A NOTE ON CONVERGENCE OF SUB-MARTINGALES[1]

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A number of conditions for convergence of sub-martingales are known today. The earliest of these in Doob's theorem [2] which requires an upper bound on the expectations of x_n. Snell [4] then generalized Doob's result by conditioning the expectations. Chow [1] proved a different generalization based on random times. It was suggested by Loève that the two results could be combined to yield a further generalization. The theorem presented here is of this nature. The proof adheres very closely to Chow's approach.

LEMMA. *Let* $(x_n, n \geqq 1, \mathfrak{A}_n)$ *be a sub-martingale sequence. Then for all* k, n, m, $k \leqq n \leqq m$:

$$E(I_{(C_{kn}-D_{nm})}x_m \mid \mathfrak{A}_k) \geqq P(C_{kn} \mid \mathfrak{A}_k), \text{ a.s.}$$

where $C_{kn} = \{x_i \geqq 1$ *for some* i *such that* $k \leqq i < n\}$ *and* $D_{kn} = \{x_i \leqq 0$ *for some* i *such that* $k \leqq i < n\}$.

PROOF. Define t_1 and t_2 by

$$\begin{aligned} t_1 &= \inf\{i : x_i \geqq 1\} && \text{on } C_{kn} \\ &= n && \text{on } C_{kn}^c, \\ t_2 &= \inf\{i : x_i \leqq 0\} && \text{on } D_{nm}C_{kn} \\ &= n && \text{on } C_{kn}^c \\ &= m && \text{on } C_{kn} - D_{nm}C_{kn}. \end{aligned}$$

Then t_1 and t_2 are $(\mathfrak{A}_n)$-times and $m \geqq t_2 \geqq n \geqq t \geqq k$, hence $E(x_{t_2} \mid \mathfrak{A}_{t_1}) \geqq E(x_{t_1} \mid \mathfrak{A}_{t_1})$ or $E(x_{t_2} \mid \mathfrak{A}_k) = E(E(x_{t_2} \mid \mathfrak{A}_{t_1}) \mid \mathfrak{A}_k) \geqq E(x_{t_1} \mid \mathfrak{A}_k)$, since $\mathfrak{A}_k \subset \mathfrak{A}_{t_1}$. Also $I_{D_{nm}}x_{t_2} \leqq 0$, therefore

$$\begin{aligned} E(I_{(C_{kn}-D_{nm})C_{kn}}x_m \mid \mathfrak{A}_k) &= E(I_{D_{kn}}x_{t_2} \mid \mathfrak{A}_k) - E(I_{D_{nm}C_{kn}}x_{t_2} \mid \mathfrak{A}_k) \\ &\geqq E(I_{C_{kn}}x_{t_1} \mid \mathfrak{A}_k) \geqq E(I_{C_{kn}} \mid \mathfrak{A}_k) = P(C_{kn} \mid \mathfrak{A}_k). \end{aligned}$$

DEFINITION. A random variable is called an $(\mathfrak{A}_n)$-time (martingale time, stopping time) iff $\{t = n\} \varepsilon \mathfrak{A}_n$ (Loève [3], p. 530).

THEOREM. *Let* $[x_n, \mathfrak{A}_n, n \geqq 1)$ *be a sub-martingale sequence. Let* F *be a measurable set such that for each* $(\mathfrak{A}_n)$*-time* t *there exists* $\{n_j\}_{j=1}^{\infty}$ *such that*

$$\lim_{n_j \to \infty} P(\{E(I_{[n \leqq t < \infty]}x_t^+ \mid \mathfrak{A}_{n_j}) = \infty\}F) = 0.$$

Then $\lim x_n$ *exists* a.e. *on* F.

PROOF. It will be shown that if $\lim x_n$ does not exist there exists an $(\mathfrak{A}_n)$-time

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t such that $P(\{E(I_{[n \leqq t < \infty]} x_t^+ \mid \mathfrak{A}_n) = \infty\} \cap F) > \alpha > 0$ for some α and for all n. Let $V = \{\limsup x_n \geqq 1 > 0 \geqq \liminf x_n\}$. If $\lim x_n$ does not exist on F we have $P(\{\limsup x_n \geqq b > a \geqq \liminf x_n\}F) > 0$ for some a, b. Since it is always possible to normalize the x_n without affecting their sub-martingale property, we may assume without loss of generality that if $\{x_n\}$ does not converge on F, $P(VF) > 0$. Also let ϵ_i be such that $P\{P(VF \mid \mathfrak{A}_i) \geqq \epsilon_i I_{VF}\} \geqq 1 - \delta$.

Since $P(VF \mid \mathfrak{A}_i) \to I_{VF}$ a.s., we may assume $\inf \epsilon_i = \epsilon > 0$. We may also require $\epsilon < P(VF)$.

We define n_i, m_i and A_i, B_i so that the following relations hold with C_{kn} and D_{nm} as defined above:

$$(1) \quad m_0 = 1, \quad B_0 = \Omega, \quad A_i = B_{i-1}C_{m_{i-1}n_i}, \quad B_i = A_i D_{n_i m_i}, \quad \mathfrak{A}_0 = \{\Omega, \phi\},$$

$$(2) \quad P\{P((VFB_i - VFB_iC_{m_in_{i+1}}) \mid \mathfrak{A}_l) > \epsilon/2^{i+3}\} < \delta/2^i, \quad l = 0, 1, 2, \cdots, m_i,$$

$$(3) \quad \begin{aligned} &P\{P((VFA_{i+1} - VFA_{i+1}D_{n_{i+1}m_{i+1}}) \mid \mathfrak{A}_l) > \epsilon/2^{i+3}\} < \delta/2^i, \\ &\qquad\qquad l = 0, 1, 2, \cdots, m_i, \end{aligned}$$

where n_{i+1} and m_{i+1} exist since $\bigcup_{p=m_i}^{\infty} C_{m_ip} \supset V$ and $\bigcup_{p=n_i}^{\infty} D_{n_ip} \supset V$ for all m_i, n_i, and the requirement that this be true for all $l \leqq m_i$ involves only a finite minimization. Let

$$J_l = \bigcap_{(i:m_i \geqq l)} \{P((VFB_i - VFB_iC_{m_in_{i+1}}) \mid \mathfrak{A}_l) \leqq \epsilon/2^{i+3}\},$$

$$H_l = \bigcap_{(i:m_i \geqq l)} \{P((VFA_{i+1} - VFA_{i+1}D_{n_{i+1}m_{i+1}}) \mid \mathfrak{A}_l) \leqq \epsilon/2^{i+3}\},$$

$$G_l = J_lH_l\{P(VF \mid \mathfrak{A}_l) \geqq \epsilon I_{VF}\}.$$

Then $P(G_l) \geqq 1 - \delta - 2\sum_{i=0}^{\infty} \delta/2^i = 1 - 5\delta$. Thus, noting that $B_j \varepsilon \mathfrak{A}_{m_j}$ we have for all $k > j$:

$$\begin{aligned} P(A_k \mid \mathfrak{A}_{m_j}) &\geqq P(VFB_j \mid \mathfrak{A}_{m_j}) - P((VFB_j - VFA_{j+1}) \mid \mathfrak{A}_{m_j}) \\ &\quad - \cdots - P((VFB_{k-1} - VFA_k) \mid \mathfrak{A}_{m_j}) \geqq I_{VFB_j}\epsilon - \epsilon/2^{j+3} \\ &\quad - \epsilon/2^{j+3} - \cdots - 2\epsilon/2^{k+2} \geqq \epsilon/2 \end{aligned}$$

on $G_{m_j}B_jVF$. Also

$$\begin{aligned} P(VFB_j) = P(VF) - P(VF - VFA_1) - \cdots - P(VFA_j - VFB_j) \\ \geqq P(VF) - \epsilon/2 \geqq P(VF)/2. \end{aligned}$$

δ may be selected such that $5\delta < P(VF)/4$ and hence $P(G_lVFB_j) > P(VF)/4$. Applying the lemma we obtain for $k > j$:

$$\begin{aligned} E(I_{(B_{k-1} - B_k)} x_{m_k}^+ \mid \mathfrak{A}_{m_j}) &\geqq E(I_{(A_k - B_k)} x_{m_k}^+ \mid \mathfrak{A}_{m_j}) \\ &= E(I_{B_{k-1}} E(I_{(C_{m_{k-1}n_k} - C_{m_{k-1}n_k} D_{n_km_k})} x_{m_k}^+ \mid \mathfrak{A}_{m_{k-1}}) \mid \mathfrak{A}_{m_j}) \\ &\geqq E(I_{B_{k-1}} P(C_{m_{k-1}n_k} \mid \mathfrak{A}_{m_{k-1}}) \mid \mathfrak{A}_{m_j}) = P(A_k \mid \mathfrak{A}_{m_j}), \end{aligned}$$

so let $t = m_k$ on $B_{k-1} - B_k$: $E(x_t^+ I_{[m_j \leqq t < \infty]} \mid \mathfrak{A}_{m_j}) \geqq \sum_{i \geqq j} P(A_i \mid \mathfrak{A}_{m_j}) \geqq \sum \epsilon/2 = \infty$ on $B_j G_{m_j} FV$. Also:

$$\{E(I_{[l \leqq t < \infty]} x_t^+ \mid \mathfrak{A}_l) = \infty\} \supset \{E(I_{[m_j \leqq t < \infty]} x_t^+ \mid \mathfrak{A}_{m_j}) = \infty\} \quad \text{for } l < m_j.$$

So $P(\{E(I_{[l \leqq t < \infty]} x_t^+ \mid \mathfrak{A}_l) = \infty\} F) \geqq P(\{E(I_{[m_j \leqq t < \infty]} x_t^+ \mid \mathfrak{A}_{m_j}) = \infty\} F) \geqq P(B_j G_{m_j} FV) \geqq P(VF)/4$ for all l.

REMARK. It may be noted that since the sets $\{E(I_{[n \leqq t < \infty]} x_t^+ \mid \mathfrak{A}_n) = \infty\} \cap F$ are decreasing, the condition in the theorem may be replaced by

$$\inf_n [E(I_{[n \leqq t < \infty)} x_t^+ \mid \mathfrak{A}_n)] I_F < \infty$$

a.s. It is a simple matter to check now that this theorem implies both Snell's and Chow's.

COROLLARY 1. (Chow). $E(x_t^+ I_{[t < \infty]}) < \infty \Rightarrow \lim x_n$ *exists* a.s.

This is immediate from the statement of the theorem, since $(x_n, \mathfrak{A}_n, n \geqq 0)$ form a sub-martingale with $\mathfrak{A}_0 = \{\phi, \Omega\}$.

COROLLARY 2. (Snell). $\lim x_n$ *exists* a.e. *and is finite on the set*

$$\{\lim_k \sup_n E(x_n^+ \mid \mathfrak{A}_k) < \infty\} = F.$$

PROOF. Let $F_k = \{\sup_n E(x_n^+ \mid \mathfrak{A}_k) < \infty\}$. Then $F = \bigcup_{k=1}^\infty F_k$, and it is enough to show convergence on F_k for all k. But for $n \leqq m$: $E(I_{[k \leqq t \leqq n]} x_{t \wedge n}^+ \mid \mathfrak{A}_n) \leqq E(x_m^+ I_{[k \leqq t < n]} \mid \mathfrak{A}_n)$ since $t \wedge n$ and m are $(\mathfrak{A}_n)$-times or taking the conditional expectation of both sides with respect to $\mathfrak{A}_k$: $E(I_{[k \leqq t \leqq n]} x_{t \wedge n}^+ \mid \mathfrak{A}_k) \leqq E(x_m^+ \mid \mathfrak{A}_k) \leqq \sup E(x_m^+ \mid \mathfrak{A}_k)$, so

$$E(I_{[k \leqq t < \infty]} x_t^+ \mid \mathfrak{A}_k) \leqq \sup E(I_{[k \leqq t \leqq n]} x_{t \wedge n}^+ \mid \mathfrak{A}_k) \leqq \sup_m E(x_m^+ \mid \mathfrak{A}_k),$$

and existence of the limit on F_k is proved by the hypothesis and the theorem. Finiteness is easily checked using Snell's proof.

Both Snell's and Chow's theorems are strictly contained in the theorem stated above as is shown in the following

EXAMPLE. Let x_n be defined as follows:

$$\begin{aligned} x_n &= 2 && \text{on } (\tfrac{1}{2}, 1] \\ &= 4 && \text{on } (\tfrac{1}{4}, \tfrac{1}{2}] \\ &= \vdots && \\ &= 2^n && \text{on } [-1, \tfrac{1}{2}^n]. \end{aligned}$$

$\mathfrak{A}_n = \mathfrak{F}([-1, 0], (\frac{1}{2}^n, \frac{1}{2}^{n-1}], \cdots, (\frac{1}{2}, 1])$ with uniform distribution on $[-1, 1]$. $x_n \geqq x_{n-1}$ hence $\{x_n\}$ forms a sub-martingale. Also $\lim x_n = \infty$ on $[-1, 0]$. Therefore Snell does not hold on $[-1, 0]$ and if t is defined as $t = n + 1$ on $(\frac{1}{2}^n, \frac{1}{2}^{n-1}]$, $t = \infty$ on $[-1, 0]$,

$$E(x_t I_{(t < \infty)}) = \sum_{n=1}^\infty \int_{\{t=n\}} x_n^+ \, dP = \sum_{n=1}^\infty \frac{1}{2} = \infty.$$

We may conclude that Chow's condition is not satisfied as well.
But

$$\int_{(\frac{1}{2}^k,1]} E(x_t^+ I_{[t<\infty]} \mid \mathcal{A}_k)\, dP = \int_{(\frac{1}{2}^k,1]\,[t<\infty]} x_t^+ \, dP \leqq k/2 < \infty.$$

Also on $[-1, 0]$ either $t = k$, in which case

$$\int_{[-1,0]} E(x_t^+ I_{[t<\infty]} \mid \mathcal{A}_1) = \int_{[-1,0]} x_k^+ \, dP < \infty$$

or $t = \infty$ and

$$\int_{[-1,0]} E(x_t^+ I_{[t<\infty]} \mid \mathcal{A}_1)\, dP = 0.$$

Thus, let $F_n = (0, \frac{1}{2}^n]^c$. Then $E(x_t^+ I_{[t<\infty]} \mid \mathcal{A}_n) < \infty$ a.e. on F_n and $UF_n = \Omega$, so the conditions of the theorem are satisfied.

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REFERENCES

[1] Chow, Y. S. (1963). Convergence theorem of martingales. *Z. Wahrscheinlichkeitstheorie.* **1** 340–346.

[2] Doob, J. L. (1953). *Stochastic Processes*. Wiley, New York.

[3] Loève, M. (1963). *Probability Theory* (3rd ed.). Princeton, New Jersey.

[4] Snell, J. L. (1952). Applications of martingale system theorems. *Trans. Amer. Math. Soc.* **73** 293–312.