SOME RESULTS RELATING MOMENT GENERATING FUNCTIONS AND CONVERGENCE RATES IN THE LAW OF LARGE NUMBERS

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1. Introduction and summary. Let X_N for $N = 1, 2, \cdots$ be an independent sequence of random variables with finite first absolute moments; let $a_N = \{a_{N,k} : k = 1, 2, \cdots\}$ for $N = 1, 2, \cdots$; let $A_N = 1/N \sum_{k=1}^{N} (X_k - EX_k)$; and let $S_N = \sum_{k=1}^{\infty} a_{N,k}(X_k - EX_k)$. Early work in probability dealt with the convergence (almost everywhere and in probability) to zero of the sequence A_N. More recent work has dealt with the convergence to zero of sequences of the form S_N under various assumptions on the coefficients $a_{N,k}$ and the distributions of the X_N's. In most cases the assumptions made about the X_N's have been not much stronger or weaker than the assumption of a finite upper bound on their γth absolute moments for some $\gamma \geqq 1$.

The classical result giving exponential convergence rates in the law of large numbers was established by Cramér [6] (see also [4]) and states that if the X_N's are identically distributed, and if their common moment generating function is finite in some interval about the origin, then for each $\epsilon > 0$ there exists $0 \leqq \rho < 1$ such that $P\{|A_N| \geqq \epsilon\} \leqq 2\rho^N$. Baum, Katz, and Read [3] investigated this exponential convergence further. However, their investigation was restricted to sequences of the form A_N. Koopmans [17] dealt with averages of the form $1/N \sum_{k=1}^{N} \sum_{j=-\infty}^{\infty} a_j X_{k-j}$. In [11] the exponential rate was obtained for sequences S_N provided $\sum_k |a_{N,k}| \leqq M < \infty$ for all N and $\max_k |a_{N,k}| \leqq O(1/\mathrm{N})$. A corresponding result was obtained for continuous time stochastic processes in [12]. More recently Chow [5, Section 2] obtained similar results under stronger assumptions on the moment generating functions involved. The results obtained here generalize and unify the results of [11], [12], and [5, Section 2].

The results are stated in Section 2 and proved in Section 3. Corollaries and details of the relationships between these results and previous results are contained in Section 4.

2. Statement of the main results. Let $a = \{a_k : k = 0, \pm 1, \cdots\}$ be a sequence of real numbers. Let p and q be numbers in $[1, \infty]$ satisfying $1/p + 1/q = 1$. Define $\|a\|_\infty = \sup_k \{|a_k|\}$ and for $p \,\varepsilon\, [1, \infty)$ define $\|a\|_p = [\sum_k |a_k|^p]^{1/p}$.

THEOREM A. *Let $\{X_k : k = 0, \pm 1, \cdots\}$ be a sequence of independent real valued random variables with finite first absolute moments. Suppose*

(1) *there exist positive constants M, γ, and $1 \leqq p \leqq 2$ such that for $0 < x < \infty$ and $k = 0, \pm 1, \cdots$*

$$P\{|X_k - EX_k| \geqq x\} \leqq \int_x^\infty Me^{-\gamma t^p}\, dt.$$

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Assume $||a||_2$ *and* $||a||_q$ *are finite. Then*

$$T = \lim_{a\to-\infty, b\to+\infty} \sum_{k=a}^{b} a_k(X_k - EX_k)$$

exists as an almost sure limit, and there exist positive constants C_1 *and* C_2 *depending only on* M, γ, *and* p *such that for every* $\epsilon > 0$

$$P\{T \geqq \epsilon\} \leqq \exp\left(-\min\{C_1(\epsilon/||a||_2)^2, C_2(\epsilon/||a||_q)\}\right). \tag{2}$$

Now suppose μ is a measure on the real line. If f is a real valued function on the real line we define $||f||_\infty = \inf [\lambda \,\varepsilon\, [0, \infty] | \mu\{x: |f(x)| > \lambda\} = 0]$ and for $p \,\varepsilon\, [1, \infty)$ we define $||f||_p = [\int |f|^p \, d\mu]^{1/p}$. We use the notation $\mu(a, b] = \mu\{x \mid a < x \leqq b\}$.

THEOREM B. *Let* $\{X_t : -\infty < t < \infty\}$ *be a real-valued stochastic process with independent increments. Suppose* μ *is a measure on the real line such that for all* $-\infty < s \leqq t < \infty$ *either*

(3a) $p = 1$ *and there exists* $T > 0$ *such that for all* $|\lambda| \leqq T$

$$E \exp\{\lambda[(X_t - X_s) - E(X_t - X_s)]\} \leqq \exp\{\lambda^2 \mu(s, t]\}$$

or else

(3b) $1 < p \leqq 2$ *and for all values of* λ

$$E \exp\{\lambda[(X_t - X_s) - E(X_t - X_s)]\} \leqq \exp\{(\lambda^2 + |\lambda|^q)\mu(s, t]\}.$$

(4) *Assume that* $||f||_2 < \infty$ *and* $||f||_q < \infty$ *and that for every* $\eta > 0$ *there exists a simple function* $h = \sum C_i\chi_{A_i}$ *such that* $||f - h||_2 < \eta$ *and* $||f - h||_q < \eta$ *where the* A_i*'s are disjoint intervals of the form* (a, b]. χ_{A_i} *is the set characteristic function of the set* A_i *and the sum is finite.*

Then there exist positive numbers C_1 *and* C_2 *depending only on* p *and* T *such that* $\int f(t) \, d[X_t - EX_t]$ *is well defined as a limit-in-the-mean of order* 2, *and for every* $\epsilon > 0$

$$P\{\textstyle\int f(t) \, d[X_t - EX_t] \geqq \epsilon\} \leqq \exp[-\min\{C_1(\epsilon/||f||_2)^2, C_2(\epsilon/||f||_q)^p\}]. \tag{5}$$

3. Proofs of Theorems A and B.

LEMMA 1. $|e^b - 1 - b| \leqq b^2 e^{|b|}$.

LEMMA 2. *If* $1 < p \leqq 2$, $M > 0$, $\gamma > 0$, $EX = 0$, *and* $P\{|X| \geqq x\} \leqq \int_x^\infty Me^{-\gamma t^p} \, dt$ *for* $0 < x < \infty$, *then there exists a constant* $C > 0$ *such that for all values of* t

$$Ee^{tX} \leqq e^{C(t^2+|t|^q)}.$$

PROOF.

$$\begin{aligned} Ee^{tX} &= \textstyle\int_{-\infty}^{\infty} e^{tX} \, dP = 1 + \int_{-\infty}^{\infty} (e^{tX} - 1 - tX) \, dP \\ &\leqq 1 + t^2 \textstyle\int_{-\infty}^{\infty} X^2 e^{|tX|} \, dP \leqq 1 + t^2 \int_0^\infty Mx^2 e^{|t|x} e^{-\gamma x^p} \, dx \\ &\leqq 1 + Mt^2\{\sup_{0\leqq x<\infty} [\exp[|t| \, x - \gamma x^p/2]]\} \textstyle\int_0^\infty x^2 \exp[-\gamma x^p/2] \, dx \\ &= 1 + Mt^2\{\exp[|t|^q \times (1/q) \times (2/p\gamma)^{1/(p-1)}]\} \textstyle\int_0^\infty x^2 \exp[-\gamma x^p/2] \, dx \\ &= 1 + M^* t^2 e^{\lambda|t|^q}, \end{aligned}$$

where λ and M^* are positive constants depending on M, p, and γ. Let $C = M^* + \lambda$ and obtain

$$Ee^{tX} \leqq (1 + M^*t^2)e^{\lambda|t|^q} \leqq \exp[M^*t^2 + \lambda|t|^q] \leqq \exp[C(t^2 + |t|^q)].$$

LEMMA 3. *If $M > 0$, $\gamma > 0$, $EX = 0$, and $P\{|X| \geqq x\} \leqq \int_x^\infty Me^{-\gamma t}\,dt$ for $0 < x < \infty$, then there exist positive constants C and τ such that for $|t| < \tau$ we have $Ee^{tX} \leqq e^{Ct^2}$.*

PROOF.

$$\begin{aligned} Ee^{tX} &= 1 + \int_{-\infty}^\infty (e^{tX} - 1 - tX)\,dP \\ &\leqq 1 + \int_{-\infty}^\infty t^2X^2e^{|tX|}\,dP \leqq 1 + Mt^2\int_0^\infty x^2e^{|t|x}e^{-\gamma x}\,dx \\ &= 1 + Mt^2(1/(\gamma - |t|))^3\int_0^\infty x^2e^{-x}\,dx = 1 + 2Mt^2(1/(\gamma - |t|))^3 \end{aligned}$$

for $|t| < \gamma$. Let $\tau = \gamma/2$ and $C = 16M/\gamma^3$. Then for $|t| \leqq \tau$,

$$Ee^{tX} \leqq 1 + 2Mt^2(2/\gamma)^3 = 1 + Ct^2 \leqq e^{Ct^2}.$$

REMARK. Lemmas 2 and 3 show that from the exponential behavior of the tails of a distribution one can deduce a certain type of bound on the associated moment generating function. The converse is also true. If $Ee^{tX} \leqq e^{Ct^2}$ for $|t| \leqq \tau$, then for $|t| \leqq \tau$ and $x > 0$,

$$P\{X \geqq x\} = P\{X - x \geqq 0\} \leqq Ee^{t(X-x)} \leqq e^{-tx+Ct^2} = e^{-t(x-Ct)}.$$

Thus

$$P\{X \geqq x\} \leqq \begin{Bmatrix} 1 & \text{if } x \leqq 2C\tau \\ e^{-\tau x/2} & \text{if } 2C\tau \leqq x \end{Bmatrix} \leqq [1 + e^{C\tau^2}]e^{-\tau x/2}.$$

The same bound holds on $P\{-X \geqq x\}$ so

$$P\{|X| \geqq x\} \leqq \int_x^\infty \tau(1 + e^{C\tau^2})e^{-\tau s/2}\,ds$$

which is of the proper form. Similarly, if $1 < p \leqq 2$ and

$$Ee^{tX} \leqq \exp\{C(t^2 + |t|^q)\},$$

then for all $t \geqq 0$

$$\begin{aligned} P\{|X| \geqq x\} &\leqq 2\exp[-tx + C(t^2 + t^q)] \\ &\leqq 2\exp[-t(x - C - 2Ct^{q-1})] \\ &\leqq 2, && 0 \leqq x \\ &\leqq 2\exp[-t(x/2 - 2Ct^{q-1})], && 2C \leqq x. \end{aligned}$$

Minimizing with respect to t gives

$$\begin{aligned} P\{|X| \geqq x\} &\leqq 2, && 0 \leqq x \\ &\leqq 2\exp[-1/2p(1/4Cq)^{1/(q-1)}x^p] = 2e^{-\lambda x^p}, && 2C \leqq x. \end{aligned}$$

For $t \geqq 0$ we have $\lambda t^p e^{-(\lambda/2)t^p} \leqq 1$ so that for $x \geqq \max\{2C, (2/\lambda)^{1/p}\}$ we have

$$\begin{aligned} P\{|X| \geqq x\} &\leqq 2e^{-\lambda x^p} \\ &= \int_x^\infty [e^{-\lambda t^p/2}(2p/t)][\lambda t^p e^{-\lambda t^p/2}]\,dt \\ &\leqq \int_x^\infty 2p(\lambda/2)^{1/p} e^{-\lambda t^p/2}\,dt. \end{aligned}$$

If we let $\delta = \max\{2C, (2/\lambda)^{1/p}\}$ and

$$M = \max\{2p(\lambda/2)^{1/p},\ 2[\int_\delta^\infty \exp(-\lambda t^p/2)\,dt]^{-1}\},$$

then

$$P\{|X| \geqq x\} \leqq M \int_x^\infty e^{-\lambda t^p/2}\,dt$$

for $0 < x < \infty$. The last expression is of the desired form. These relations between the form of the moment generating function of X and the tails of the distribution of X have been observed before when $p = 1$ and [14; $p - 75$, property 4] when $p = 2$.

Lemma 4. *Under the conditions of theorem A,*

(a) *T exists as an almost sure limit*

(b) *if $1 < p \leqq 2$, then $\phi(t) = \lim_{a\to-\infty, b\to\infty} \prod_{k=a}^b Ee^{a_k t(X_k - EX_k)}$ exists for all values of t and $Ee^{tT} = \phi(t)$*

(c) *if $p = 1$ and $E\exp[t(X_k - EX_k)] \leqq e^{Ct^2}$ for $|t| < \tau/\|a\|_\infty$ and for all values of k, then $\phi(t) = \lim_{a\to-\infty, b\to\infty} \prod_{k=a}^b E\exp[a_k t(X_k - EX_k)]$ exists for $|t| < \tau$, and $Ee^{tT} = \phi(t)$ for $|t| < \tau$.*

Proof. The method of proof is essentially the same as that used in the lemma of Section 2 of [11]. The reader should refer to [11] for more details. Using Lemma 2 we see that

$$E\exp[t(X_k - EX_k)] \leqq \exp[C(t^2 + |t|^q)]$$

for all t if $1 < p \leqq 2$. Let $\sigma = \{z = t + is\colon |t| < \lambda\}$ where λ is arbitrary if $1 < p \leqq 2$ and $\lambda = \tau/\|a\|_\infty$ if $p = 1$. We see that the functions $\prod_{k=a}^b E\exp[a_k z(X_k - EX_k)]$

(i) are analytic in σ

(ii) are uniformly bounded in σ, by $\exp[C(\|a\|_2^2\lambda^2 + \|a\|_q^q\lambda^q)]$ if $1 < p \leqq 2$, and by $\exp[C\|a\|_2^2\lambda^2]$ if $p = 1$ and

(iii) $|\prod_{k=a}^b E\exp[a_k z(X_k - EX_k)] - \prod_{k=-N}^N E\exp[a_k z(X_k - EX_k)]|$ for real $z \in \sigma$, $a \leqq -N$, $N \leqq b$ is bounded by $\exp[C(\|a\|_2^2\lambda^2 + \|a\|_q^q\lambda^q)][\exp\{C\lambda^2 \cdot \sum_{|k|>N} a_k^2 + C\lambda^q \sum_{|k|>N} |a_k|^q\} - 1]$ for $1 < p \leqq 2$, and is bounded by

$$\exp[C\|a\|_2^2\lambda^2][\exp\{C\lambda^2 \textstyle\sum_{|k|>N} a_k^2\} - 1]$$

for $p = 1$.

Thus $\{\prod_{k=a}^b E\exp[a_k z(X_k - EX_k)]\}$ is Cauchy for real z and has some limit, say $\phi(z)$. By Vitali's theorem [20, p. 168], the limit $\phi(z)$ exists for all $z \in \sigma$ and ϕ is analytic in σ. By the equivalence theorem and its Corollary 1 on page 251 of [18] it follows that T exists as an almost sure limit and that $\phi(is)$ is its characteristic function. Because λ was arbitrary when $1 < p \leqq 2$, we have shown that the limit $\phi(t)$ exists for all real t when $1 < p \leqq 2$, and for all real t satisfying

$|t| < \tau/\|a\|_\infty$ when $p = 1$. It remains to show that $\phi(t) = Ee^{tT}$ for these values of t; this can be done by comparing coefficients in the expansions of $\phi(is)$ and $\phi(t)$.

PROOF OF THEOREM A. Putting together Lemmas 2, 3, and 4, we see that the limit T exists almost everywhere and has moment generating function $\phi(t)$ bounded, when $p = 1$ by $\exp\{C\|a\|_2^2t^2\}$ for all $|t| \leqq \tau/\|a\|_\infty$, and when $1 < p \leqq 2$ by $\exp\{C[\|a\|_2^2t^2 + \|a\|_q^q|t|^q]\}$ for all t. For $t \geqq 0$ we have

$$P\{T \geqq \epsilon\} = P\{T - \epsilon \geqq 0\} \leqq Ee^{(T-\epsilon)t} = e^{-\epsilon t}\phi(t).$$

If $p = 1$, we get for $|t| \leqq \tau/\|a\|_\infty$

$$P\{T \geqq \epsilon\} \leqq e^{-\epsilon t} \exp [C\|a\|_2^2t^2].$$

We see that this last expression is minimized when $t = \epsilon/2C\|a\|_2^2$. Letting $t_0 = \min\{\epsilon/2C\|a\|_2^2, \tau/\|a\|_\infty\}$ we get when $p = 1$

$$\begin{aligned} P\{T \geqq \epsilon\} &\leqq \exp[-t_0(\epsilon - C\|a\|_2^2t_0)] \leqq e^{-(\epsilon/2)t_0} \\ &= \exp(-\epsilon/2 \min\{\epsilon/2C\|a\|_2^2, \tau/\|a\|_\infty\}) \\ &= \exp(-\min\{C_1(\epsilon/\|a\|_2)^2, C_2(\epsilon/\|a\|_\infty)^1\}). \end{aligned}$$

If $1 < p \leqq 2$, we have for all non-negative values of t

$$\begin{aligned} P\{T \geqq \epsilon\} &\leqq e^{-\epsilon t} \exp[C(\|a\|_2^2t^2 + \|a\|_q^qt^q)] \\ &= \exp\{-t[(\epsilon/2 - C\|a\|_2^2t) + (\epsilon/2 - C\|a\|_q^qt^{q-1})]\}. \end{aligned}$$

Let $t_0 = \min\{\epsilon/4C\|a\|_2^2, [\epsilon/4C\|a\|_q^q]^{1/(q-1)}\}$. Then

$$\begin{aligned} P\{T \geqq \epsilon\} &\leqq e^{-(\epsilon/2)t_0} \\ &= \exp(-(\epsilon/2)\min\{\epsilon/4C(1/\|a\|_2^2), (\epsilon/4C)^{1/(q-1)}(1/\|a\|_q^p)\}) \\ &= \exp[-\min\{C_1(\epsilon/\|a\|_2)^2, C_2(\epsilon/\|a\|_q)^p\}]. \end{aligned}$$

Note that C and τ depended only on M, γ, and p so the same is true for C_1 and C_2.

PROOF OF THEOREM B. For notational convenience we assume that

$$E(X_t - X_s) = 0$$

for all $-\infty < s \leqq t < \infty$, i.e. that the mean of the process has already been subtracted off.

Let $f_n = \sum_i C_{n,i}\chi_{n,i}$ be a sequence of simple functions such that $\|f_n - f\|_2 \to 0$ and $\|f_n - f\|_q \to 0$. Here $\chi_{n,i}$ is the set characteristic function of $\Delta_{n,i} = (a_{n,i-1}, a_{n,i}]$ where $-\infty \leqq \cdots < a_{n,k-1} < a_{n,k} < a_{n,k+1} < \cdots \leqq \infty$. We see that $C_{n,i}$ is the value of f_n on $\Delta_{n,i}$ and note that all but a finite number of $C_{n,i}$ can be taken equal to zero for each n. (We can always approximate f by simple functions f_n such that $\|f_n - f\|_2 \to 0$ and $\|f_n - f\|_q \to 0$. It takes more (more even than μ σ-finite) to assure that the sets involved in the definition of the f_n's can be taken to be of the form (a, b]. It is sufficient for the latter part of assumption (4) to assume that either f is continuous or to assume that $(-\infty, \infty) = \sum I_i$ where each I_i is of the form (a, b] and has finite measure. The problem of the existence of approximating simple functions of this special form was overlooked in [12].)

For each n we form the stochastic integral

$$Y_n = \int f_n(t)\, dX_t = \sum_i C_{n,i}[X_{a_{n,i}} - X_{a_{n,i-1}}].$$

(See [7; pages 426–428] for a reference on stochastic integrals.) Fix n and m and suppose the partitions of the real line used in the definitions of f_n and f_m have been merged so that $a_{n,i} = a_{m,i}$ for all i. Then by the assumption of independent increments

$$E(Y_n - Y_m)^2 = \sum_i (C_{n,i} - C_{m,i})^2 E[X_{a_{n,i}} - X_{a_{n,i-1}}]^2.$$

Note that

$$\begin{aligned} 2 + \lambda^2 E(X_{a_{n,i}} - X_{a_{n,i-1}})^2 &\leqq E \exp[\lambda(X_{a_{n,i}} - X_{a_{n,i-1}})] \\ &\quad + E \exp[-\lambda(X_{a_{n,i}} - X_{a_{n,i-1}})] \\ &\leqq 2 \exp\{(\lambda^2 + |\lambda|^q)\mu(a_{n,i-1}, a_{n,i}]\}, \quad 1 < p \leqq 2, \\ &\leqq 2 \exp\{\lambda^2 \mu(a_{n,i-1}, a_{n,i}]\} \quad \text{for} \quad |\lambda| \leqq T, \quad p = 1. \end{aligned}$$

In either case, if $\mu(a_{n,i-1}, a_{n,i}] < \infty$, then for sufficiently small values of λ we have

$$2 + \lambda^2 E(X_{a_{n,i}} - X_{a_{n,i-1}})^2 \leqq 2\{1 + 3\lambda^2 \mu(a_{n,i-1}, a_{n,i}]\}$$

so that

$$E(X_{a_{n,i}} - X_{a_{n,i-1}})^2 \leqq 3\mu(a_{n,i-1}, a_{n,i}].$$

This last inequality clearly holds when $\mu(a_{n,i-1}, a_{n,i}] = \infty$. Thus

$$\begin{aligned} E(Y_n - Y_m)^2 &\leqq 3 \sum_i (C_{n,i} - C_{m,i})^2 \mu(a_{n,i-1}, a_{n,i}] \\ &= 3\|f_n - f_m\|_2^2. \end{aligned}$$

However $\|f_n - f_m\|_2^2 \leqq [\|f_n - f\|_2 + \|f - f_m\|_2]^2 \to 0$ as $n, m \to \infty$ so that $\{Y_n\}$ is a cauchy sequence in L_2. This implies that its L_2 limit $\int f(t)\, dX_t$ exists and that $E[\int f(t)\, dX_t - Y_n]^2 \to 0$, so that for every real number ϵ

$$P\{\int f(t)\, dX_t \geqq \epsilon\} \leqq \lim_{m\to\infty} \limsup_{n\to\infty} P\{Y_n \geqq \epsilon - 1/m\}.$$

For $\lambda \geqq 0$, $\epsilon > 0$, and $m > 1/\epsilon$, we have

$$\begin{aligned} P\{Y_n \geqq \epsilon - 1/m\} &\leqq \exp[-\lambda(\epsilon - 1/m)] E e^{\lambda Y_n} \\ &= \exp[-\lambda(\epsilon - 1/m)] \prod_i E \exp[\lambda C_{n,i}[X_{a_{n,i}} - X_{a_{n,i-1}}]]. \end{aligned}$$

When $p = 1$ we may assume that $|C_{n,i}| \leqq \|f\|_\infty$ for all n and i. Then for $\lambda \leqq T/\|f\|_\infty$ we have

$$P\{Y_n \geqq \epsilon - 1/m\} \leqq \exp\,[-\lambda(\epsilon - 1/m)]\,\exp\,[\lambda^2 \textstyle\sum_i C_{n,i}^2 \mu\{\Delta_{n,i}\}]$$
$$= \exp\,[-\lambda(\epsilon - 1/m)]\,\exp\,[\lambda^2 \|f_n\|_2^2].$$

This is minimized when $\lambda = (\epsilon - 1/m)/2\|f_n\|_2^2$. If we set

$$\lambda_0 = \min\,\{T/\|f\|_\infty\,,\,(\epsilon - 1/m)/2\|f_n\|_2^2\}$$

we get

$$P\{Y_n \geqq \epsilon - 1/m\} \leqq \exp\,[-\lambda_0(\epsilon - 1/m)/2]$$
$$= \exp\,[(-(m\epsilon - 1)/2m)\min\,\{T/\|f\|_\infty\,,\,(m\epsilon - 1)/2m\|f_n\|_2^2\}].$$

Taking "$\lim_{m\to\infty} \limsup_{n\to\infty}$" of the above expression gives

$$P\{\textstyle\int f(t)\,dX_t \geqq \epsilon\} \leqq \exp\,[-\min\,\{\tfrac{1}{4}(\epsilon/\|f\|_2)^2,\ T/2(\epsilon/\|f\|_\infty)^1\}].$$

When $1 < p \leqq 2$ we obtain for all $\lambda \geqq 0$,

$$P\{Y_n \geqq \epsilon - 1/m\}$$
$$\leqq \exp\,[-\lambda(\epsilon - 1/m)]\,\exp\,[[\lambda^2 \textstyle\sum_i C_{n,i}^2 + \lambda^q \sum_i |C_{n,i}|^q]\mu\{\Delta_{n,i}\}]$$
$$= \exp\,[-\lambda(\epsilon - 1/m) + (\lambda^2\|f_n\|_2^2 + \lambda^q\|f_n\|_q^q)]$$
$$= \exp\,\{-\lambda[((m\epsilon - 1)/2m - \lambda\|f_n\|_2^2) + ((m\epsilon - 1)/2m - \lambda^{q-1}\|f_n\|_q^q)]\}.$$

If we set $\lambda_0 = \min\,\{(m\epsilon - 1)/4m\|f_n\|_2^2\,,\,((m\epsilon - 1)/4m\|f_n\|_q^q)^{1/(q-1)}\}$, then

$$P\{Y_n \geqq \epsilon - 1/m\} \leqq \exp\,[-\lambda_0(m\epsilon - 1)/2m]$$
$$= \exp\,[-\min\,\{\tfrac{1}{8}((m\epsilon - 1)/m\|f_n\|_2)^2,\ 2(\tfrac{1}{4})^p((m\epsilon - 1)/m\|f_n\|_q)^p].$$

Again, taking "$\lim_{m\to\infty} \limsup_{n\to\infty}$" in the above expression gives

$$P\{\textstyle\int f(t)\,dX_t \geqq \epsilon\} \leqq \exp\,[-\min\,\{\tfrac{1}{8}(\epsilon/\|f\|_2)^2,\ 2(\tfrac{1}{4})^p(\epsilon/\|f\|_q)^p].$$

4. Corollaries and relationships between these theorems and previous results. Hypothesis (3) of Theorem B could have been given in terms of upper bounds on the probabilities in the tails of the distributions of the stochastic increments $(X_t - X_s) - E(X_t - X_s)$ to obtain a condition similar to hypothesis (1) of Theorem A. However, doing so would tend to obscure the nature of the theorem.

Inequality (2) of Theorem A can be obtained from Theorem B via Lemmas 1 and 2 if one defines the process $\{X_t^*: -\infty < t < \infty\}$ by

$$X_t^* = \textstyle\sum_{0 \leqq k \leqq t} X_k - \sum_{t<k<0} X_k$$

and sets $\mu(\mathrm{a}, \mathrm{b}]$ equal to a constant times the number of integers in the interval $(\mathrm{a}, \mathrm{b}]$. A generalization of Theorem A can, in fact, be obtained from Theorem B. The constants obtained in Lemmas 2 and 3 depend on M, γ, and p from hypothesis (1) of Theorem A. If M and γ were allowed to vary with k when $1 < p \leqq 2$ (allow only M to vary when $p = 1$), then these constants would depend on k. Then we let $\mu\{k\}$ take this value, say d_k, and change our definition of $\|a\|_p$ to $\|a\|_p = [\sum_k |a_k|^p\, d_k]^{1/p}$ for $1 \leqq p < \infty$ and to $\|a\|_\infty = \sup\,\{|a_k| : d_k > 0\}$ for $p = \infty$.

Since $\|a\|_2^2 \leqq \|a\|_1\|a\|_\infty$, we see that Theorem 1 of [11], which states that $P\{T \geqq \epsilon\} \leqq \exp\{-C/\|a\|_\infty\}$ when $\|a\|_1$ is bounded and the X_k's satisfy a condition equivalent to condition (1) with $p = 1$, is a corollary to our Theorem A. Theorem A is actually more general than Theorem 1 of [11] since $\|a\|_1$ need not be finite to use Theorem A.

Let $\{X_k : k = 0, \pm 1, \cdots\}$ be a sequence of independent random variables; let $a^{(n)} = \{a_{n,k} : k = 0, \pm 1, \cdots\}$ for $n = 1, 2, \cdots$ be a sequence of sequences of real numbers; and define $T_n = \sum_k a_{n,k}(X_k - EX_k)$. We obtain the following corollaries to Theorem A.

Corollary 1. *If the sequence* $\{X_k\}$ *satisfies condition* (1) *of Theorem* A, *if* $1 \leqq p \leqq 2$, *if* $\|a^{(n)}\|_2$ *and* $\|a^{(n)}\|_q$ *are finite for all* n, *and if the sums*

$$\sum_{n=1}^{\infty} \exp(-t/\|a^{(n)}\|_2^2) \quad \text{and} \quad \sum_{n=1}^{\infty} \exp(-t/\|a^{(n)}\|_q^p)$$

are finite for every $t > 0$, *then* T_n *exists as an* a.e. *limit for each* n *and for every* $\epsilon > 0$,

$$\sum_1^\infty P[|T_n| \geqq \epsilon] < \infty$$

so that $\lim_{n\to\infty} T_n = 0$ a.e.

Corollary 2. *If* $\{X_k\}$ *satisfies condition* (1) *of Theorem* A, *if* $1 \leqq p \leqq 2$, *if* $\|a^{(n)}\|_2$ *and* $\|a^{(n)}\|_q$ *are finite for all* n, *if either*

$$\|a^{(n)}\|_2^2 = o(1/\log n) \quad \text{and} \quad \|a^{(n)}\|_q^p = o(1/\log n)$$

or if the sums

$$\sum_{n=1}^{\infty} \exp(-t/\|a^{(n)}\|_2^2) \quad \text{and} \quad \sum_{n=1}^{\infty} \exp(-t/\|a^{(n)}\|_q^p)$$

are finite for every $t > 0$, *and if* $C_n = \sum_k a_{n,k}EX_k$ *converges for each* n *and* $C_n \to C$, *then* $T_n^* = \sum_k a_{n,k}X_k$ *exists as an* a.e. *limit for each* n *and* $\lim_{n\to\infty} T_n^* = C$ a.e. These are extensions of Theorems 1 and 2 of [5]. Their proofs are very simple applications of Theorem A and are essentially the same as the proofs given by Chow in [5].

Note that one can not gain anything using these methods by assuming that $p > 2$, or even by assuming that the X_k's are uniformly bounded. The bound provided when $p = 2$ seems to be very closely related to the workings of the central limit theorem.

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