USE OF INTER-BLOCK INFORMATION TO OBTAIN UNIFORMLY BETTER ESTIMATORS

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1. Introduction and summary. Yates (1939, 1940) suggested use of information about treatment differences contained in differences of block totals. The procedure given by Yates for three dimensional lattice designs (1939) and for balanced incomplete block (BIB) designs was adopted by Nair (1944) for partially balanced incomplete block (PBIB) designs and was later generalized by Rao (1947) for use with any incomplete block design.

The procedure is called recovery of inter-block information and consists of the following stages. The method of least-squares is applied to both intra- and inter-block contrasts, assuming that the value of ρ, the ratio of the inter-block variance to the intra-block variance is known. This gives the so called "normal" equations for combined estimation. The equations involve ρ which is estimated from the observations by equating the error sum of squares (intra-block) and the adjusted block sum of squares in the standard analysis of variance to their respective expected values. This estimate is substituted for ρ in the normal equations and the combined estimates are obtained by solving these equations. A priori, the inter-block variance is expected to be larger than the intra-block variance and hence it is customary to use the above estimator of ρ, truncated at unity.

The error sum of squares in the inter-block analysis has at times been used in place of the adjusted block sum of squares (Yates (1939) for three dimensional lattice designs, Graybill and Deal (1959) for BIB designs).

If ρ were known, the combined estimators would have all the good properties of least-squares estimates. Since only an estimate of ρ is used, the properties of the combined estimators have to be critically examined. One would expect these to depend on the type of estimator of ρ used. To use the combined estimator of a treatment contrast with confidence one would like to know if it is unbiased and if its variance is smaller than that of the corresponding intra-block estimator, uniformly in ρ.

The question of unbiasedness has been examined by some authors. Graybill and Weeks (1959) showed that for a BIB design, the combined estimator of a treatment contrast based on the Yates' estimator of ρ in its untruncated form is unbiased. Graybill and Seshadri (1960) proved the same with Yates' estimator of ρ in its usual truncated form, again for BIB designs. Roy and Shah (1962) showed that for any incomplete block design, if the estimator of ρ is the ratio of quadratic forms of a special type, the corresponding combined estimators of treatment contrasts are unbiased. The customary estimator of ρ (as given by

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Yates (1939) and Rao (1947)) is of the above type and hence gives rise to unbiased combined estimators.

The variance of the combined estimators has also been examined by some authors. Yates (1939) used the method of numerical integration to show that for a three dimensional lattice design with 27 treatments and with 6 replications or more, the combined estimator of a treatment contrast has variance smaller than that of the intra-block estimator, uniformly in ρ. For a BIB design for which the number of blocks exceeds the number of treatments by at least 10 (or by 9 if in addition, the number of degrees of freedom for intra-block error is not less than 18), Graybill and Deal (1959) used the exact expression for the variance to establish this property of the combined estimators. In both the cases, the estimator of ρ is based on the inter-block error and thus differs from the usual one based on the adjusted block sum of squares. For BIB designs, Seshadri (1963) gave yet another estimator of ρ which gives rise to more precise combined estimators provided that the number of treatments exceeds 8.

Roy and Shah (1962) gave an expression for the variance of the combined estimator based on any estimator of ρ belonging to the class described above. Shah (1964) used this expression to show that the combined estimator of any treatment contrast in any incomplete block design has variance smaller than that of the corresponding intra-block estimator if ρ does not exceed 2.

The question that now arises is whether a combined estimator for a treatment contrast can be constructed which is "uniformly better" than the intra-block estimator, in the sense of having a smaller variance for all values of ρ. It is shown in Section 4 that for a linked block (LB) design with 4 or 5 blocks, recovery of inter-block information by the Yates-Rao procedure may even result in loss of efficiency for large values of ρ.

A method of constructing a certain estimator of ρ, applicable to any incomplete block design for which the association matrix has a nonzero latent root of multiplicity $p > 2$, is presented in Section 3. For any treatment contrast belonging to a sub-space associated with the multiple latent root, the combined estimator based on this estimator of ρ is shown to be uniformly better than the intra-block estimator if and only if $(p - 4) \times (e_0 - 2) \geqq 8$, where e_0 is the number of degrees of freedom for error (inter-block). For almost all well-known designs, the association matrix has multiple latent roots and this method can therefore be applied to many of the standard designs, at least for some of the treatment contrasts.

It may be noted that, in general, this estimator of ρ is different from the customary one given by Yates (1939) and Rao (1947). For LB designs however, this estimator of ρ coincides with the customary one. It is shown here that, for a LB design, the usual procedure of recovery of inter-block information gives uniformly better combined estimators for *all* treatment constrasts if the number of blocks exceeds 5. As was pointed out before, if the number of blocks is 4 or 5 and if ρ is large, recovery of inter-block information by the usual procedure results in loss of efficiency.

Using the above method, we obtain an estimator of ρ which produces a combined estimator uniformly better than the intra-block estimator for any treatment contrast for the following designs: (i) a BIB design with more than five treatments (ii) a simple lattice design with sixteen treatments or more and (iii) a triple lattice design with nine treatments or more. Applications to some other two-associate partially balanced incomplete block designs and to inter- and intra-group balanced designs have also been worked out in Sections 4 and 5. A computational procedure for obtaining the estimate of ρ has been given for each case.

2. Preliminaries. Consider an incomplete block design with b blocks of k plots each involving v treatments each replicated r times, having the $v \times b$ matrix $\mathbf{N} = ((n_{ij}))$ as the incidence matrix. Any linear function of observations which is orthogonal to each of the block totals will be called an intra-block contrast. Obviously, we can construct $b(k - 1)$ mutually orthogonal intra-block contrasts. Also, if a contrast in observations is a function of block totals only, we shall call it an inter-block contrast. We can construct $(b - 1)$ mutually orthogonal inter-block contrasts. Without loss of generality we may assume that these contrasts are normalised, i.e. the sum of squares of co-efficients is unity.

We assume that the joint distribution of these $b(k - 1)$ contrasts is multivariate normal where, the expected value of any contrast is obtained by replacing in the contrast every observation by the corresponding treatment parameter, the variance of any intra (inter)-block contrast is $\sigma_0^2(\sigma_1^2)$ and the covariances are all zero. Thus we may call $\sigma_0^2(\sigma_1^2)$ intra (inter)-block variance per plot or simply intra (inter)-block variance. Evidently, if these assumptions hold for one set of mutually orthogonal normalised inter- and intra-block contrasts they will hold for any other. We also assume that $\rho = \sigma_1^2/\sigma_0^2 \geqq 1$.

Let $\mathbf{B} = \{B_1, B_2, \cdots, B_b\}$, $\mathbf{T} = \{T_1, T_2, \cdots, T_v\}$ and $\boldsymbol{\theta} = \{\theta_1, \theta_2, \cdots, \theta_v\}$ be the column-vectors of block totals, treatment totals and treatment parameters respectively. By G, we shall denote the total of all observations. Let further,

$$\begin{aligned} \mathbf{Q} &= \mathbf{T} - k^{-1}\mathbf{NB} \qquad & \mathbf{Q}_1 &= k^{-1}\mathbf{NB} - (rG/bk)\mathbf{E}_{v,1} \\ \mathbf{C} &= r\mathbf{I} - k^{-1}\mathbf{NN}' \qquad & \mathbf{C}_1 &= k^{-1}\mathbf{NN}' - (r^2/bk)\mathbf{E}_{v,v} \end{aligned} \tag{2.1}$$

where $\mathbf{E}_{m,n}$ denotes a $(m \times n)$ matrix with all elements unity.

It is known (see, for example, Rao (1947)) that minimum variance unbiased linear estimates of treatment effects $\boldsymbol{\theta}$ based on intra-block contrasts only, are obtained from the equations

$$\mathbf{C}\boldsymbol{\theta} = \mathbf{Q}. \tag{2.2}$$

These shall be called the intra-block normal equations for estimation or simply the intra-block equations. If the ratio $\rho = \sigma_1^2/\sigma_0^2$ is known, both intra-block and inter-block contrasts can be used together, and minimum variance linear unbiased estimates in this case are obtained from the equations

$$(\mathbf{C} + \rho^{-1}\mathbf{C}_1)\boldsymbol{\theta} = (\mathbf{Q} + \rho^{-1}\mathbf{Q}_1). \tag{2.3}$$

These shall be called the combined normal equations for estimation or simply the combined equations. The solution of these equations will be denoted by $\bar{\boldsymbol{\theta}}(\rho)$. When ρ is not known, an estimate ρ^* for ρ is substituted in (2.3) and $\bar{\boldsymbol{\theta}}(\rho^*)$ is taken as an estimate for $\boldsymbol{\theta}$.

Since we shall consider connected designs only, the matrix $\mathbf{C}$ has exactly one latent root zero. Hence the matrix $\mathbf{NN}'$ has a latent root rk with multiplicity one and all other latent roots will be smaller than rk.

Let $\mathbf{p}_s$, $s = 1, 2, \cdots, q$ be a set of orthonormal latent vectors of $\mathbf{NN}'$ corresponding to the q positive latent roots $\phi_1, \phi_2, \cdots, \phi_q$, all smaller than rk. Let $\mathbf{p}_s$, $s = q + 1, \cdots, v - 1$, be a set of $v - 1 - q$ normalised orthogonal vectors each orthogonal to $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_q$ and also to $\mathbf{E}_{v,1}$. Now we define

$$
\begin{aligned}
x_{0s} &= k^{\frac{1}{2}}(rk - \phi_s)^{-\frac{1}{2}}\mathbf{Q}'\mathbf{p}_s & s &= 1, 2, \cdots, q \\
&= r^{-\frac{1}{2}}\mathbf{Q}'\mathbf{p}_s & s &= q + 1, \cdots, v - 1.
\end{aligned}
\tag{2.4}
$$

It can be seen that x_{0s}, $s = 1, 2, \cdots, v - 1$ are mutually orthogonal normalised intra-block contrasts. Hence we can find $e_0 = b(k - 1) - v + 1$ mutually orthogonal normalised intra-block contrasts each orthogonal to x_{01}, $x_{02}, \cdots, x_{0,v-1}$. These we may denote by z_{0s}, $s = 1, 2, \cdots, e_0$. We also define

$$
x_{1s} = (k\phi_s)^{-\frac{1}{2}}\mathbf{B}'\mathbf{N}'\mathbf{p}_s \qquad s = 1, 2, \cdots, q.
\tag{2.5}
$$

These can be seen to be mutually orthogonal normalised inter-block contrasts and hence we can find $e_1 = b - 1 - q$ mutually orthogonal normalised inter-block contrasts each orthogonal to $x_{11}, \cdots, x_{1q}$. We denote these by z_{1s}, $s = 1, 2, \cdots, e_1$.

It follows from our assumptions that

$$
\begin{aligned}
E(x_{0s}) &= a_{0s}\tau_s & &\text{for } s = 1, 2, \cdots, v - 1. \\
E(x_{1s}) &= a_{1s}\tau_s & &\text{for } s = 1, 2, \cdots, q \\
E(z_{0s}) &= 0 & &\text{for } s = 1, 2, \cdots, e_0 \\
E(z_{1s}) &= 0 & &\text{for } s = 1, 2, \cdots, e_1
\end{aligned}
\tag{2.6}
$$

where

$$
\begin{aligned}
a_{0s} &= (r - \phi_s/k)^{\frac{1}{2}} & &\text{for } s = 1, 2, \cdots, q \\
&= r^{\frac{1}{2}} & &\text{for } s = q + 1, \cdots, v - 1 \\
a_{1s} &= (\phi_s/k)^{\frac{1}{2}} & &\text{for } s = 1, 2, \cdots, q \\
\tau_s &= \boldsymbol{\theta}'\mathbf{p}_s & &s = 1, 2, \cdots, v - 1.
\end{aligned}
\tag{2.7}
$$

It also follows from our assumptions that x_{0s}, $s = 1, 2, \cdots, v - 1$ and z_{0s}, $s = 1, 2, \cdots, e_0$ are all uncorrelated each having variance σ_0^2 and x_{1s}, $s = 1, 2, \cdots, q$ and z_{1s}, $s = 1, 2, \cdots, e_1$ are all uncorrelated each with variance σ_1^2.

We note that $\tau_1, \tau_2, \cdots, \tau_{v-1}$ are linearly independent parametric contrasts. It is easily seen that Equations (2.2) are equivalent to $\tau_s = t_s$ where

$$t_s = x_{0s}/a_{0s} \qquad s = 1, 2, \cdots, v-1. \tag{2.8}$$

Equations (2.3) are equivalent to $\tau_s = \bar{t}_s(\rho)$ where

$$\begin{aligned} \bar{t}_s(\rho) &= (\rho a_{0s} x_{0s} + a_{1s} x_{1s})/(\rho a_{0s}^2 + a_{1s}^2) & s &= 1, 2, \cdots, q \\ &= x_{0s}/a_{0s} & s &= q+1, \cdots, v-1. \end{aligned} \tag{2.9}$$

It is easily verified that

$$\begin{aligned} V(t_s) &= \sigma_0^2/a_{0s}^2 & s &= 1, 2, \cdots, v-1 \\ V\{\bar{t}_s(\rho)\} &= \rho\sigma_0^2/(\rho a_{0s}^2 + a_{1s}^2) & s &= 1, 2, \cdots, q \\ &= \sigma_0^2/a_{0s}^2 & s &= q+1, \cdots, v-1. \end{aligned} \tag{2.10}$$

The procedure commonly adopted is to substitute for ρ in (2.9) some estimate ρ^* which is a function of observations. Such an estimate of τ_s will be denoted by $\bar{t}_s(\rho^*)$. In this case, the variance of $\bar{t}_s(\rho^*)$ will depend upon the estimate ρ^* used.

Consider a statistic P of the form

$$P = \frac{aS_1 + \sum_{s=1}^{q} b_s z_s^2}{S_0} + d \tag{2.11}$$

where $z_s = x_{0s} - a_{0s}x_{1s}/a_{1s}$, $s = 1, 2, \cdots, q$,

$$S_0 = \sum_{s=1}^{e_0} z_{0s}^2, \qquad S_1 = \sum_{s=1}^{e_1} z_{1s}^2 \tag{2.12}$$

and $a, b_1, \cdots, b_q, d$ are some constants. One may choose $a, b_1, \cdots, b_q$ and d suitably and define

$$\begin{aligned} \rho^* &= P & &\text{if } P \geqq 1 \\ &= 1 & &\text{otherwise.} \end{aligned} \tag{2.13}$$

It is easily seen that for $s = 1, 2, \cdots, q$

$$\bar{t}_s(\rho^*) - \bar{t}_s(\rho) = [c_s/a_{0s}(1 + \rho c_s)]w_s$$

where

$$\begin{aligned} c_s &= a_{0s}^2/a_{1s}^2 = (rk - \phi_s)/\phi_s \\ w_s &= [(\rho^* - \rho)/(1 + \rho^* c_s)]z_s. \end{aligned} \tag{2.14}$$

We note that $V(z_s) = \sigma_0^2 + c_s\sigma_1^2$, $s = 1, 2, \cdots, q$.

It is shown in Roy and Shah (1962) that if ρ^* is of the form (2.13), $\bar{t}_s(\rho^*)$ is unbiased for τ_s and its variance is given by

$$V\{\bar{t}_s(\rho^*)\} = V\{\bar{t}_s(\rho)\} + [c_s^2/a_{0s}^2(1 + \rho c_s)^2]E(w_s^2). \tag{2.15}$$

It is also shown in Roy and Shah (1962) that in this case, the combined estimators of τ_s and $\tau_{s'}(s \neq s')$ are uncorrelated. Now, any treatment contrast τ can be expressed as

$$\tau = m_1\tau_1 + m_2\tau_2 + \cdots + m_{v-1}\tau_{v-1} \tag{2.16}$$

where $m_1, m_2, \cdots, m_{v-1}$ are some constants. Let $\bar{t}(\rho^*) = \sum_{s=1}^{v-1} m_s\bar{t}_s(\rho^*)$ denote the combined estimator of τ when ρ^* is used as an estimate for ρ. If ρ^* is of the form (2.13), $V\{\bar{t}(\rho^*)\}$ is given by

$$V\{\bar{t}(\rho^*)\} = V\{\bar{t}(\rho)\} + \sum_{s=1}^{q} c_s^2 m_s^2 E(w_s^2)/a_{0s}^2(1 + \rho c_s)^2 \tag{2.17}$$

where $\bar{t}(\rho) = \sum_{s=1}^{v-1} m_s\bar{t}_s(\rho)$ denotes the combined estimator of τ when ρ is known.

3. Construction of combined estimators with uniformly smaller variance. The variance of a combined estimator of τ_s is given by (2.15). In this section we shall construct a suitable estimator ρ^* and evaluate this variance in terms of incomplete Beta functions.

Suppose the association matrix $\mathbf{NN}'$ has a latent root ϕ of multiplicity $p(p > 2)$. Without loss of generality we may say that the positive latent roots of $\mathbf{NN}'$ are $rk, \phi_1, \phi_2, \cdots, \phi_p, \phi_{p+1}, \cdots, \phi_q$ where, $\phi_s = \phi$ for $s \leqq p$ and $\phi_s \neq \phi$ for $s > p$. Denote the common value of $a_{01}, a_{02}, \cdots, a_{0p}$ by $\bar{a}_0$, of $a_{11}, a_{12}, \cdots, a_{1p}$ by $\bar{a}_1$ and of $c_1, c_2, \cdots, c_p$ by $\bar{c}$. Also let

$$\sum_{s=1}^{p} z_s^2 = Z. \tag{3.1}$$

We take ρ^* as defined in (2.13) where, to obtain P we shall put

$$\begin{aligned} a &= 0 \\ b_s &= e_0/\bar{c}p && \text{for } s = 1, 2, \cdots, p \\ &= 0 && \text{for } s = p+1, \cdots, q \\ d &= -1/\bar{c} \end{aligned} \tag{3.2}$$

in the defining Equation (2.11). This gives

$$\begin{aligned} \rho^* &= e_0Z/\bar{c}pS_0 - 1/\bar{c} && \text{if } S_0 \leqq KZ \\ &= 1 && \text{otherwise} \end{aligned} \tag{3.3}$$

where

$$K = e_0/p(1 + \bar{c}) \tag{3.4}$$

and $\bar{c}$ is the common value of $c_1, c_2, \cdots, c_p$.

It can be easily seen that this gives us

$$(3.5)\qquad \begin{aligned} w_s &= (z_s/\bar{c})\{1 - p(1 + \bar{c}\rho)S_0/e_0 Z\} && \text{if } S_0 \leqq KZ \\ &= (z_s/\bar{c})\{1 - p(1 + \bar{c}\rho)K/e_0\} && \text{otherwise} \end{aligned}$$

for $s = 1, 2, \cdots, p$.

Evidently, $w_1, w_2, \cdots, w_p$ are identically distributed and hence $E(w_1^2) = E(w_2^2) = \cdots, = E(w_p^2) = E(p^{-1}\sum_{s=1}^{p} w_s^2)$. Using (3.5) one gets

$$(3.6)\qquad \begin{aligned} \frac{1}{p}\sum_{s=1}^{p} w_s^2 &= \frac{1}{p\bar{c}^2} Z - \frac{2(1 + \bar{c}\rho)}{e_0\,\bar{c}^2} S_0 + \frac{p(1 + \bar{c}\rho)^2}{e_0^2\bar{c}^2} \cdot \frac{S_0^2}{Z} && \text{if } S_0 \leqq KZ \\ &= \frac{1}{p\bar{c}^2} Z - \frac{2(1 + \bar{c}\rho)}{e_0\,\bar{c}^2} KZ + \frac{p(1 + \bar{c}\rho)^2}{e_0^2\,\bar{c}^2} K^2 Z && \text{otherwise.} \end{aligned}$$

We shall now use the following lemma.

LEMMA 3.1. *Let S and Z be two independent random variables, S/σ_s^2 being a χ^2 with e_s* d.f. *and Z/σ_z^2 being a χ^2 with e_z* d.f. *Let $m \leqq e_z/2 + 1$ be a positive number and let $K > 0$ be a given constant. Consider a function $F(S, Z, m, K)$ defined by*

$$(3.7)\qquad \begin{aligned} F(S, Z, m, K) &= S^m Z^{1-m} && \textit{if } S \leqq KZ \\ &= K^m Z && \textit{otherwise.} \end{aligned}$$

The expectation of $F(S, Z, m, K)$ is given by

$$(3.8)\qquad \begin{aligned} E\{F(S, Z, m, K)\} &= \sigma_z^2 K^m e_z I_x(\tfrac{1}{2}e_z + 1, \tfrac{1}{2}e_s) \\ &+ \frac{(e_z + e_s)\sigma_z^2}{(\sigma_z^2/\sigma_s^2)^m} \cdot \frac{B(\tfrac{1}{2}e_s + m, \tfrac{1}{2}e_z - m + 1)}{B(\tfrac{1}{2}e_s, \tfrac{1}{2}e_z)} I_{1-x}(\tfrac{1}{2}e_s + m, \tfrac{1}{2}e_z - m + 1) \end{aligned}$$

where $x = \sigma_z^2/(\sigma_s^2 + K\sigma_z^2)$, $B(p, q)$ denotes the Beta function with arguments p and q and $I_x(p, q)$ denotes the corresponding incomplete Beta function.

PROOF. The joint distribution of S and Z is given by

$$A \cdot \exp\{-(S/2\sigma_s^2) - (Z/2\sigma_z^2)\} S^{(e_s/2)-1} Z^{(e_z/2)-1}\, dS\, dZ$$

where $1/A = \Gamma(e_s/2)\Gamma(e_z/2)(2\sigma_s^2)^{e_s/2}(2\sigma_z^2)^{e_z/2}$.

Consider a transformation from S, Z to U, V given by

$$S/Z = U, \qquad S/2\sigma_s^2 + Z/2\sigma_z^2 = V.$$

The Jacobian is given by

$$\partial(S, Z)/\partial(U, V) = V(U/2\sigma_s^2 + 1/2\sigma_z^2)^{-2}.$$

Hence the joint distribution of U and V is given by

$$A \cdot e^{-V} V^{(e_s+e_z)/2-1}\, dV\, U^{(e_s/2)-1}(U/2\sigma_s^2 + 1/2\sigma_z^2)^{(e_s+e_z)/2}\, dU.$$

It is easily seen that

$$\begin{aligned} F(S, Z, m, K) &= VU^m(U/2\sigma_s^2 + 1/2\sigma_z^2)^{-1} && \text{if } U \leqq K \\ &= VK^m(U/2\sigma_s^2 + 1/2\sigma_z^2)^{-1} && \text{otherwise.} \end{aligned}$$

Hence we have

$$E\{F(S, Z, m, K)\} = A\left\{\int_{V=0}^{\infty} e^{-V}\cdot V^{(e_s+e_z)/2}\, dV\right\}$$
$$\times\left\{\int_{U=0}^{K} \frac{U^{(e_s/2)+m-1}\, dU}{(U/2\sigma_s^2 + 1/2\sigma_z^2)^{(e_s+e_z+2)/2}} + \int_{U=K}^{\infty} \frac{K^m U^{(e_s/2)-1}\, dU}{(U/2\sigma_s^2 + 1/2\sigma_z^2)^{(e_s+e_z+2)/2}}\right\}.$$

The integral in V is $\Gamma(e_s + e_z + 2)/2)$. Using the substitution $f = \sigma_s^2/(\sigma_s^2 + \sigma_z^2 U)$ and integrating w.r.t. f the result is readily obtained.

In the lemma, we set $S = S_0$, $Z = Z$, $\sigma_s^2 = \sigma_0^2$, $\sigma_z^2 = \sigma_0^2(1 + \bar{c}\rho)$, $e_s = e_0$ and $e_z = p$. Using (3.6) and (3.7) we get

$$\frac{1}{p}\sum_{s=1}^{p} w_s^2 = \frac{1}{p\bar{c}^2} F(S_0, Z, 0, K) - \frac{2(1 + \bar{c}\rho)}{e_0\, \bar{c}^2} F(S_0, Z, 1, K) + \frac{p(1 + \bar{c}\rho)^2}{e_0^2\, \bar{c}^2} F(S_0, Z, 2, K). \tag{3.9}$$

An application of Lemma 3.1, gives us

$$E\left(\frac{1}{p}\sum_{s=1}^{p} w_s^2\right) = \frac{(1 + \bar{c}\rho)\sigma_0^2}{\bar{c}^2}\left\{1 + X(X - 2)I_x\left(\frac{p+2}{2}, \frac{e_0}{2}\right) - 2I_{1-x}\left(\frac{e_0+2}{2}, \frac{p}{2}\right) + \frac{p(e_0 + 2)}{e_0(p - 2)} I_{1-x}\left(\frac{e_0+4}{2}, \frac{p-2}{2}\right)\right\}, \tag{3.10}$$

where $X = (1 + \bar{c}\rho)/(1 + \bar{c})$.

It is easily seen that $x = p/(p + e_0 X)$.

The following expression for variance follows from (2.15) and (3.10):

$$V(\bar{t}_s(\rho^*)) = \frac{\sigma_0^2}{\bar{a}_0^2(1 + \bar{c}\rho)}\left\{1 + \bar{c}\rho + X(X - 2)I_x\left(\frac{p+2}{2}, \frac{e_0}{2}\right) - 2I_{1-x}\left(\frac{e_0+2}{2}, \frac{p}{2}\right) + \frac{p(e_0 + 2)}{e_0(p - 2)} I_{1-x}\left(\frac{e_0+4}{2}, \frac{p-2}{2}\right)\right\} \quad \text{for } s = 1, 2, \cdots, p. \tag{3.11}$$

We note that as $\rho \to \infty$, $X \to \infty$ and $x \to 0$. It is easy to prove that

$$\lim_{x\to 0}\{V(\bar{t}_s(\rho^*))\} = \sigma_0^2/\bar{a}_0^2 \qquad \text{for } s = 1, 2, \cdots, p. \tag{3.12}$$

It can also be shown that for $s = 1, 2, \cdots, p$,

$$\frac{dV(\bar{t}_s(\rho^*))}{dx} = -\frac{\sigma_0^2}{\bar{a}_0^2(1 + \bar{c})(1 - x)^2}\left\{\frac{p(1 - x)^2}{e_0 x^2} I_x\left(\frac{p+2}{2}, \frac{e_0}{2}\right) + \frac{2e_0}{p} I_{1-x}\left(\frac{e_0+2}{2}, \frac{p}{2}\right) - \frac{e_0+2}{p-2} I_{1-x}\left(\frac{e_0+4}{2}, \frac{p-2}{2}\right)\right\}. \tag{3.13}$$

We note that this is always negative if $2\, e_0(p - 2) \geqq p(e_0 + 2)$ or equivalently if $(p - 4)(e_0 - 2) \geqq 8$. On the other hand if $(p - 4)(e_0 - 2) < 8$, the derivative is positive for sufficiently small values of x. An examination of (3.12) and (3.13) leads to the following.

(1) If $(p - 4)(e_0 - 2) \geqq 8$, for $s = 1, 2, \cdots, p$, $V(\bar{t}_s(\rho^*)) < V(t_s)$ uniformly in ρ. This is a consequence of the fact that $V(\bar{t}_s(\rho^*))$ increases with ρ and reaches the limit $V(t_s) = \sigma_0^2/\bar{a}_0^2$ as $\rho \to \infty$.

(2) If $p > 2$ and $(p - 4)(e_0 - 2) < 8$, for $s = 1, 2, \cdots, p$, $V(\bar{t}_s(\rho^*) > V(t_s)$ when ρ is sufficiently large; i.e. the combined estimator considered does *not* have uniformly smaller variance as compared with the intra-block estimator. This follows from the fact that for sufficiently large values of ρ, $V(t_s(\rho^*))$ decreases with ρ. Thus the limit as $\rho \to \infty$ (which coincides with the variance of the intra-block estimator) is reached from above. It is shown in Shah (1964) that for values of ρ not exceeding 2, the combined estimator has smaller variance.

In view of (2.17), it is clear that the above analysis holds for any treatment contrast τ which is of the form $\tau = \sum_{s=1}^{p} 1_s \tau_s$ where $1_1, 1_2, \cdots 1_p$ are some constants. It is clear that such treatment contrasts form a vector space, call it V, with the following properties:

(i) rank $V = p$,

(ii) variance of the intra-block estimator of any normalised contrast in V is the same,

(iii) for any pair of mutually orthogonal treatment contrast of which at least one belongs to V, their intra-block estimators are uncorrelated.

We thus have the following theorem:

THEOREM 3.1. *Consider an incomplete block design for which the association matrix has a non-zero latent root (other than rk) of multiplicity p, let* $\mathbf{m}$ *be a latent vector associated with this root. Let ρ^* be the estimator of ρ constructed as in* (3.3) *based on this latent root. Let $\tau = \mathbf{m}'\boldsymbol{\theta}$, t its intra-block estimator, and $\bar{t}(\rho^*)$ the combined estimator using ρ^*. Then,*

$$V(\bar{t}(\rho^*)) < V(t) \text{ for all values of } \rho \tag{3.14}$$

provided that

$$(p - 4)(e_0 - 2) \geqq 8. \tag{3.15}$$

Further, if

$$p > 2 \text{ and } (p - 4)(e_0 - 2) < 8, \tag{3.16}$$

$$V(\bar{t}(\rho^*)) > V(t) \quad \text{for sufficiently large value of } \rho. \tag{3.17}$$

4. Uniformly better combined estimators for all treatment contrasts in a certain class of designs. A general procedure for constructing a combined estimator of a treatment contrast with variance uniformly smaller than that of the intra-block estimator was developed in Section 3. In this section and in the succeeding one we discuss applications of this procedure to some well-known designs.

A combined estimator of a treatment contrast will be said to be "uniformly better" if its variance is smaller than that of the intra-block estimator for all values of ρ. Further, any statement relating to combined estimators will apply only to those treatment contrasts on which inter-block information is available.

Let us denote by D_1, the class of incomplete block designs for which the association matrix has only one non-zero latent root (other than rk). We shall use Theorem 3.1 to construct a uniformly better combined estimator for *any* treatment contrast for any incomplete block design belonging to the class D_1.

For any design belonging to the class D_1, we shall denote by ϕ, the non-zero latent root of $\mathbf{NN}'$ (other than rk). Thus the multiplicity of ϕ is given by $q =$ (rank $\mathbf{NN}'$) $- 1$. As before, let $\tau_1, \tau_2, \cdots, \tau_q$ denote the canonical contrasts corresponding to ϕ. If $(q - 4)(e_0 - 2) \geqq 8$, we can apply Theorem 3.1 to obtain a uniformly better combined estimator for a treatment contrast τ of the form $\tau = \sum_{b=1}^{q} m_s\tau_s$. As is evident from (2.6) for $\tau_{q+1}, \tau_{q+2}, \cdots, \tau_{v-1}$, the $(v - 1 - q)$ canonical contrasts corresponding to the zero root of $\mathbf{NN}'$, no inter-block information is available. Hence, $\bar{t}_s(\rho^*) \equiv t_s$ for $s = q + 1, q + 2, \cdots, v - 1$. Now any treatment contrast τ is a linear combination of $\tau_1, \tau_2, \cdots, \tau_{v-1}$ and for ρ^* defined by (2.13), $\bar{t}_s(\rho^*)$ and $\bar{t}_{s'}(\rho^*)$ are uncorrelated for $s \neq s'$. It follows from Theorem 3.1 that for any treatment contrast τ which admits of inter-block information, $V(\bar{t}(\rho^*)) < V(t)$ for all values of ρ.

To compute ρ^* defined by (2.13), we note that for a design in the class D_1, $Z = \sum_{s=1}^{q} z_s^2$ may be expressed in the form

$$Z = \bar{c}(1 + \bar{c})\{(\mathbf{T}'\mathbf{T}/r - G^2/bk) - (2\mathbf{T}' - r\boldsymbol{\theta}^{*\prime})\boldsymbol{\theta}^*\} \tag{4.1}$$

where $\boldsymbol{\theta}^*$ is any solution of (2.2), the intra-block normal equations and as before, $\bar{c} = (rk/\phi) - 1$. Thus, ρ^* may be written down as

$$\begin{aligned} \rho^* &= \frac{\phi}{rk - \phi}\left(\frac{Z}{qs_0^2} - 1\right) && \text{if } \frac{Z}{s_0^2} > \frac{rkq}{\phi} \\ &= \quad 1 && \text{otherwise} \end{aligned} \tag{4.2}$$

where s_0^2 denotes the intra-block error mean square.

For a design in the class D_1, ϕ and q may be evaluated in the following manner. Since the association matrix is symmetric, the sum of the latent roots is equal to the sum of the diagonal elements, and the sum of squares of the latent roots is equal to the sum of squares of all the elements. Since only non-zero latent roots are rk and ϕ with multiplicities 1 and q respectively, we have

$$q\phi = r(v - k) \tag{4.3}$$

and

$$q\phi^2 = vr^2 + \sum_{j \neq j'} \lambda_{jj'}^2 - r^2k^2 \tag{4.4}$$

where $\lambda_{jj'}$ denotes the number of blocks containing both the treatments j and j'. We thus have the following theorem.

THEOREM 4.1. *Consider an incomplete block design belonging to the class* D_1. *When* ρ^* *as given by* (4.2) *is used, for* any *treatment contrast* τ

$$V(\bar{t}(\rho^*)) < V(t) \qquad \text{for all values of } \rho \tag{4.5}$$

provided that

$$(q - 4)(e_0 - 2) \geqq 8 \tag{4.6}$$

where q is obtained from equations (4.3) *and* (4.4). *Further, if* $q > 2$ *and* $(q - 4)(e_0 - 2) < 8$, (4.5) *does not hold for all values of* ρ.

Applications of this theorem to some well-known designs are given below.

(i) Applications to BIB designs and to LB designs: It is implied in Bose (1949) that for a BIB design the association matrix is of full rank and has only one latent root other than rk. Thus any BIB design belongs to the class D_1. Since $q = v - 1$, it follows that inter-block information is available for all treatment contrasts and hence when (4.6) holds we get uniformly better combined estimators for all treatment contrasts.

For all BIB designs with more than 5 treatments, the condition (4.6) holds. It may be noted that the estimator of ρ given by (4.2) differs from the customary one proposed by Yates (1940).

Linked block (LB) designs were obtained by Youden (1951) by dualising the BIB designs. It is shown by Roy and Laha (1956) that for a LB design the association matrix has a non-zero latent root (other than rk) of multiplicity $(b - 1)$. Since rank $\mathbf{NN}' \leqq b$, all other latent roots must be zero. Thus a LB design belongs to the class D_1.

Condition (4.6), which in this case amounts to $(b - 5)(e_0 - 2) \geqq 8$, holds for all LB designs with $b \geqq 6$.

It may be checked that R, the untruncated form of the customary estimator of ρ, can be expressed as

$$R = \frac{e_0\, k \left\{S_1 + \sum_{s=1}^{q} \phi_s z_s^2/rk\right\}}{v(r - 1)\, S_0} - \frac{v - k}{v(r - 1)}. \tag{4.7}$$

For a LB design $e_1 = b - 1 - q = 0$ and $\phi_1 = \phi_2 = \cdots = \phi_q = r(v - k)/(b - 1)$. Consequently $\bar{c} = (rk - \phi)/\phi = v(r - 1)/(v - k)$. Substituting these in (4.3) and (4.7) it is readily seen that the customary estimator of ρ coincides with the one given (4.2). It is also easily seen that linked blocks are the only designs for which these two estimators coincide.

It follows from Theorem 4.1 that for a LB design the traditional combined estimators are uniformly better than the intra-block estimators if $b \geqq 6$, but not so if $b = 4$ or 5.

(ii) Applications to PBIB designs with two associate classes: Now we shall search for PBIB designs with two associate classes belonging to the class D_1. We shall adopt the standard definition and notation for these designs as given in Bose and Connor (1952).

Connor and Clatworthy (1954) have shown that the association matrix of a

PBIB design with two associate classes has exactly two distinct latent roots other than rk. From the values of these roots given there it can be deduced that a necessary and sufficient condition for one of the roots to be zero (i.e. for the design to belong to the class D_1) is:

$$(4.8)\qquad (r - \lambda_1)(r - \lambda_2)/(\lambda_1 - \lambda_2) = p_{12}^1(r - \lambda_2) - p_{12}^2(r - \lambda_1).$$

Evidently if $b < v$, rank $\mathbf{NN}' < v$. Consequently zero is a latent root.

For any two-associate PBIB design in D_1, ϕ and q obtained from (4.3) and (4.4) turn out to be

$$(4.9)\qquad \begin{aligned} \phi &= \{v(r^2 + n_1\lambda_1^2 + n_2\lambda_2^2) - r^2k^2\}/r(v - k),\\ q &= r^2(v - k)^2/\{v(r^2 + n_1\lambda_1^2 + n_2\lambda_2^2) - r^2k^2\}. \end{aligned}$$

Two-associate PBIB designs have been classified by Bose, Clatworthy and Shrikhande (1954) as (1) group divisible: (a) singular, (b) semiregular, (c) regular, (2) triangular, (3) Latin square type, (4) simple and (5) cyclic.

(1) For a group divisible (GD) design Bose and Connor (1952) have shown that the two distinct latent roots of $\mathbf{NN}'$ are $(r - \lambda_1)$ and $(rk - v\lambda_2)$ with multiplicities $m(n - 1)$ and $m - 1$ respectively. Thus a GD design belongs to the class D_1 if either $r - \lambda_1 = 0$ i.e., if the design is singular or if $rk - v\lambda_2 = 0$ i.e. if the design is semi-regular. For a regular GD design $r > \lambda_1$ and $rk > v\lambda_2$ and hence no regular GD belongs to D_1. In a singular GD design uniformly better combined estimators are obtained if $(m - 5)(e_0 - 2) \geqq 8$. A corresponding condition in the case of a semi-regular GD is $(m(n - 1) - 4)(e_0 - 2) \geqq 8$.

For the next two types, use of (4.8) gives the following conditions on the parameters which ensure that they belong to the class D_1. In each case to apply Theorem 4.1, Condition (4.6) may be verified with the help of q given by Equations (4.9).

(2) In a triangular design defined by Bose and Shimamoto (1952), $v = \frac{1}{2}n(n - 1)$, $n_1 = 2(n - 2)$, $n_2 = \frac{1}{2}(n - 2)(n - 3)$, $p_{12}^1 = (n - 2)$. A necessary and sufficient condition for a triangular design to belong to the class D_1 is that $r = 2\lambda_1 - \lambda_2$ or $(n - 3)\lambda_2 - (n - 4)\lambda_1$.

(3) For a Latin square type design with i constraints $v = n^2$, $n_1 = i(n - 1)$, $n_2 = (n - 1)(n - i + 1)$ and $p_{11}^1 = i(i - 3) + n$. A Latin square type design belongs to the class D_1 if and only if $r = (i - n)(\lambda_1 - \lambda_2) + \lambda_2$ or $i(\lambda_1 - \lambda_2) + \lambda_2$. In particular, the simple lattice is a two associate PBIB design of the Latin square type with $i = 2$. Since $\lambda_1 = 1$ and $\lambda_2 = 0$ the above condition is satisfied. Condition (4.6) holds for $n > 3$. Similarly the triple lattice is a two associate PBIB design of the Latin square type with $i = 3$. Since $\lambda_1 = 1$ and $\lambda_2 = 0$ the design again belongs to the class D_1. Condition (4.6) in this case is satisfied for $n > 2$.

5. Some other applications. In this section we shall consider two applications of Theorem 3.1 where uniformly better combined estimators will be constructed only for treatment contrasts of a certain type.

(i) Inter- and intra-group balanced (IIGB) designs: IIGB designs were first introduced by Nair and Rao (1942). An IIGB design with equal number of replications for all treatments may be defined as follows. In an incomplete block design let there be m groups of treatments, there being v_i treatments in the ith group. Let each pair of treatments in the ith group occur in λ_{ii} blocks and let each pair of treatments one of which belongs to the ith group and the other to the jth group occur in λ_{jj} blocks. Such a design is called an IIGB design.

The association matrix $\mathbf{NN}'$ is given by

$$\mathbf{NN}' = \begin{bmatrix} (r-\lambda_{11})\mathbf{I}_{v_1} + \lambda_{11}\mathbf{E}_{v_1v_1} & \lambda_{12}\mathbf{E}_{v_1v_2} & \cdots & \lambda_{1m}\mathbf{E}_{v_1v_m} \\ \lambda_{12}\mathbf{E}_{v_2v_1} & (r-\lambda_{22})\mathbf{I}_{v_2} + \lambda_{22}\mathbf{E}_{v_2v_2} & \cdots & \lambda_{2m}\mathbf{E}_{v_2v_m} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{1m}\mathbf{E}_{v_mv_1} & \lambda_{2m}\mathbf{E}_{v_mv_2} & \cdots & (r-\lambda_{mm})\mathbf{I}_{v_m} + \lambda_{mn}\mathbf{E}_{v_mv_m} \end{bmatrix}.$$

It is easily seen that the vector of co-efficients $\mathbf{m}$ corresponding to any treatment contrast $\tau = \mathbf{m}'\boldsymbol{\theta}$ involving treatments from the ith group only is a latent vector of $\mathbf{NN}'$ corresponding to a latent root of multiplicity $(v_i - 1)$. By Theorem 3.1, we can construct uniformly better combined estimator for any intragroup contrast involving treatments from the ith group only provided that $(v_i - 5)(e_0 - 2) \geqq 8$ (we consider only those groups for which $r - \lambda_{ii} \neq 0$). An estimator of ρ as in (3.3) may be computed as follows. Let θ_1, $\theta_2 \cdots$, θ_{v_i} denote the treatment effect parameters for the treatments in the ith group. Let further $\theta_1^*, \cdots, \theta_{v_i}^*$ denote the solutions (corresponding to these treatments) of the intra-block equations and let $\theta_1', \theta_2', \cdots, \theta_{v_i}'$ the corresponding part of the solutions of the inter-block normal equations namely $\mathbf{C}_1\boldsymbol{\theta} = \mathbf{Q}_1$. We shall put $d_j = \theta_j^* - \theta_j'$. To obtain ρ^*, we substitute in (3.3)

$$\begin{aligned} p &= v_i - 1, \\ Z &= \frac{(rk - r + \lambda_{ii})}{k}\left\{\sum_{j=1}^{v_i} d_j^2 - \frac{(\sum d_j)^2}{v_i}\right\} \\ \bar{c} &= \frac{rk - r\lambda_{ii}}{r - \lambda_{ii}}. \end{aligned} \tag{5.2}$$

(ii) Regular GD designs: Bose and Connor (1952) have shown that a GD design is a special case of an IIGB design where, $\lambda_{ii} = \lambda_1$, $v_i = n$ and $\lambda_{ij} = \lambda_2$ for all $i, j = 1, 2, \cdots, m$ $(i \neq j)$. The association matrix of a GD design is obtained by substituting these in the right hand side of (5.1).

It is easy to check that the vector of co-efficients $\mathbf{m}$ corresponding to any treatment contrast $\tau = \mathbf{m}'\boldsymbol{\theta}$ involving treatments all from the same group is a latent vector of $\mathbf{NN}'$ corresponding to the root $(r - \lambda_1)$. Thus the vector space of treatment contrasts associated with $(r - \lambda_1)$ consists of all intra-group contrasts and this has rank $m(n - 1)$. In this section we consider only regular GD designs so that $r - \lambda_1 \neq 0$.

If $(m(n - 1) - 4)(e_0 - 2) \geqq 8$, by Theorem 3.1, we can construct uniformly better combined estimator for any intra-group treatment contrast.

The following computational procedure may be adopted to obtain the estimate of ρ as defined by (3.3). Let θ_{ij} denote the treatment effect parameter for the j-th treatment in the i-th group. Let further θ^*_{ij} and θ'_{ij} $(i = 1, 2, \cdots, m, j = 1, 2, \cdots, n)$ denote the respective solutions of the intra-block and the inter-block normal equations. Put $d_{ij} = \theta^*_{ij} - \theta'_{ij}$. The estimate ρ^* is obtained by substituting in (3.3)

$$p = m(n-1)$$

$$Z = \frac{rk - r + \lambda_1}{k} \left\{ \sum_i \sum_j d_{ij}^2 - \frac{\sum_i (\sum_j d_{ij})^2}{n} \right\}$$

$$\bar{c} = \frac{rk - r + \lambda_1}{r - \lambda_1}.$$

When $(m - 5)(e_0 - 4) \geqq 8$, a similar procedure can be adopted to obtain uniformly better combined estimators of inter-group contrasts.

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