VARIATIONS OF THE NON-CENTRAL t AND BETA DISTRIBUTIONS[1]

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1. Introduction. Many functions of independent normally distributed random variables have become classical; for example, Student's t, the beta, chi-square, and F distributions [1]. These classical distributions are related to zero mean normal variables. Some extensions to the case when some of the component variables are non-central such as the non-central F [12] and non-central beta [3] have also become widely known. Much of the work on the non-central variates involves expressing their distribution functions as certain infinite sums, and tabulating these functions, etc. [4], [8], [13]. In many engineering applications it is necessary to obtain simple (in terms of possible instrumentation) expressions which are good approximations to the mode, moments, etc. This paper is directed in this vein.

Two functions of random variables where all the component functions are non-central will be considered. With X normal $(\xi, 1)$ and Y an independent non-central chi variable with N degrees of freedom we let

$$U_1 = X/Y \tag{1.1}$$

and

$$U_2 = X/(X^2 + Y^2)^{\frac{1}{2}} = U_1/(1 + U_1^2)^{\frac{1}{2}}. \tag{1.2}$$

If Y were central then U_2^2 would be a certain non-central beta and $N^{\frac{1}{2}}U_1$ would be a non-central t with non-centrality ξ and N degrees of freedom. In addition to statistical applications these variables enjoy certain engineering applications in the treatment of random vectors in $N + 1$ dimensional spaces [5], [11]. In this instance U_2 is a direction cosine of such a vector. Another engineering application occurs when we consider a function of time that is non-linear in a parameter α and embedded in Gaussian noise. The posterior distribution of statistics and optimum estimates of such a parameter can often be described or approximated through considerations of variables like U_1 and U_2.

In general, neither the frequency nor the moments of U_1 and U_2 can be expressed except in terms of an integral or infinite series. Therefore, approximations to the mean, variance and other moments are obtained in terms of elementary or well tabulated functions. Asympototic expressions for the "tails" of the distribution are also obtained. Maximum likelihood estimates of one-sample cases are of interest in engineering applications, hence approximations to the peak of the density functions are derived.

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In the case when Y has one degree of freedom, i.e. a non-central normal variable, the density functions can be reduced to expressions containing only error functions and elementary functions. Therefore, in the last section graphs of some of these density functions are given for this special case.

2. A generalization of the non-central *t*. The frequency function of Y, $g(y)$, is given in [6], p. 138. It is 0 for $y \leqq 0$, and is otherwise given by

$$g(y) = \kappa(y/\kappa)^{N/2} \exp[-(y^2 + \kappa^2)/2] I_{(N-2)/2}(y\kappa). \tag{2.1}$$

(I_n is the modified Bessel function of the first kind and nth order.) The random variable in this section is U_1 the quotient of a non-central normal and a non-central chi, hence a generalization of the non-central t. This variate has the following properties:

(i) The asymptotic expression for the probability density, $h_1(u)$, as $|u| \to \infty$ is

$$h_1(u) \sim K/|u|^{N+1} \tag{2.2}$$

where

$$K = (2/\pi)^{\frac{1}{2}} \left(\frac{1}{2}\right)^{N/2} \exp(-\kappa^2/2) \int_0^\infty \omega^N \exp[-(\omega - \xi)^2/2]\, d\omega. \tag{2.3}$$

(ii) The mode, denoted u_{10}, is given by

$$\begin{aligned} u_{10} &= (\xi/\kappa)\{1 - \tfrac{1}{2}/[1 + \xi^2/(\xi^2 + \kappa^2) + \kappa^2/(N + 3)]\} && \text{for } \kappa \gg 1.0 \\ &= \xi/(N + 1)^{\frac{1}{2}} && \text{for } \kappa \ll 1.0 \ll |\xi| \\ &= (\xi/2)\Gamma[(N + 2)/2]/\Gamma[(N + 3)/2] && \text{for } \kappa, |\xi| \ll 1.0. \end{aligned} \tag{2.4}$$

(iii) The expectation is (for $N > 1$)

$$\begin{aligned} E(U_1) &= (\xi/\kappa)[1 - (N - 3)/2\kappa^2 - (N - 3)(N - 5)/8\kappa^4 - \cdots] \\ & \qquad\qquad && \text{for } \kappa > 1 \\ &= (\xi/2^{\frac{1}{2}})\exp(-\kappa^2)A_N[1 + ((N - 1)\kappa^2/2N) + \cdots] && \text{for } \kappa < 1 \end{aligned} \tag{2.5}$$

where

$$A_N = \Gamma[(N - 1)/2]/\Gamma(N/2).$$

(iv) The variance is (for $N > 2$)

$$\begin{aligned} \mathrm{Var}(U_1) &= 1/\kappa^2[1 + (\xi^2/\kappa^2)] \\ &\quad - (1/\kappa^4)[(N - 4) + (N - 3)(\xi^2/\kappa^2)] + \cdots && \text{for } \kappa > 1 \\ &= \exp(-\kappa^2/2)\{[(1 + \xi^2)/(N - 2) - (\xi^2/2)A_N^2 \exp(-\kappa^2/2)] \\ &\quad + (\kappa^2/2)[(1 + \xi^2)/N - (\xi^2/2)A_N^2(N - 1)/N \\ &\quad \cdot \exp(-\kappa^2/2)] + \cdots\} && \text{for } \kappa < 1. \end{aligned} \tag{2.6}$$

To obtain the above results we first derive a general expression for h_1. The probability density of the quotient of two independent variables is given by $\int_0^\infty yg(y)f(uy)dy$, where f and g are the respective densities of the numerator and denominator. Using this and (2.1), we obtain for $h_1(u)$

$$h_1(u) = (1/2\pi)^{\frac{1}{2}}\exp(-\kappa^2/2)\kappa^{(2-N)/2}\int_0^\infty z^{(N+2)/2}\exp(-z^2/2) \cdot\exp[-(uz - \xi)^2/2]I_{(N-2)/2}(\kappa z)\, dz. \tag{2.7}$$

It can easily be shown that

$$|I_\alpha(x) - (x/2)^\alpha| \leqq 2\exp(x)(x/2)^{\alpha+2};$$

hence replacing the Bessel function $I_\alpha(x)$ by $(x/2)^\alpha$ in (2.7) and denoting the result by $h_1^*(u)$ we observe immediately that

$$|h_1(u) - h_1^*(u)| = O(1/|u|^{N+3}). \tag{2.8}$$

Changing the variable of integration to ω, $\omega = uz$, in the expression for $h_1^*(u)$ we obtain

$$h_1^*(u) = k(u)u^{-(N+1)} \tag{2.9}$$

where

$$k(u) = (2/\pi)^{\frac{1}{2}}2^{-N/2}\exp(-\kappa^2/2) \cdot\int_0^\infty \omega^N \exp(-\omega^2/2u^2)\exp[-(\omega - \xi)^2/2]\, d\omega. \tag{2.10}$$

Expanding $\exp(-\omega^2/2u^2)$ in a power series about zero and integrating term by term we obtain

$$\begin{aligned} k(u) &= (2/\pi)^{\frac{1}{2}}\left(\frac{1}{2}\right)^{N/2}\exp(-\kappa^2/2) \cdot\int_0^\infty \omega^N\exp\{-[(\omega - \xi)^2/2]\}\, d\omega + O(1/u^2) \\ &= K + O(1/u^2). \end{aligned} \tag{2.11}$$

Combining (2.8), (2.9), and (2.11) we obtain the desired result, namely (2.2).

When $\xi = \kappa = 0$ the expression for $h_1(u)$ reduces to

$$h_1(u) = (1/\pi)^{\frac{1}{2}}\{\Gamma[(N + 1)/2]/\Gamma(N/2)\}(1 + u^2)^{-(N+1)/2} \tag{2.12}$$

which is the well known Student's distribution.

If we replace u by $k\xi$ $(k > 0)$ in (2.7) it can be seen that $h_1(k\xi) > h_1(-k\xi)$ for $\xi \neq 0$ and $h_1(u) = h_1(-u)$ if and only if $\xi = 0$. That is, $h_1(u)$ is symmetrical about zero if $\xi = 0$ and in fact has its only maximum at $u = 0$. Furthermore, if $\xi \neq 0$ $h_1(u)$ has its maximum at a value of u which has the same sign as ξ. It can be shown that $h_1(u)$ is not symmetrical about any value of u if $\xi \neq 0$.

It is instructive to obtain approximate values of h_1 and u_{10}, the mode, when κ is large. Since the probability density function has a continuous derivative the mode occurs at a relative maximum where this derivative vanishes. In the case $\xi = 0$ it has only one relative maximum and we shall assume it has only one in general. We observe that in the integral in (2.7) an ϵ- neighborhood about the origin (ϵ independent of κ) can be deleted with a negligible effect on the integral. With this ϵ fixed we can use the asymptotic expansion for the Bessel function with good accuracy for κ sufficiently large. The resulting expression for h_1 is

$$(2.13)\quad h_1(u) \cong (1/2\pi)^{\frac{1}{2}}\kappa^{(N-2)/2} \int_0^\infty z^{(N+1)/2} \cdot \exp[-(z-\kappa)^2/2] \exp[-(uz-\xi)^2/2]\, dz.$$

Since κ is large we can approximate the integral resulting in

$$(2.14)\quad h_1(u) \cong \kappa^{\frac{3}{2}}(1+u^2)^{-(N+2)/2}[1+(u\xi/\kappa)]^{(N+1)/2}\exp[-(\xi-u\kappa)^2/2(1+u^2)].$$

We now use standard techniques to obtain the mode of the above distribution. Using a first approximation of ξ/κ we obtain a second approximation given by (2.4). It can be seen that for κ large the correction term is small and this second approximation is sufficient.

When κ is small, we obtain an approximation to h_1 by making the upper limit on the integral in (2.7) a finite value A. By picking A sufficiently large (this can be done independent of κ) the portion neglected is made arbitrarily small. With A fixed we replace for the Bessel function the first term of its power series. The error introduced here is arbitrarily small for sufficiently small κ. The resultant expression is

$$(2.15)\quad h_1(u) \cong \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\exp\{-\kappa^2/2 - [\xi^2/(1+u^2)]\}}{2^{N/2}\Gamma(N/2)(1+u^2)^{(N+\frac{1}{2})}} \cdot \int_0^\infty z^N \exp\left\{-\frac{1}{2}\left[z - \frac{\xi u}{(1+u^2)^{\frac{1}{2}}}\right]^2\right\} dz.$$

If $|\xi|$ is large compared to 1 we can approximate the integral resulting in

$$(2.16)\quad h_1(u) \cong \left[\frac{\xi u}{(1+u^2)}\right]^N \frac{2\exp\{-\frac{1}{2}[\kappa^2 - \xi^2/(1+u^2)]\}}{2^{N/2}\Gamma(N/2)(1+u^2)^{\frac{1}{2}}}.$$

Again using standard techniques we obtain the mode given in (2.4) for the above restrictions on κ and ξ.

Similarly if $|\xi|$ is small compared to 1 we obtain an approximation to u_{10} by using Newton's method on the derivative of (2.15) beginning at $u = 0$. The resulting expression is that given in (2.4).

The moments of the distribution, especially the expectation and the variance are of considerable interest. It can easily be shown that only the first $N - 1$ moments exist. Expression for the moments of h_1 cannot, in general, be reduced to elementary functions; therefore, in the cases when $|\xi|$ is large or small compared to 1, approximations in terms of elementary functions are given.

Since x and y are independent

$$E(U_1^n) = E[(x/y)^n] = E(x^n)E(y^{-n}). \tag{2.17}$$

$E(x^n)$ is well known and $E(y^{-n})$ will be denoted $M_{-n}(N, \kappa)$. (For a discussion of the moments of $g(y)$, including negative moments, see Reference 10.) Therefore,

$$E(U_1^n) = M_{-n}(N, \kappa)(1/\pi)^{\frac{1}{2}} \sum_{l=0}^{[n/2]} \binom{n}{2l} \xi^{n-2l}\Gamma\left(l + \frac{1}{2}\right), \tag{2.18}$$

where [ ] means the "greatest integer contained in." In particular, the first few moments are

$$\begin{aligned} E(U_1) &= \theta_1 = \xi M_{-1}(N, \kappa), \\ E(U_1^2) &= (\xi^2 + 1)M_{-2}(N, \kappa), \end{aligned} \tag{2.19}$$

and

$$E(U_1^3) = \xi(\xi^2 + 3)M_{-3}(N, \kappa).$$

The moments about the mean, θ_1, are obtained using the value of θ_1, given above;

$$\begin{aligned} E[(U_1 - \theta_1)^n] = \left(\frac{1}{\pi}\right)^{\frac{1}{2}} \sum_{l=0}^{n} (-1)^{n-l} M_{-1}^{n-l}(N, \kappa) \\ \cdot \left[\sum_{k=0}^{l/2} \binom{l}{2k} 2^k \xi^{n-2k} \Gamma\left(k + \frac{1}{2}\right)\right]. \end{aligned} \tag{2.20}$$

In particular the first few moments about the mean are

$$\begin{aligned} E(U_1 - \theta_1) &= 0, \\ E[(U_1 - \theta_1)^2] &= \mathrm{Var}(U_1) = (\xi^2 + 1)M_{-2}(N, \kappa) - \xi^2 M_{-1}^2(N, \kappa), \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} E[(U_1 - \theta_1)^3] = 2\xi^2 M_{-1}^3(N, \kappa) - 3\xi(\xi^2 + 1)M_{-1}(N, \kappa)M_{-2}(N, \kappa) \\ + \xi(\xi^2 + 3)M_{-3}(N, \kappa). \end{aligned}$$

The power series and asymptotic expansion, in κ, of $M_{-n}(N, \kappa)$ are (from (10))

$$M_{-n}(N, \kappa) = 2^{-n/2} \exp(-\kappa^2/2) \sum_{r=0}^{\infty} \frac{\Gamma[(2r + N - n)/2]}{r!\,\Gamma[(2r + N)/2]} (\kappa^2/2)^r$$

and

$$M_{-n}(N, \kappa) \sim (1/\kappa)^n \left[1 - \frac{n(N - n - 2)}{2\kappa^2} + \frac{n(n - 2)(N - n - 2)(N - n - 4)}{2!(2\kappa^2)^2} + \cdots\right],$$

respectively. Using these expressions in (2.19) and (2.21) we obtain the expressions for the expectation and variance given in (2.5) and (2.6).

3. A variation of the beta distribution. Let X and Y and U_2 be as defined in Section 1. The non-central Beta distribution is the quotient of a non-central chi-square and a chi-square variable. Our variate U_2 is the square root of such a variate where the numerator has only one degree of freedom and the denominator has been generalized to be non-central. U_2 is therefore called a variation of the non-central beta variables. It will be shown that this variate has the following properties:

(i) The asymptotic expression for the probability density, $h_2(u)$, as $|u| \to 1$ is

$$h_2(u) \sim (1 - u^2)^{(N-2)/2}(K/u^{N+1}), \tag{3.2}$$

where K is given in (2.3).

(ii) The mode of h_2, denoted u_{20}, is given by

$$\begin{aligned} u_{20} &= [\xi/(\xi + \kappa^2)^{\frac{1}{2}}]\{1 - (\tfrac{1}{2})[1 + (\xi^2/\kappa^2) + (\xi^2 + \kappa^2)/(N + 3)]^{-1}\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \kappa \gg 1 \\ &= \xi/(\xi^2 + N + 1)^{\frac{1}{2}} \qquad\qquad\qquad\qquad\qquad \text{for } \kappa \ll 1 \ll |\xi| \\ &= \xi\Gamma[(N + 2)/2]\{\xi^2\Gamma^2[(N + 2)/2] + 4\Gamma^2[(N + 3)/2]\}^{-1} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \kappa,\ |\xi| \ll 1.0. \end{aligned} \tag{3.3}$$

(iii) The expectation is

$$\begin{aligned} E(U_2) = {} & [\xi/(\xi^2 + \kappa^2 + N - 1)^{\frac{1}{2}}][1 - (\tfrac{1}{2}/(\xi^2 + \kappa^2 + N - 1))] \\ & + O(\kappa^4) + O(\xi^4) + O(\kappa)O(|\xi|^3) + O(\kappa^2)O(\xi^2). \end{aligned} \tag{3.4}$$

(iv) The variance is

$$\begin{aligned} \operatorname{Var}(U_2) = {} & (\kappa^2 + N - 1)/(\xi^2 + \kappa^2 + N - 1)^2 \\ & + O(\kappa^6) + O(\xi^2)O(\kappa^4) + O(\xi^4)O(\kappa^2). \end{aligned} \tag{3.5}$$

To obtain the above results we first derive a general expression for h_2. It can easily be shown that if X and Y are independent with probability densities f and g respectively that the probability density, $h_2(u)$, of U_2 is zero for $|u| \geqq 1$ is otherwise given by

$$h_2(u) = 1/(1 - u^2)^{\frac{3}{2}} \int_0^\infty yg(y)f[uy/(1 - u^2)^{\frac{1}{2}}]\, dy. \tag{3.6}$$

Hence in our case it is zero for $|u| \geqq 1$ and is otherwise

$$\begin{aligned} h_2(u) = {} & \frac{\kappa \exp(-\kappa^2/2)}{(2\pi)^{\frac{1}{2}}(1 + u^2)^{\frac{1}{2}}\kappa^{N/2}} \int_0^\infty y^{(N/2)+1} \\ & \exp\left(-\frac{y^2}{2}\right) \exp\left[-\left(\frac{1}{2}\right)\left(\frac{uy}{1 - u^2} - \xi^2\right)^2\right] dy. \end{aligned} \tag{3.7}$$

If we replace u by $k\xi$ $(k > 0)$ in the above, it may easily be shown that $h_2(k\xi) > h_2(-k\xi)$ for $0 < k\xi < 1$ and $h_2(u) = h_2(-u)$ for $\xi = 0$. That is, $h_2(u)$ is symmetrical about zero if $\xi = 0$ and in fact has its only maximum at $u = 0$. Furthermore, if $\xi \neq 0$, $h_2(u)$ has its maximum at a value of u which has the same sign as ξ. It may also be shown that $h_2(u)$ is not symmetrical about any value for u for $\xi \neq 0$.

It can easily be shown that the corresponding density functions of the variables U_1 and U_2 are related by

$$h_2(u) = [1/(1 - u^2)^{\frac{3}{2}}]h_1[u/(1 - u^2)^{\frac{1}{2}}] \qquad \text{for } |u| < 1. \tag{3.8}$$

From this expression and (2.2) we obtain the asymptotic expression for $h_2(u)$ when u goes to ± 1.0 given by (3.2). Similarly with (2.4) we obtain the expression for the value of u when $h_2(u)$ is a maximum given by (3.3).

Approximations to the mean and variance of this distribution are computed for the case ξ and κ are both large compared to 1. The mean is given by

$$E(U_2) = \int_0^\infty \int_{-\infty}^\infty [x/(x^2 + y^2)^{\frac{1}{2}}]f(x)g(y)\,dx\,dy, \tag{3.9}$$

where f and g are the same as in the previous section. The above integration is approximated by expanding $x/(x^2 + y^2)^{\frac{1}{2}}$ in a Taylor series with a remainder in x about ξ. Using the first four terms and performing the integration over x we obtain

$$E(U_2) = \int_0^\infty [\xi/(\xi^2 + y^2)^{\frac{1}{2}} - 3\xi y^2/2(\xi^2 + y^2)^{\frac{5}{2}} + 3R]g(y)\,dy. \tag{3.10}$$

The remainder term, R, is easily shown to be bounded by $1/2y^4$.

In order to perform the y-integration we expand $1/(\xi^2 + y^2)^{\frac{1}{2}}$ and $y^2/(\xi^2 + y^2)^{\frac{5}{2}}$ in Taylor series with a remainder about M_1 (the first moment of $g(y)$). After performing the integration we obtain

$$\begin{aligned} E(U_2) = {} & \frac{\xi}{(\xi^2 + M_1^2)^{\frac{1}{2}}} + \frac{(2M_1^2 - \xi^2)(M_2 - M_1^2) - 3M_1^2}{2(\xi^2 + M_1^2)^{\frac{5}{2}}} \\ & + 3R_2\xi(M_2 - M_1^2) + \xi R_1(M_3 - 3M_2M_1 + 2M_1^3) + \int_0^\infty 3Rg(y)\,dy. \end{aligned} \tag{3.11}$$

The remainder terms R_1 and R_2 are dominated in the following way: $R_1 \leqq 1/3\xi^4$, $R_2 \leqq 1/|\xi|^5$. Using the asymptotic expressions for the M_n's and the appropriate dominating terms for the R's we obtain (3.4).

To obtain the variance of h_2 we first compute the second moment about zero of h_2, $E(U_2^2)$ in the same way we obtained the expression for $E(U_2)$. The variance is obtained from these expressions and the relation $\operatorname{Var}(U_2) = E(U_2^2) - E^2(U_2)$, resulting in

$$\operatorname{Var}(U_2) = [M_1^2/(\xi^2 + M_1^2)^2][1 + O(\kappa^4) + O(\kappa^6/\xi^2)]. \tag{3.12}$$

Using the asymptotic expression for M_1 we obtain the desired expression given in (3.5).

4. Reduced forms for two dimensions. It is instructive to consider the two variables in the previous sections in the case when N is one. In this case the density functions can be reduced to error functions and hence easily computed and plotted. The approximations for the peak of these functions can then be checked and a general idea of the shape of these families of density functions obtained.

In this reduced form the cumulative distribution functions of these variables are closely related to Ruben's W function [14], Owen's T [9] and Nicholson's V [7]. These are all concerned with the probabilities of certain infinite sectors under a circular normal distribution: Ruben's W is the probability content of a sector bounded by an arbitrary line through the center of the distribution and a ray drawn from a point C_0 on the x-axis at an angle θ. The probability under this area is denoted $W(c_0, \theta)$ where c_0 is the distance between C_0 and the center of the distribution. Writing the cumulative distribution function of U_2, $H_2(U)$, more explicitly as $H_2(U/\xi, \kappa)$, then for $\xi \leqq 0$,

$$
\begin{aligned}
1 - H_2(u/\xi, \kappa) &= W[(\kappa^2 + \xi^2)^{\frac{1}{2}}, \cos^{-1}u - \tan^{-1}(-\kappa/\xi)] \\
&\quad + W[(\kappa^2 + \xi^2)^{\frac{1}{2}}, \cos^{-1}u + \tan^{-1}(-\kappa/\xi)] \\
&\qquad\qquad \text{for } \cos^{-1}u \geqq \tan^{-1}(-\kappa/\xi) \\
&= W[(\kappa^2 + \xi^2)^{\frac{1}{2}}, \cos^{-1}u + \tan^{-1}(-\kappa/\xi)] \\
&\quad - W[(\kappa^2 + \xi^2)^{\frac{1}{2}}, \tan^{-1}(-\kappa/\xi) - \cos^{-1}u] \\
&\qquad\qquad \text{for } \cos^{-1}u \leqq \tan^{-1}(-\kappa/\xi)
\end{aligned}
\tag{4.1}
$$

(for each u in the range $(-1, 1)$, $\cos^{-1}u$ is to be interpreted as the angle, θ, in the range $(0, \pi)$ satisfying $\cos\theta = U$). In particular, $1 - H_2(U/-c_0, 0) = 2W(c_0, \cos^{-1}u)$, where $c_0 \geqq 0$ by definition, (or, equivalently, $H_2(\cos\theta/-c_0, 0) = 1 - 2W(c_0, \theta)$). H_1 enjoys a similar relationship since H_1 and H_2 are related by (for $|u| \leqq 1$),

$$H_2(u) = H_1(u/(1 - u^2)^{\frac{1}{2}}. \tag{4.2}$$

The relationship between Owen's T, Nicholson's V and Ruben's W are discussed in [14] and hence H_1 and H_2 can be related to these. The resulting expressions are more cumbersome than the above and hence are omitted.

Consider the random variable U_1 of the form $U_1 = X/|Y|$, where X and Y are independent Gaussian random variables with means ξ and κ respectively and unit variances. The probability density of U_1 is obtained from (2.7), resulting in

$$
\begin{aligned}
h_1(u) = (1/2\pi) \int_0^\infty z \left\{ \exp\left[-\left(\frac{1}{2}\right)(z - \kappa)^2\right] \right. \\
\left. + \exp\left[-\left(\frac{1}{2}\right)(z + \kappa)^2\right] \right\} \exp\left[-\left(\frac{1}{2}\right)(uz - \xi)^2\right] dz.
\end{aligned}
\tag{4.3}
$$

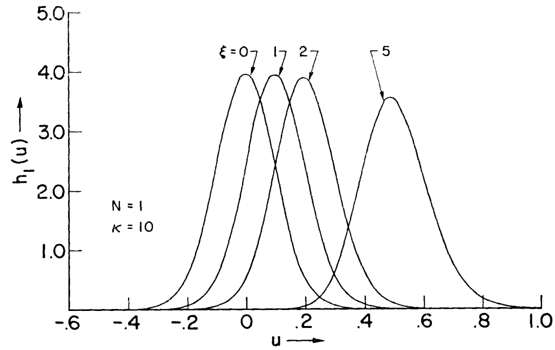

FIG. 1. Probability density function of the generalized non-central t with one degree of freedom.

It can easily be seen that the expectation does not exist. The asymptotic expression and the mode of $h_1(u)$ are obtained by setting N equal to 1 in (2.2), (2.3) and (2.4).

In order to compute $h_1(u)$ for various values of ξ and κ we reduce (4.3) to an expression in terms of the error function. That is

$$\begin{aligned} h_1(u) = {} & \{\exp[-(\tfrac{1}{2})(\xi^2 + \kappa^2)]/\pi(1 + u^2)\} \\ & \cdot\{1 + (\pi/8)^{\frac{1}{2}}[V_1(1 + 2 \operatorname{erf} V_1)\exp(V_1^2/2) \\ & + V_2(1 + 2 \operatorname{erf} V_2)\exp(V_2^2/2)]\}, \end{aligned} \tag{4.4}$$

where

$$V_1 = (u\xi + \kappa)/(1 + u^2)^{\frac{1}{2}}, \qquad V_2 = (u\xi - \kappa)/(1 + u^2)^{\frac{1}{2}}.$$

The probability density $h_1(u)$ can now be computed and typical results are shown in Figure 1.

Consider now the random variable of the form $U_2 = X/(X^2 + Y^2)^{\frac{1}{2}}$ where as before X and Y are independent Gaussian variables. The probability density is obtained as in Section 3 and is zero for $|u| \geqq 1$ and is otherwise given by

$$\begin{aligned} h_2(u) = {} & 1/2\pi(1 - u^2)^{\frac{3}{2}} \int_0^\infty z\,\{\exp[-(z - \kappa)^2/2] + \exp[-(z + \kappa)^2/2]\} \\ & \cdot\exp\{-(\tfrac{1}{2})\,[uz/(1 - u^2)^{\frac{1}{2}} - \xi]^2\}\,dz. \end{aligned} \tag{4.5}$$

The asymptotic expression, the mode, the expectation and the variance are obtained by setting N equal to one in (3.2), (3.3), (3.4) and (3.5).

$h_2(u)$ can also be expressed in terms of error functions, the expression being zero for $|u| \geqq 1$ and otherwise

$$\begin{aligned} h_2(u) = {} & (1/\pi)(1 - u^2)^{-\frac{1}{2}}\exp[-(\tfrac{1}{2})(\xi^2 + \kappa^2)] \\ & \cdot\{1 + (\pi/8)^{\frac{1}{2}}[B_1(1 + 2 \operatorname{erf} B_1)\exp(B_1^2/2) \\ & + B_2(1 + 2 \operatorname{erf} B_2)\exp(B_2^2/2)]\}, \end{aligned} \tag{4.6}$$

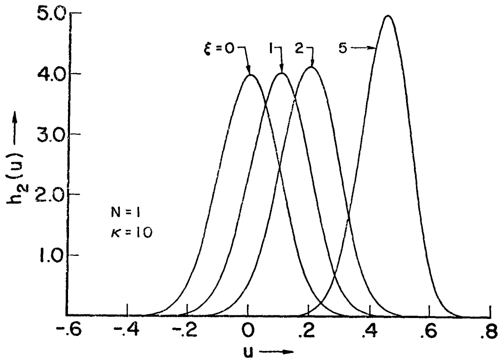

FIG. 2. Probability density function of a variation of the non-central β with one degree of freedom.

where

$$B_1 = \xi u + (1 - u^2)^{\frac{1}{2}}\kappa, \qquad B_2 = \xi u - (1 - u^2)^{\frac{1}{2}}\kappa.$$

The probability density was computed for various values of ξ and κ using (4.6) and the results are shown in Figure 2.

The above curves can be taken as an indication of the form these density functions will have for values of N other than 1.

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