h \geqq 1. \tag{1.7}$$

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In consequence of the well-known estimate

$$(1.8)\qquad \left|\sum_{j=1}^{N} e^{i2\pi h\alpha j}\right| \leqq \frac{1}{2\|h\alpha\|},$$

in which $\|x\|$ denotes the distance from x to the nearest integer, and Weyl's criterion, the sequence $x_j = \alpha j$ is uniformly distributed modulo 1, provided α is irrational. In fact, if x_j is of the form

$$(1.9)\qquad x_j = \alpha_0 + \alpha_1 j + \alpha_2 j^2 + \cdots + \alpha_n j^n,$$

in which at least one $\alpha_\nu (1 \leqq \nu \leqq n)$ is irrational, then the sequence x_j is uniformly distributed modulo 1 [1], p. 60.

The autocorrelation function $\psi(\tau)$ of a sequence gives information on the extent of linear dependency among the successive terms of the sequence. The autocorrelation function $\psi(\tau)$ is defined by

$$(1.10)\qquad \psi(\tau) = \lim_{N\to\infty} (1/N) \sum_{j=1}^{N} \rho(x_j)\, \rho(x_{j+\tau}),$$

when the indicated limit exists. Let the sequence x_j be uniformly distributed modulo 1; then, after use of (1.4), (1.10) becomes

$$(1.11)\qquad \psi(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \left(\sum_{k=1}^{\infty} \frac{\sin 2\pi k x_j}{\pi k}\right)\left(\sum_{\nu=1}^{\infty} \frac{\sin 2\pi \nu x_{j+\tau}}{\pi \nu}\right).$$

Since when $\{x\} = 0$, the Fourier series for $\rho(x)$ has the value zero while $\rho(x) = \frac{1}{2}$, it is necessary to establish (1.11).
Let

$$(1.12)\qquad \sum_{j=1}^{N} \rho(x_j)\rho(x_{j+\tau}) = \sum_{j=1}^{N} \left(\sum_{k=1}^{\infty} \frac{\sin 2\pi k x_j}{\pi k}\right)\left(\sum_{\nu=1}^{\infty} \frac{\sin 2\pi \nu x_{j+\tau}}{\pi \nu}\right) + \sum_{j=1}^{N} e_j;$$

then $|e_j| \leqq \frac{1}{4}$ and $e_j = 0$ when $\{x_j\} > 0$ and $\{x_{j+\tau}\} > 0$. The number of times $e_j \neq 0$ cannot exceed the number of times $\{x_j\} = 0$ plus the number of times $\{x_{j+\tau}\} = 0$, which is $o(N)$. Hence,

$$(1.13)\qquad \sum_{j=1}^{N} \rho(x_j)\rho(x_{j+\tau}) = \sum_{j=1}^{N} \left(\sum_{k=1}^{\infty} \frac{\sin 2\pi k x_j}{\pi k}\right)\left(\sum_{\nu=1}^{\infty} \frac{\sin 2\pi \nu x_{j+\tau}}{\pi \nu}\right) + o(N),$$

and (1.11) is established.

A concomitant function normally considered is the integrated power spectrum $\Lambda(\omega)$, defined by

$$(1.14)\qquad \Lambda(\omega) = \psi(0)\omega + \sum_{\tau=1}^{\infty} \psi(\tau)\ (\sin 2\pi\tau\omega/\pi\tau),$$

whenever the series converges. The function

$$(1.15)\qquad \phi(\omega) = \Lambda'(\omega),$$

when it exists, is called the power spectrum or spectral density of the sequence.

One of the desiderata of a random number generator is that the sequence generated possess a specified spectral density. A sequence whose spectral density is constant, and hence whose autocorrelation function has the form

$$\psi(\tau) = 0, \qquad \tau \geqq 1, \tag{1.16}$$

is termed *white*. Since it is possible to produce a sequence with almost any desired spectral density, although not necessarily uniformly distributed, as the image under a *simple* transformation of a white sequence, it becomes important to construct generators of white sequences [1], pp. 138–142.

The limit of (1.10) may be written as an asymptotic relation; namely,

$$\sum_{j=1}^{N} \rho(x_j)\, \rho(x_{j+\tau}) = \psi(\tau)N + w(N), \qquad w(N) = o(N). \tag{1.17}$$

In particular, the definition of a white sequence is equivalent to the asymptotic form

$$\sum_{j=1}^{N} \rho(x_j)\, \rho(x_{j+\tau}) = o(N), \qquad \tau \geqq 1. \tag{1.18}$$

This raises the question of a more accurate investigation of the behavior of the sum, in particular, the determination of further terms of its asymptotic expansion.

Let a be a positive integer and let j run over the set $a \leqq j \leqq a + N$; then the asymptotic relation (1.17) takes the form

$$\sum_{j=a}^{a+N} \rho(x_j)\, \rho(x_{j+\tau}) = \psi(\tau)N + w(N, a). \tag{1.19}$$

An investigation of the behavior of the term $w(N, a)$ as a function of both N and a may now be made. It would be especially important to determine when effective estimates of $w(N, a)$ may be given which are uniform in a.

In the practical use of sequences uniformly distributed modulo 1, it is important to recognize the limitations imposed by a digital computer. The most important limitation, from the viewpoint of employing precise mathematical statements concerning infinite sequences generated by use of the properties of irrational numbers, is the restriction to a finite number of digits in the representation of a number. For the purpose of random number generation, this distinction can be profound. If an irrational number is desired, a rational approximation must be employed and the effect of the replacement carefully evaluated. For such use one must undertake independent investigations of the properties of any proposed random number generator especially with regard to the effect of round-off error on the desired result [3], p. 37. It is shown in this paper that the sequence

$$x_j = \alpha j^2, \qquad \alpha \text{ irrational}, \tag{1.20}$$

is white. This suggests that the sequence given by

$$x_j = (a/m)\, j^2, \qquad (a, m) = 1,\ a, m \text{ integral} \tag{1.21}$$

may possibly constitute a satisfactory random number generator for appropriate choice of a m and for a finite range of j. The author has investigated this sequence and determined the joint probability distribution and autocorrelation functions of x_j, $x_{j+\tau}$. The author has prepared a paper to present the results of this research.

2. Analytical discussion. The difficulty inherent in the formulation of (1.11) for the computation of the correlation function lies in the inability normally to effect the summation in closed form of the trigonometric series. However, if it were possible to perform the indicated limiting operation first, the difficulty would vanish. That that is possible is guaranteed by the following theorem, whose proof is the subject of the analytical discussion.

THEOREM 1. *Let x_j $(j \geqq 1)$ be a sequence uniformly distributed modulo 1, then the autocorrelation function $\psi(\tau)$ exists and is given by*

$$\psi(\tau) = \sum_{k=1}^{\infty} \sum_{\nu=1}^{\infty} (1/\pi^2 k\nu) \lim_{N\to\infty} (1/N) \sum_{j=1}^{N} (\sin 2\pi k x_j)(\sin 2\pi \nu x_{j+\tau})$$

whenever the indicated limit exists. The summation over k and ν may be performed in either order.

PROOF. It will be convenient to introduce the function

$$F(\tau, k, \nu) = \lim_{N\to\infty} (1/N) \sum_{j=1}^{N} (\sin 2\pi k x_j)\ (\sin 2\pi \nu x_{j+\tau}), \tag{2.1}$$

in which it is assumed that the limit exists. From (1.11) one has

$$\psi(\tau) = \sum_{k=1}^{M-1} \sum_{\nu=1}^{M-1} [F(\tau, k, \nu)/\pi^2 k\nu] + \lim_{N\to\infty} R_{N,M}\,, \tag{2.2}$$

in which

$$\begin{aligned} R_{N,M} = \frac{1}{N}\sum_{j=1}^{N}\Bigg[&\left(\sum_{k=M}^{\infty}\frac{\sin 2\pi k x_j}{\pi k}\right)\left(\sum_{\nu=M}^{\infty}\frac{\sin 2\pi \nu x_{j+\tau}}{\pi\nu}\right) \\ &+\left(\sum_{k=1}^{M-1}\frac{\sin 2\pi k x_j}{\pi k}\right)\left(\sum_{\nu=M}^{\infty}\frac{\sin 2\pi \nu x_{j+\tau}}{\pi\nu}\right) \\ &+\left(\sum_{k=M}^{\infty}\frac{\sin 2\pi k x_j}{\pi k}\right)\left(\sum_{\nu=1}^{M-1}\frac{\sin 2\pi \nu x_{j+\tau}}{\pi\nu}\right)\Bigg]. \end{aligned} \tag{2.3}$$

One now has

$$\begin{aligned} |R_{N,M}| \leqq \frac{1}{N}\sum_{j=1}^{N}\Bigg[&\left|\sum_{k=M}^{\infty}\frac{\sin 2\pi k x_j}{\pi k}\right|\left|\sum_{\nu=M}^{\infty}\frac{\sin 2\pi \nu x_{j+\tau}}{\pi\nu}\right| \\ &+\left|\sum_{k=1}^{M-1}\frac{\sin 2\pi k x_j}{\pi k}\right|\left|\sum_{\nu=M}^{\infty}\frac{\sin 2\pi \nu x_{j+\tau}}{\pi\nu}\right|+\left|\sum_{k=M}^{\infty}\frac{\sin 2\pi k x_j}{\pi k}\right|\left|\sum_{\nu=1}^{M=1}\frac{\sin 2\pi \nu x_{j+\tau}}{\pi\nu}\right|\Bigg]. \end{aligned} \tag{2.4}$$

In order to proceed with the estimation of $|R_{N,M}|$, the following lemma is required.

LEMMA.

$$\left|\sum_{l=M}^{\infty} \frac{\sin 2\pi lx}{\pi l}\right| \leqq \min\left(1, \frac{1}{2\pi M\|x\|}\right).$$

The following standard theorem derived from Abel's transformation of series will be used:

$$a_l \geqq 0 \downarrow, \left|\sum_{p=M}^{l} b_p\right| \leqq B \Rightarrow \left|\sum_{l=M}^{\infty} a_l b_l\right| \leqq a_M B. \tag{2.5}$$

Also standard is the following estimate:

$$\left|\sum_{p=M}^{l} \sin 2\pi px\right| \leqq \frac{1}{2\|x\|}. \tag{2.6}$$

The use of the above theorem and the estimate of (2.6) yields

$$\left|\sum_{l=M}^{\infty} \frac{\sin 2\pi lx}{\pi l}\right| \leqq \frac{1}{2M\|x\|}. \tag{2.7}$$

One has

$$\sum_{l=M}^{\infty} \frac{\sin 2\pi lx}{\pi l} = \sum_{l=1}^{\infty} \frac{\sin 2\pi lx}{\pi l} - \sum_{l=1}^{M-1} \frac{\sin 2\pi lx}{\pi l}, \tag{2.8}$$

and hence

$$\left|\sum_{l=M}^{\infty} \frac{\sin 2\pi lx}{\pi l}\right| \leqq \frac{1}{2} + \left|\sum_{l=1}^{M-1} \frac{\sin 2\pi lx}{\pi l}\right|. \tag{2.9}$$

In the case $2\pi M\|x\| > 1$, the lemma is clear. Accordingly, assume $2\pi M\|x\| \leqq 1$. One has

$$\left|\sum_{l=1}^{M-1} \frac{\sin 2\pi lx}{\pi l}\right| \leqq \sum_{l=1}^{M-1} \frac{|\sin 2\pi l\|x\||}{\pi l} \leqq \sum_{l=1}^{M-1} 2\|x\| < 2M\|x\| \leqq 1/\pi; \tag{2.10}$$

Therefore,

$$\left|\sum_{l=M}^{\infty} \frac{\sin 2\pi lx}{\pi l}\right| < \frac{1}{2} + 1/\pi < 1. \tag{2.11}$$

The lemma follows from (2.7) and (2.11). Further, (2.8) yields

$$\left|\sum_{l=1}^{M-1} \frac{\sin 2\pi lx}{\pi l}\right| \leqq \frac{1}{2} + \left|\sum_{l=M}^{\infty} \frac{\sin 2\pi lx}{\pi l}\right| \tag{2.12}$$

and hence, by use of the lemma

$$\left|\sum_{l=1}^{M-1} \frac{\sin 2\pi lx}{\pi l}\right| < \frac{3}{2}. \tag{2.13}$$

Application of the above lemma and (2.13) to (2.4) yields

$$(2.14)\quad \begin{aligned}|R_{N,M}| < (1/N) \sum_{j=1}^{N} [&\min(1, 1/2\pi M\|x_j\|) \min(1, 1/2\pi M\|x_{j+\tau}\|)\\ &+ \frac{3}{2}\min(1, 1/2\pi M\|x_j\|) + \frac{3}{2}\min(1, 1/2\pi M\|x_{j+\tau}\|)].\end{aligned}$$

The Cauchy inequality applied to the sums in (2.14) yields

$$(2.15)\quad \begin{aligned}|R_{N,M}| < &\left[\frac{1}{N}\sum_{j=1}^{N}\min\left(1, \frac{1}{4\pi^2M^2\|x_j\|^2}\right)\right]^{\frac{1}{2}} \cdot \left[\frac{1}{N}\sum_{j=1}^{N}\min\left(1, \frac{1}{4\pi^2M^2\|x_{j+\tau}\|^2}\right)\right]^{\frac{1}{2}}\\ &\frac{3}{2}\left[\frac{1}{N}\sum_{j=1}^{N}\min\left(1, \frac{1}{4\pi^2M^2\|x_j\|^2}\right)\right]^{\frac{1}{2}} + \frac{3}{2}\left[\frac{1}{N}\sum_{j=1}^{N}\min\left(1, \frac{1}{4\pi^2M^2\|x_{j+\tau}\|^2}\right)\right]^{\frac{1}{2}}.\end{aligned}$$

It is now necessary to employ another lemma.

LEMMA. *If the sequence x_j is uniformly distributed modulo 1, $u \geqq 0$, $b > 0$, then*

$$(1/N) \sum_{j=1}^{N} \min(u^2, 1/b^2\|x_j\|^2) < 4ub^{-1} + o(1).$$

Let $a > 1$ be integral and even. Divide the interval [0, 1] by the points of division sa^{-1}, $0 \leqq s < a$. The sum $\sum_{j=1}^{N} \min(u^2, 1/b^2\|x_j\|^2)$ is dissected into

$$\sum_{s=0} \min(u^2, 1/b^2\|x_j\|^2) + \cdots + \sum_{s=a-1} \min(u^2, 1/b^2\|x_j\|^2),$$

in which each sum is taken over all x_j satisfying

$$(2.16)\quad sa^{-1} \leqq \{x_j\} < sa^{-1} + a^{-1}.$$

The number of such x_j is $Na^{-1} + o(N)$. For the sums corresponding to $s = 0$, $s = a - 1$, the estimate u^2 is used. In the terms corresponding to $1 \leqq s \leqq a/2 - 1$, one has $\|x_j\| \geqq sa^{-1}$. For the terms corresponding to $a/2 \leqq s \leqq a - 2$, let $\sigma = a - s - 1$; then $1 \leqq \sigma \leqq a/2 - 1$. For this range of s, one has $\{x_j\} = 1 - \|x_j\|$, and hence, the inequality $\|x_j\| > \sigma a^{-1}$ is valid. Hence,

$$(2.17)\quad \sum_{j=1}^{N} \min\left(u^2, \frac{1}{b^2\|x_j\|^2}\right) < \{Na^{-1} + o(N)\}\left\{2u^2 + 2\sum_{s=1}^{a/2-1} \min\left(u^2, \frac{a^2}{b^2s^2}\right)\right\}.$$

The function $\min(u^2, a^2b^{-2}s^{-2})$ is monotonically decreasing; hence

$$(2.18)\quad \sum_{s=1}^{a/2-1} \min\left(u^2, \frac{a^2}{b^2s^2}\right) < \int_0^\infty \min\left(u^2, \frac{a^2}{b^2v^2}\right) dv = \frac{2au}{b}.$$

One now obtains, from (2.17) and (2.18),

$$(2.19)\quad (1/N) \sum_{j=1}^{N} \min(u^2, 1/b^2\|x_j\|^2) < 4u/b + 2u^2/a + o(1).$$

Since a is arbitrary, one may choose a so large that $2u^2a^{-1} < \epsilon/2$ ($\epsilon > 0$) and then N so that $|o(1)| < \epsilon/2$; hence, $2u\, a^{-1} + o(1) = o(1)$ for large N and the lemma follows.

A simple consequence of the definition is that $x_{j+\tau}$ is uniformly distributed modulo 1 if x_j is; hence, applying the above lemma to (2.15), one has $R_{N,M} = O(M^{-\frac{1}{2}})$ uniformly in N. Since M may be chosen arbitrarily large, the theorem follows.

COROLLARY 1. *If $x_{j+\tau} + kx_j$ is uniformly distributed modulo 1 for all $\nu > 0$, $k > 0$, $\tau > 0$, and $\nu x_{j+\tau} - kx_j$ is uniformly distributed modulo 1 for all $\nu > 0$, $k > 0$, $(k \neq \nu)$, $\tau > 0$, then*

$$\psi(\tau) = \sum_{k=1}^{\infty} (1/2\pi^2k^2) \lim_{N\to\infty} (1/N) \sum_{j=1}^{N} \cos 2\pi k\, (x_{j+\tau} - x_j).$$

From the identity

$$\sin 2\pi k x_j \sin 2\pi \nu x_{j+\tau} = \tfrac{1}{2}\cos 2\pi(\nu x_{j+\tau} - kx_j) - \tfrac{1}{2}\cos 2\pi(\nu x_{j+\tau} + kx_j). \tag{2.20}$$

one has

$$\begin{aligned} \lim_{N\to\infty} (1/N) \sum_{j=1}^{N} \sin 2\pi k x_j \sin 2\pi \nu x_{j+\tau} &= \lim_{N\to\infty} (1/2N) \sum_{j=1}^{N} \cos 2\pi(\nu x_{j+\tau} - kx_j) \\ &\quad - \lim_{N\to\infty} (1/2N) \sum_{j=1}^{N} \cos 2\pi(\nu x_{j+\tau} + kx_j). \end{aligned} \tag{2.21}$$

Since $\nu x_{j+\tau} + kx_j$ is uniformly distributed modulo 1, Weyl's criterion yields

$$\lim_{N\to\infty} (1/2N) \sum_{j=1}^{N} \cos 2\pi(\nu x_{j+\tau} + kx_j) = 0. \tag{2.22}$$

Also since $\nu x_{j+\tau} - kx_j$ is uniformly distributed modulo 1 for $k \neq \nu$, one has

$$\lim_{N\to\infty} (1/2N) \sum_{j=1}^{N} \cos 2\pi(\nu x_{j+\tau} - kx_j) = 0, \qquad k \neq \nu. \tag{2.23}$$

The corollary now follows.

COROLLARY 2. *$\nu x_{j+\tau} + kx_j$ is uniformly distributed modulo 1 for all $\nu > 0$, $k > 0$, $\tau > 0$, and $\nu x_{j+\tau} - kx_j$ is uniformly distributed modulo 1 for all $\nu > 0$, $k > 0$, $\tau > 0$, implies $\psi(\tau) = 0$, $\tau > 0$.*

This is an immediate consequence of Corollary 1 since $x_{j+\tau} - x_j$ is now assumed uniformly distributed modulo 1.

COROLLARY 3. *$\nu x_{j+\tau} + kx_j$ is uniformly distributed modulo 1 for all $\nu > 0$, $k > 0$, $\tau > 0$, $\nu x_{j+\tau} - kx_j$ is uniformly distributed modulo 1 for all $\nu > 0$, $k > 0$ $(k \neq \nu)$, $\tau > 0$, and*

$$\lim_{N\to\infty} (1/N) \sum_{j=1}^{N} \cos 2\pi k(x_{j+\tau} - x_j) = 1 \Rightarrow \psi(\tau) = \tfrac{1}{12}, \; \tau \geqq 0.$$

This follows directly from Corollary 1.

COROLLARY 4. *$\nu x_{j+\tau} + kx_j$ is uniformly distributed modulo 1 for all $\nu > 0$, $k > 0$, $\tau > 0$, $\nu x_{j+\tau} - kx_j$ is uniformly distributed modulo 1 for all $\nu > 0$, $k > 0$ $(k \neq \nu)$, $\tau > 0$, and*

$$\lim_{j\to\infty} (x_{j+\tau} - x_j) = 0,\ \tau \geqq 0, \Rightarrow \psi(\tau) = \tfrac{1}{12},\ \tau \geqq 0.$$

This follows from Corollary 3 on observing that

$$\lim_{j\to\infty} \cos 2\pi k(x_{j+\tau} - x_j) = 1 \tag{2.24}$$

and using a well-known lemma on arithmetic means.

3. Applications. An immediate example of a white sequence is given by

$$x_j = \alpha j^2, \qquad \alpha \text{ irrational.} \tag{3.1}$$

The result follows from Corollary 2 and Weyl's theorem on the uniform distribution of polynomials quoted after (1.9). This sequence may be suitable as a random number generator.

The sequence

$$x_j = \alpha j, \qquad \alpha \text{ irrational} \tag{3.2}$$

is not white. Introduce the function

$$\sigma(x) = \int_0^x \rho(u)\, du, \tag{3.3}$$

then $\sigma(x)$ has period 1, is continuous, and satisfies

$$0 \leqq \sigma(x) \leqq \tfrac{1}{8}. \tag{3.4}$$

Further, the Fourier expansion of $\sigma(x)$ is

$$\sigma(x) = \tfrac{1}{12} - \sum_{k=1}^{\infty} (\cos 2\pi kx/2\pi^2k^2). \tag{3.5}$$

One now has the following theorem.

THEOREM 2. *If $x_j = \alpha j$, α irrational, then $\psi(\tau) = \frac{1}{12} - \sigma(\alpha\tau)$.*

All conditions of Corollary 1 are met and

$$\lim_{N\to\infty} (1/N) \sum_{j=1}^{N} \cos 2\pi k(x_{j+\tau} - x_j) = \cos 2\pi k\alpha\tau. \tag{3.6}$$

Hence,

$$\psi(\tau) = \sum_{k=1}^{\infty} (\cos 2\pi k\alpha\tau/2\pi^2k^2). \tag{3.7}$$

Comparison of (3.7) with (3.5) yields the result of the theorem.

The integrated power spectrum is obtained from (1.14).

$$\Lambda(\omega) = \frac{1}{12}\,\omega + \sum_{\tau=1}^{\infty} \left\{\frac{1}{12} - \sigma(\alpha\tau)\right\} \frac{\sin 2\pi\tau\omega}{\pi\tau}. \tag{3.8}$$

It is clear that the sequence of (3.2) does not possess a spectral density.

A theorem on trigonometric sum estimation is the following.

THEOREM 3. *$f'(x)$ is monotonic, $f'(x) \geqq \epsilon > 0$ or $-f'(x) \geqq \epsilon > 0$ for all $1 \leqq x \leqq N$, $|f(N) - f(1)| \leqq B$ implies $\sum_{j=1}^{N} e^{i2\pi f(j)} = O(1 + B + \epsilon^{-1})$.*

Let

$$R = \sum_{j=1}^{N-1} e^{i2\pi f(j)} - \int_1^N e^{i2\pi f(x)}\, dx, \tag{3.9}$$

then $R = \sum_{j=1}^{N-1} R_j$,

$$R_j = \int_j^{j+1} \{e^{i2\pi f(j)} - e^{i2\pi f(x)}\}\, dx. \tag{3.10}$$

One has

$$|R_j| \leqq \int_j^{j+1} |e^{i2\pi f(j)} - e^{i2\pi f(x)}|\, dx \leqq 2\pi \int_j^{j+1} |f(j) - f(x)|\, dx, \tag{3.11}$$

$$|R_j| \leqq 2\pi\, |f(j+1) - f(j)|. \tag{3.12}$$

(3.12) follows from the monotonicity of $f(x)$. Thus

$$|R| \leqq 2\pi \sum_{j=1}^{N-1} |f(j+1) - f(j)| = 2\pi \left|\sum_{j=1}^{N-1} f(j+1) - f(j)\right| = O(B). \tag{3.13}$$

Hence,

$$\sum_{j=1}^{N} e^{i2\pi f(j)} = \int_1^N e^{i2\pi f(x)}\, dx + O(1 + B). \tag{3.14}$$

Since

$$\int_1^N e^{i2\pi f(x)}\, dx = \int_1^N \frac{1}{i2\pi f'(x)}\, de^{\,i2\pi f(x)}, \tag{3.15}$$

one has, on application of the second mean value theorem to the real and imaginary parts of the integral,

$$\int_1^N e^{i2\pi f(x)}\, dx = O(\epsilon^{-1}). \tag{3.16}$$

The theorem now follows.

Application of Theorem 3 and Weyl's criterion to the sequence

$$x_j = j^\sigma, \qquad 0 < \sigma < 1, \tag{3.17}$$

establishes its uniform distribution modulo 1. In fact, it further follows from Theorem 3 that $\nu(j + \tau)^\sigma + kj^\sigma$ is uniformly distributed modulo 1 for all $\nu > 0$, $k > 0$, $\tau > 0$, and $\nu(j + \tau)^\sigma - kj^\sigma$ is uniformly distributed modulo 1 for all $\nu > 0$, $k > 0$ $(k \neq \nu)$, $\tau > 0$. One also has

$$\lim_{j\to\infty} \{(j + \tau)^\sigma - j^\sigma\} = 0, \qquad 0 < \sigma < 1. \tag{3.18}$$

Thus all conditions of Corollary 4 are satisfied and one has the following theorem.

THEOREM 4. $x_j = j^\sigma, 0 < \sigma < 1 \Rightarrow \psi(\tau) = \frac{1}{12}, \tau \geqq 0$.

The sequence of Theorem 4 thus exhibits complete positive correlation. The integrated power spectrum is

$$\Lambda(\omega) = \tfrac{1}{24}, \qquad 0 < \omega < 1. \tag{3.19}$$

The total energy of the sequence is thus concentrated at $\omega = 0$.

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