TESTING AGAINST ORDERED ALTERNATIVES IN MODEL I ANALYSIS OF VARIANCE; NORMAL THEORY AND NONPARAMETRIC

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1. Introduction. In the present paper we extend Bartholomew's (1959a) and (1961b) results for testing against ordered alternatives in the one-way analysis of variance to more general linear models. Theorem 1 is the fundamental result on which the applications rely; it represents a generalization of Bartholomew's result.

A number of well known results are presented as special cases of the theorem: (i) Bartholomew's (1959a) likelihood ratio (LR) test for equality of means against ordered alternatives in the one-way layout (Section 5). (ii) Chacko's (1963) rank analog of this test for equal sample sizes is presented (Section 8) and the distribution theory is here extended to the case of unequal sample sizes.

Moreover, the following new results are presented: (i) LR tests against ordered alternatives in the two-way layout (Section 4) and in general complete layouts (Section 6). (ii) LR tests against ordered alternatives in incomplete layouts are illustrated with the Latin square (Section 7) and independence of test statistics for nested hypothesis is there observed. (iii) A rank analog of the test of Section 4 is proposed (Section 9) and its distribution theory and Pitman efficiency are presented. (iv) Extensions to asymptotically nonparametric tests against ordered alternatives in the general linear model are indicated (Section 11).

In Section 10, Bartholomew's (1961a) results for the case of partially ordered alternatives in the one way layout are similarly extended to other layouts and to the non-parametric case.

The technique of the examples in Sections 4–7 illustrates an extremely simple technique for generating LR tests against ordered alternatives in general linear models. It should be noted that when σ^2 is unknown the LR tests differ from the tests proposed by Kudô (1963).

The examples also illustrate how (provided certain sample sizes are required to be equal) the distribution of the LR test statistic for testing against ordered alternatives depends in a natural manner on a two parameter family of distributions. Percentage points of these $\bar{B}[k, N]$ distributions are given in Table 1. These play the role that F distributions fill for testing against unordered alternatives.

Lemma 1 provides the bridge between the normal theory and nonparametric results by showing that the probabilities $p_{m,k}(n_1, \cdots, n_k)$ have identical values for two distinct covariance matrices. While the proof is elementary in nature, this result may be interesting in its own right.

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The relationship of Theorem 1 to Nuesch's (1966) result is discussed in Section 12. In particular, an error which invalidates a key statement of Nuesch is pointed out.

Section 3 merely presents notation to put the examples of Sections 4–7, 11 in proper perspective.

Finally, it is felt that this paper will provide a unifying influence on the results which have preceded it.

I—GENERAL THEORY

2. Statement of general results. Let $\bar{Z}_i = (1/n_i) \sum_{\alpha=1}^{n_i} Z_{i\alpha}$ for $i = 1, \cdots, k$ where the $Z_{i\alpha}$'s are arbitrary real numbers. Let $\hat{\xi}_1 \leqq \cdots \leqq \hat{\xi}_k$ denote the values of $\xi_1, \cdots, \xi_k$ that minimize $\sum_{i=1}^{k} n_i(\bar{Z}_i - \xi_i)^2$ subject to the condition $\xi_1 \leqq \cdots \leqq \xi_k$. Define $\bar{Z}_{[t,s]} = (n_t\bar{Z}_t + \cdots + n_s\bar{Z}_s)/(n_t + \cdots + n_s)$ for $1 \leqq t < s \leqq k$.

The "amalgamation process" by which $\hat{\xi}_1 \leqq \cdots \leqq \hat{\xi}_k$ are found is described in Bartholomew (1959a); and is assumed well known. It yields integers $t_1, \cdots, t_m$ with each $t_j > 0$ and $t_1 + \cdots + t_m = k$ for which

$$\hat{\xi}_{\tau_j+1} = \cdots = \hat{\xi}_{\tau_{j+1}} = \bar{Z}_{[\tau_j+1,\tau_{j+1}]} \qquad \text{for} \quad j = 0, 1, \cdots, m-1$$

where $\tau_0 = 0$ and $\tau_j = t_1 + \cdots + t_j$ for $j = 1, \cdots, m$. For notational convenience we denote the m distinct $\hat{\xi}_i$'s by $\bar{Z}_{[t_j]}$ for $j = 1, \cdots, m$. We also define $N_{[t_j]} = n_{\tau_{j-1}+1} + \cdots + n_{\tau_j}$. This notation is from Chacko (1963).

LEMMA 1. *Let $(n_1^{\frac{1}{2}}\bar{Z}_1, \cdots, n_k^{\frac{1}{2}}\bar{Z}_k)$ have a multivariate $N(0, \Sigma)$ distribution where either*

CASE 0. $\Sigma = \sigma^2 I$ or

CASE 1. $\Sigma = \sigma^2\|\delta_{ij} - (n_in_j)^{\frac{1}{2}}/N\|$ *where* $N = n_1 + \cdots + n_k$.

Let $p_{m,k}(n_1, \cdots, n_k)$ denote the probability that the amalgamation process when applied to the $\bar{Z}_i$'s produces exactly m distinct $\hat{\xi}_i$'s. Then $p_{m,k}(n_1, \cdots, n_k)$ has the same value for Case 1 as it does for Case 0.

PROOF. Let us denote the region in the sample space leading to the m distinct $\hat{\xi}_i$'s of $\bar{Z}_{[t_1]}, \cdots, \bar{Z}_{[t_m]}$ by $\mathfrak{X}(t_1, \cdots, t_m)$. Let $\mathfrak{X}^*(m) = \cup\, \mathfrak{X}(t_1, \cdots, t_m)$ where the union is taken over all regions in the sample space leading to exactly m distinct estimates. Then $p_{m,k}(n_1, \cdots, n_k) = P(\mathfrak{X}^*(m)) = \Sigma P(\mathfrak{X}(t_1, \cdots, t_m))$. Now Chacko (1963) shows that the region $\mathfrak{X}(t_1, \cdots, t_m)$ is defined by the inequalities

$$\bar{Z}_{[t_j]} - \bar{Z}_{[t_{j+1}]} < 0, \quad \text{for } j = 1, \cdots, m-1,$$

$$\bar{Z}_{[\tau_{j-1}+1,\tau_{j-1}+i]} - \bar{Z}_{[t_j]} > 0, \quad \text{for } i = 1, \cdots, t_j - 1$$

and $j = 1, \cdots, m$. It is straightforward, but tedious, to show that the linear forms involved in these inequalities have the same distribution in Case 1 as in Case 0. This completes the proof of the lemma; however, note that the first $m - 1$ linear forms listed above are independent of the remaining $\sum_{j=1}^{m} (t_j - 1)$ linear forms. This latter fact is used in the proof of Theorem 1.

For equal n_i's the probabilities $p_{m,k}(n_1, \cdots, n_k)$ are denoted simply $p_{m,k}$.

See Bartholomew (1959a) for known values of $p_{m,k}(n_1, \cdots, n_k)$ and Chacko (1963) for a simple general formula for $p_{m,k}$. See also Miles (1959).

DEFINITION. Let χ_a^2 and $\beta(a, b)$ denote chi-square and beta random variables respectively. Let W be a non-negative random variable. If for all $t > 0$

$$P(W > t) = \sum_{m=2}^{k} p_{m,k}(n_1, \cdots, n_k) P(\beta((m-1)/2, (N-m)/2) > t)$$

we say that W has the $\bar{B}[n_1, \cdots, n_k; k, N]$ distribution. If for all $t > 0$

$$P(W > t) = \sum_{m=2}^{k} p_{m,k}(n_1, \cdots, n_k) P(\chi_{m-1}^2 > t)$$

we say that W has the $\bar{D}[n_1, \cdots, n_k; k]$ distribution. If $n_1 = \cdots = n_k$ we call these the $\bar{B}[k, N]$ and $\bar{D}[k]$ distributions respectively.

See Bartholomew (1959a) and (1959b) for percentage points of $\bar{D}[n_1, \cdots, n_k; k]$ distributions.

THEOREM 1. *Let* $(n_1^{\frac{1}{2}}\bar{Z}_1, \cdots, n_k^{\frac{1}{2}}\bar{Z}_k)$ *have a multivariate* $N(0, \Sigma)$ *distribution where either*

CASE 0. $\Sigma = \sigma^2 I$ *or*

CASE 1. $\Sigma = \sigma^2\|\delta_{ij} - (n_i n_j)^{\frac{1}{2}}/N\|$. *Define*

$$S = \inf_{\xi_k \leqq \cdots \leqq \xi_1} \sum_{i=1}^{k} n_i(\bar{Z}_i - \xi_i)^2$$

and

$$Q = \sum_{i=1}^{k} n_i(\bar{Z}_i - \bar{Z}_{[1,k]})^2.$$

Let T be a random variable such that T/σ^2 has a χ_ν^2 distribution; and let T be independent of $\bar{Z}_1 - \bar{Z}_{[1,k]}, \cdots, \bar{Z}_k - \bar{Z}_{[1,k]}$. *Then* $(Q - S)/\sigma^2$ *has the* $\bar{D}[n_1, \cdots, n_k; k]$ *distribution and* $(Q - S)/(T + Q)$ *has the* $\bar{B}[n_1, \cdots, n_k; k, \nu + k]$ *distribution.*

PROOF. Let $\mathfrak{X}(t_1, \cdots, t_m)$ and $\mathfrak{X}^*(m)$ be as in the proof of Lemma 1. Define

$$SS(t_1, \cdots, t_m) = \sum_{j=1}^{m} \sum_{i=\tau_{j-1}+1}^{\tau_j} n_i(\bar{Z}_i - \bar{Z}_{[t_j]})^2.$$

Although the $m, t_1, \cdots, t_m$ that appear in the definition of S are random variables, we stress that when they appear in $\mathfrak{X}(t_1, \cdots, t_m)$, $\mathfrak{X}^*(m)$ and $SS(t_1, \cdots, t_m)$, the quantities $m, t_1, \cdots, t_m$ are to be regarded as a set of fixed constants. Note that $SS(k)$ is identically equal to Q.

We now show that $SS(t_1, \cdots, t_m)$ and $Q - SS(t_1, \cdots, t_m)$ are independent and are distributed as $\sigma^2\chi_{k-m}^2$ and $\sigma^2\chi_{m-1}^2$ respectively. First let

$$V^{m\times 1} = (N_{[t_1]}^{\frac{1}{2}}\bar{Z}_{[t_1]}, \cdots, N_{[t_m]}^{\frac{1}{2}}\bar{Z}_{[t_m]})'$$

and let

$$\Omega^{m\times m} = \|\delta_{ij} - (\omega_i\omega_j)^{\frac{1}{2}}\|$$

where $\omega_j = N_{[t_j]}/N$ for $j = 1, \cdots, m$ and δ_{ij} is the Kronecker delta. Then $Q - SS(t_1, \cdots, t_m) = V'\Omega V$ where V is $N(0, \sigma^2\,\Sigma_v)$ with

$$\Sigma_v = \Omega \qquad \text{if Case 1}$$

or

$$\Sigma_v = I \qquad \text{if Case 0.}$$

Let Γ be an $m \times m$ orthogonal matrix with first row $\omega_1^{\frac{1}{2}}, \cdots, \omega_m^{\frac{1}{2}}$. Then

$$Q - SS(t_1, \cdots, t_m) = V'\Gamma'\Gamma\Omega\Gamma'\Gamma V = (\Gamma V)' \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & 1 \end{pmatrix} (\Gamma V)$$

where ΓV is $N(0, \sigma^2\Gamma\Sigma_v\Gamma')$. However, we have

$$\Gamma\Sigma_v\Gamma' = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & 1 \end{pmatrix} \text{ in Case 1}$$

$$\Gamma\Sigma_v\Gamma' = I \qquad \text{in Case 0.}$$

In either case $Q - SS(t_1, \cdots, t_m)$ is seen to be distributed as $\sigma^2\chi^2_{m-1}$. That $SS(t_1, \cdots, t_m)$ and $Q - SS(t_1, \cdots, t_m)$ are independent follows as usual by orthogonality. Hence $SS(t_1, \cdots, t_m)$ has a $\sigma^2\chi^2_{k-m}$ distribution. Thus for all $t > 0$ we have

$$\begin{aligned} P((Q - S)/\sigma^2 > t) &= \textstyle\sum_{\text{all}\,\mathfrak{X}(t_1,\cdots,t_m)} P(\mathfrak{X}(t_1, \cdots, t_m) \quad \text{and} \quad (Q - S)/\sigma^2 > t) \\ &= \textstyle\sum_{\text{all}\,\mathfrak{X}(t_1,\cdots,t_m)} P(\mathfrak{X}(t_1, \cdots, t_m)) \\ &\quad \cdot P((Q - SS(t_1, \cdots, t_m))/\sigma^2 > t) \\ &= \textstyle\sum_{\text{all}\,\mathfrak{X}(t_1,\cdots,t_m)} P(\mathfrak{X}(t_1, \cdots, t_m))P(\chi^2_{m-1} > t) \\ &= \textstyle\sum_{m=2}^{k} P(\mathfrak{X}^*(m))P(\chi^2_{m-1} > t) \\ &= \textstyle\sum_{m=2}^{k} p_{m,k}(n_1, \cdots, n_k)P(\chi^2_{m-1} > t). \end{aligned}$$

The crucial step of this argument used the independence of the linear forms defining the region $\mathfrak{X}(t_1, \cdots, t_m)$ and $Q - SS(t_1, \cdots, t_m)$. See the proof of Lemma 1.

Similarly for all $t > 0$ we have

$$\begin{aligned} &P((Q - S)/(T + Q) > t) \\ &\quad = \textstyle\sum_{\text{all}\,\mathfrak{X}(t_1,\cdots,t_m)} P(\mathfrak{X}(t_1, \cdots, t_m) \quad \text{and} \quad (Q - SS(t_1, \cdots, t_m))/(T + Q) > t) \\ &\quad = \textstyle\sum_{\text{all}\,\mathfrak{X}(t_1,\cdots,t_m)} P(\mathfrak{X}(t_1, \cdots, t_m))P((Q - SS(t_1, \cdots, t_m))/(T + Q) > t) \\ &\quad = \textstyle\sum_{m=2}^{k} p_{m,k}(n_1, \cdots, n_k)P(\beta((m - 1)/2, (\nu + k - m)/2) > t). \end{aligned}$$

This completes the proof of the theorem.

TABLE I
Upper 5% and 1% values of the $\bar{B}[k, N]$ distribution (Interpolate linearly in $1/N$ in the columns of the table)

N	$k = 3$		$k = 4$		$k = 5$		$k = 6$	
	5%	1%	5%	1%	5%	1%	5%	1%
4	.950	1.000	*	*	*	*	*	*
5	.811	.953	.895	.986	*	*	*	*
6	.687	.878	.773	.929	.831	.960	*	*
7	.590	.800	.671	.857	.728	.895	.769	.922
8	.514	.728	.590	.787	.643	.828	.683	.858
9	.455	.665	.525	.724	.574	.764	.612	.795
10	.408	.610	.472	.668	.518	.708	.553	.738
11	.369	.563	.429	.618	.471	.657	.504	.687
12	.337	.522	.392	.575	.432	.612	.461	.641
13	.310	.486	.362	.537	.398	.573	.423	.600
14	.287	.455	.335	.503	.370	.538	.397	.564
15	.267	.427	.312	.474	.345	.506	.370	.532
16	.250	.403	.292	.447	.323	.478	.347	.503
18	.221	.361	.259	.401	.287	.430	.308	.453
20	.198	.327	.233	.364	.258	.391	.277	.412
22	.180	.299	.211	.333	.234	.358	.252	.377
24	.164	.287	.193	.307	.214	.330	.231	.348
27	.146	.246	.171	.275	.190	.296	.205	.312
30	.131	.222	.154	.248	.170	.268	.184	.283
40	.098	.168	.115	.188	.128	.203	.138	.215
50	.078	.135	.092	.151	.102	.164	.110	.173
∞	.000	.000	.000	.000	.000	.000	.000	.000

II. NORMAL THEORY

3. Introduction and notation. Suppose

$$\Omega: \mathbf{Y}^{n\times 1} = \mathbf{X}'\boldsymbol{\beta}^{p\times 1} + \mathbf{e}^{n\times 1}$$

where $\mathbf{e}$ has a $N(\mathbf{O}, \sigma^2\mathbf{I})$ distribution. Let $S(\mathbf{Y}, \boldsymbol{\beta}) = (\mathbf{Y} - \mathbf{X}'\boldsymbol{\beta})'(\mathbf{Y} - \mathbf{X}'\boldsymbol{\beta})$. The maximum likelihood estimates (*MLE's*) of β and σ^2 under Ω will be denoted by $\hat{\boldsymbol{\beta}}$ and $\hat{\sigma}^2$. We note that $\hat{\sigma}^2 = S(\mathbf{Y}, \hat{\boldsymbol{\beta}})/n$ where $\hat{\boldsymbol{\beta}}$ minimizes $S(\mathbf{Y}, \boldsymbol{\beta})$ among all values of β that are allowed by Ω. Let ω denote a set of assumptions that further restrict the theoretical mean of $\mathbf{Y}$. Let $\hat{\hat{\boldsymbol{\beta}}}$ and $\hat{\hat{\sigma}}^2$ denote the *MLE's* of $\boldsymbol{\beta}$ and σ^2 under ω. Again $\hat{\hat{\sigma}}^2 = S(\mathbf{Y}, \hat{\hat{\boldsymbol{\beta}}})/n$ where $\hat{\hat{\boldsymbol{\beta}}}$ minimizes $S(\mathbf{Y}, \boldsymbol{\beta})$ among all values of β that are allowed by ω. Thus the level α likelihood ratio (*LR*) test of ω against $\Omega - \omega$ rejects if

$$B = (\hat{\hat{\sigma}}^2 - \hat{\sigma}^2)/\hat{\hat{\sigma}}^2$$

exceeds the upper α per cent point of its null distribution. In the usual setup (see Scheffé (1959)) the random vector $\mathbf{Y}$ takes values in n-dimensional Euclidean space V_n, the assumptions Ω restrict the theoretical mean of $\mathbf{Y}$ to a subspace V_r of dimension r, and ω further restricts the theoretical mean to a subspace V_{r-q} of dimension $r - q$; thus $V_{r-q} \subset V_r \subset V_n$. Moreover, B has a beta distribution with parameters $q/2$ and $(n - r)/2$ under ω.

If σ^2 is known, the level α LR-test rejects ω if

$$D = n(\hat{\hat{\sigma}}^2 - \hat{\sigma}^2)$$

exceeds the upper α per cent point of its null distribution; and this null distribution is $\sigma^2\chi_q^2$ in the Scheffé setup.

It is the purpose of this paper to consider testing hypotheses when Ω restricts the theoretical mean to a subset (not a subspace) of V_n. This is accomplished by assuming that certain order relationships are satisfied by some of the parameters. If $\hat{\hat{\boldsymbol{\beta}}}$, $\hat{\hat{\sigma}}^2$ and $\hat{\boldsymbol{\beta}}$, $\hat{\sigma}^2$ still denote the MLE's of $\boldsymbol{\beta}$ and σ^2 under ω and Ω respectively, then we still have

$$\hat{\hat{\sigma}}^2 = S(\mathbf{Y}, \hat{\hat{\boldsymbol{\beta}}})/n \quad \text{and} \quad \hat{\sigma}^2 = S(\mathbf{Y}, \hat{\boldsymbol{\beta}})/n \tag{1}$$

where $\hat{\hat{\boldsymbol{\beta}}}$ and $\hat{\boldsymbol{\beta}}$ minimize $S(\mathbf{Y}, \boldsymbol{\beta})$ among all values of $\boldsymbol{\beta}$ allowed by ω and Ω respectively. Also the level α LR-test of ω against $\Omega - \omega$ still rejects ω when

$$\bar{B} = (\hat{\hat{\sigma}}^2 - \hat{\sigma}^2)/\hat{\hat{\sigma}}^2 \tag{2}$$

exceeds the upper α per cent point of its null distribution; however, this distribution is no longer beta.

If σ^2 is known, the level α LR-test still rejects ω if

$$\bar{D} = n(\hat{\hat{\sigma}}^2 - \hat{\sigma}^2) \tag{3}$$

exceeds the upper α per cent point of its null distribution; however, this distribution is no longer $\sigma^2\chi_q^2$.

4. Notation. Derivation of results for the two-way layout. Consider the following model for the two-way layout with one observation per cell. Let

$$Y_{ij} = \mu + \alpha_i + \beta_j + e_{ij} \qquad (i = 1, \cdots, I \text{ and } j = 1, \cdots, J)$$

where $\alpha. = \beta. = 0$ and the e_{ij}'s are independent identically distributed (iid) $N(0, \sigma^2)$. Now

$$S(\mathbf{Y}, \boldsymbol{\beta}) = IJ(Y.. - \mu)^2 + J\textstyle\sum_i(Y_i. - Y.. - \alpha_i)^2 + I\sum_j(Y._j - Y.. - \beta_j)^2 \\ + \textstyle\sum_i\sum_j(Y_{ij} - Y_i. - Y._j + Y..)^2.$$

We now indicate our notation for the present problem. No new definitions will be given in later sections; however, the generalizations of the notation of this section should be obvious. Let

$$SS_\alpha = J\textstyle\sum_i(Y_i. - Y..)^2,$$

$$SS_\beta = I\textstyle\sum_j(Y._j - Y..)^2,$$

$$SS_E = \text{the error sum of squares},$$
$$SAS_\alpha = \inf_{\alpha_1 \leqq \cdots \leqq \alpha_I} J \sum_i (Y_{i\cdot} - Y_{\cdot\cdot} - \alpha_i)^2,$$
$$SAS_\beta = \inf_{\beta_1 \leqq \cdots \leqq \beta_J} I \sum_j (Y_{\cdot j} - Y_{\cdot\cdot} - \beta_j)^2,$$

where SAS_α is read "the sum of amalgamated squares for the α effect". We further define

$$H_{0,\alpha} : \alpha_1 = \cdots = \alpha_I$$
$$H_{1,\alpha} : \alpha_1 \leqq \cdots \leqq \alpha_I \qquad \text{with at least one inequality strict.}$$
$$H_{2,\alpha} : \alpha_1 \neq \cdots \neq \alpha_I$$

and similarly for $H_{0,\beta}$, $H_{1,\beta}$, and $H_{2,\beta}$.

From (1), (2) and the above expression for $S(\mathbf{Y}, \boldsymbol{\beta})$ we can immediately write LR-test statistic for testing $H_{0,\alpha}$ against $H_{1,\alpha}$ as

$$\bar{B} = (SS_\alpha - SAS_\alpha)/(SS_E + SS_\alpha); \tag{4}$$

and by Theorem 1 this test statistic has a $\bar{B}[I, (I-1)J+1]$ distribution under the null hypothesis. (Apply Theorem 1 with $k = I$, $n_i = J$, $\bar{Z}_i = Y_{i.} - \mu$, Case 0, $\xi_i = \alpha_i$, or $\xi_i = d_i - Y\cdot\cdot + \mu$, $T = SS_E$ and $\nu = (I-1)(J-1)$.

If σ^2 is known the LR-test statistic for testing $H_{0,\alpha}$ against $H_{1,\alpha}$ is

$$\bar{D} = (SS_\alpha - SAS_\alpha)/\sigma^2$$

and by Theorem 1 this statistic has a $\bar{D}[I]$ distribution under the null hypothesis.

5. Derivation of Bartholomew's results for the one-way layout. This section gives a quick presentation of Bartholomew's (1959a) and (1961b) main results. We chose to present this example second because the introduction of notation seemed most natural in the two-way layout. Consider the following model. Let

$$Y_{ij} = \mu + \alpha_i + e_{ij} \qquad (i = 1, \cdots, I \text{ and } j = 1, \cdots, J_i)$$

where the e_{ij}'s are iid $N(0, \sigma^2)$ and $n = J_1 + \cdots + J_I$. Now

$$S(\mathbf{Y}, \boldsymbol{\beta}) = n(\bar{Y} - \mu)^2 + \textstyle\sum_i J_i(Y_{i.} - \bar{Y} - \alpha_i)^2 + \sum_i \sum_i (Y_{ij} - Y_{i.})^2.$$

From (1) and (2) and this expression for $S(\mathbf{Y}, \boldsymbol{\beta})$ we can immediately write the LR-test statistic for testing $H_{0,\alpha}$ against $H_{1,\alpha}$ as

$$\bar{B} = (SS_\alpha - SAS_\alpha)/(SS_E + SS_\alpha). \tag{5}$$

When $J_1 = \cdots = J_I = J$ this statistic has, by Theorem 1, the $\bar{B}[I, IJ]$ distribution. For unequal sample sizes $\bar{B}$ has the $\bar{B}[J_1, \cdots, J_I; I, n]$ distribution. (Apply Theorem 1 with $k = I$, $n_i = J_i$, $\bar{Z}_i = Y_{i.} - \mu$, Case 0, $\xi_i = \alpha_i$, $T = SS_E$ and $\nu = n - I$.)

If σ^2 is known the LR-test statistic for testing $H_{0,\alpha}$ against $H_{1,\alpha}$ is

$$\bar{D} = (SS_\alpha - SAS_\alpha)/\sigma^2$$

and by Theorem 1 this statistic has a $\bar{D}[J_1, \cdots, J_I; I]$ distribution.

6. Extension to other complete layouts. It is now clear that for other complete layouts the LR-test statistic for testing $H_{0,\alpha}$ against $H_{1,\alpha}$ (the manner in which $\alpha_1, \cdots, \alpha_I$, satisfying $\alpha. = 0$, enter into the model for the complete layout is assumed clear) is given by

$$\bar{B} = (SS_\alpha - SAS_\alpha)/(SS_E + SS_\alpha).$$

Moreover, an application of Theorem 1 will still show $\bar{B}$ to have a $\bar{B}[n_1, \cdots, n_I; I, \nu + I]$ distribution for appropriate choices of $n_1, \cdots, n_I$ and for ν equal to the degrees of freedom of SS_E.

7. Derivation of results for the Latin square. Consider the following model for the Latin square design. Let

$$Y_{ij(k)} = \mu + \alpha_i + \beta_i + \gamma_k + e_{ij(k)} \qquad (i, j, k = 1, \cdots, n)$$

where $\alpha. = \beta. = \gamma. = 0$ and the $e_{ij(k)}$'s are iid $N(0, \sigma^2)$. We write $ij(k)$ to denote the fact that values for any two of the indices determine the value for the third. Now

$$\begin{aligned} S(\mathbf{Y}, \boldsymbol{\beta}) = {} & n^2(Y_{..(.)} - \mu)^2 + n \textstyle\sum_i (Y_{i.(.)} - Y_{..(.)} - \alpha_i)^2 \\ & + n \textstyle\sum_j (Y_{.j(.)} - Y_{..(.)} - \beta_j)^2 + n \textstyle\sum_k (Y_{..(k)} - Y_{..(.)} - \gamma_k)^2 \\ & + \textstyle\sum_i \sum_j (Y_{ij(k)} - Y_{i.(.)} - Y_{.j(.)} - Y_{..(k)} + 2Y_{..(.)})^2. \end{aligned}$$

Again from (1), (2) and the expression for $S(\mathbf{Y}, \boldsymbol{\beta})$ we can write the LR-test statistic for testing $H_{0,\alpha}$ against $H_{1,\alpha}$ as

$$\bar{B} = (SS_\alpha - SAS_\alpha)/(SS_E + SS_\alpha). \tag{6}$$

By Theorem 1 this statistic has a $\bar{B}[n, (n-1)^2 + 1]$ distribution under the null hypothesis. (Apply Theorem 1 with $k = n$, $n_i = n$, $\bar{Z}_i = Y_{i.(.)} - \mu$, Case 0, $\xi_i = \alpha_i$, $T = SS_E$ and $\nu = (n-1)(n-2)$.)

Suppose now that we have performed the LR-test for $H_{0,\alpha}$ against $H_{1,\alpha}$ and that the test accepted $H_{0,\alpha}$. Suppose we now wish to test $H_{0,\alpha} \cap H_{0,\beta}$ against $H_{0,\alpha} \cap H_{1,\beta}$. Again from (1), (2) and the expression for $S(\mathbf{Y}, \boldsymbol{\beta})$ we immediately write

$$\bar{B} = (SS_\beta - SAS_\beta)/(SS_E + SS_\alpha + SS_\beta). \tag{7}$$

By Theorem 1 this statistic has a $\bar{B}[n, (n-1)^2 + n]$ distribution under the null hypothesis. (Apply Theorem 1 with $k = n$, $n_i = n$, $\bar{Z}_j = Y_{.j(.)} - \mu$, Case 0, $\xi_j = \beta_j$, $T = SS_E + SS_\alpha$ and $\nu = (n-1)(n-2) + (n-1) = (n-1)^2$.) By decomposing the sample space into $\mathfrak{X}(t_1, \cdots, t_m)$'s for both statistics and by using the independence of the sum and ratio of independent chi-squares, it is easy to show that the test statistics (6) and (7) are independent, i.e., we have independent tests for the nested hypotheses.

Similarly if we perform the LR-test of $H_{0,\alpha}$ against $H_{1,\alpha}$ using (6) and then perform the LR-test of $H_{0,\beta} \cap H_{0,\beta}$ against $H_{0,\alpha} \cap H_{2,\beta}$ using $\bar{B} = SS_\beta/(SS_E + SS_\alpha + SS_\beta)$ the two tests would be independent.

(Finally we remark that the independence indicated here also holds for the models of Section 6.)

III. NONPARAMETRIC

8. The one-way layout. In the one-way layout of Section 5 assume only that the e_{ij}'s are iid random variables having continuous cdf F. Chacko (1963) proposed the following adaptation of the Kruskal and Wallis (1952) test. Replace the observations by their overall ranks, apply the amalgamation process to the average column ranks $\bar{R}_i$, and reject $H_{0,\alpha}$ in favor of $H_{1,\alpha}$ for large values of

$$\bar{H} = (12/(n(n+1))) \textstyle\sum_{j=1}^{m} N_{[t_j]} (\bar{R}_{[t_j]} - (n+1)/2)^2. \tag{8}$$

Chacko was able to derive the distribution theory of $\bar{H}$ only when $J_1 = \cdots = J_I$. However, the obvious asymptotic version of Theorem 1 with $k = I$, $n_i = J_i$, $\bar{Z}_i = (12^{\frac{1}{2}}/n)(\bar{R}_i - (n+1)/2)$, Case 1, and $\sigma^2 = 1$ shows that $\bar{H}$ is asymptotically $\bar{D}[J_1, \cdots, J_I; I]$. Chacko's computation of Pitman efficiency carries over to this case.

9. The two-way layout with one observation per cell. Let $Y_{ji} = \mu + \alpha_i + \beta_j + U_{ji}$ ($i = 1, \cdots, I$ and $j = 1, \cdots, J$) where $\alpha. = \beta. = 0$ and the U_{ji}'s are iid random variables having continuous cdf F. We propose the following modification of Friedman's (1937) χ_r^2-test.

STEP 1. Replace each observation Y_{ji} by r_{ji}, its rank in the jth row. Let $\bar{R}_i = (1/J) \sum_j r_{ji}$.

STEP 2. Apply the amalgamation process to the $\bar{R}_i$'s to obtain m distinct quantities $\bar{R}_{[t_1]}, \cdots, \bar{R}_{[t_m]}$.

STEP 3. Reject $H_{0,\alpha}$ in favor of $H_{1,\alpha}$ for large values of the statistic

$$\bar{\chi}_r^2 = (12J/(I(I+1))) \textstyle\sum_{j=1}^{m} t_j(\bar{R}_{[t_j]} - (I+1)/2)^2. \tag{9}$$

Applying the asymptotic version of Theorem 1 with $k = I$, $n_i = J$, $\bar{Z}_i = (12J/(I(I+1)))^{\frac{1}{2}}(\bar{R}_i - (I+1)/2)$, Case 1 and $\sigma^2 = 1$ we find that asymptotically $\bar{\chi}_r^2$ has the $\bar{D}[I]$ distribution.

Using the method of Chacko (1963), it is easy to show that the Pitman asymptotic relative efficiency of the $\bar{\chi}_r^2$-test with respect to the $\bar{B}$-test for the usual translation alternatives (see Andrews (1954)) is $(I/(I+1))12\sigma_F^2 \cdot [\int F'(t)\, dF(t)]^2$.

IV. EXTENSIONS MISCELLANEOUS

10. Partially ordered alternatives. Let H_ξ^* denote an alternative hypothesis whose order restrictions specify a partial ordering of $\xi_1, \cdots, \xi_k$; for example $\xi_1 \leqq \xi_2$; $\xi_1 \leqq \xi_3$; $\xi_2 \leqq \xi_4$; $\xi_3 \leqq \xi_4$; $\xi_5, \cdots, \xi_k$. Let

$$SGAS_\xi = \inf_{(\xi_1, \cdots, \xi_k) \varepsilon H_\xi^*} \textstyle\sum_{i=1}^{k} n_i(\bar{Z}_i - \xi_i)^2.$$

We will call the process by which the minimizing $\hat{\xi}_i$'s are obtained the "generalized amalgamation process"; and $SGAS_\xi$ we will call the "sum of generalized amalgamated squares for ξ. We define $p^*_{m,k}(n_1, \cdots, n_k)$ to be the probability that m distinct $\hat{\xi}_i$'s result when the generalized amalgamation process is applied to the $\bar{Z}_i$'s; note that $p^*_{m,k}(n_1, \cdots, n_k)$ depends on H_ξ^*. $\bar{D}^*[n_1, \cdots, n_k; k]$ and $\bar{B}^*[n_1, \cdots, n_k; k, N]$ distributions are obtained by putting *'s on the probabilities in our old definitions.

We now present a result of Van Eeden (1958) in the form it is presented by Bartholomew (1961a). The result requires a distinction between *essential* and *inessential* restrictions; essential restrictions being those that remain when all those which are redundant are struck out. Let the s essential restrictions be labeled $R_1, \cdots, R_s$ where, without loss of generality, we take R_s to be $\xi_1 \leqq \xi_2$.

THEOREM (*van Eeden*) *If* $\bar{H}_\xi^*$ *denotes the set of essential restrictions* $R_1, \cdots, R_{s-1}$ *and if* $\hat{\xi}_1', \cdots, \hat{\xi}_k'$ *is the point where* $\sum_{i=1}^k n_1(\bar{Z}_i - \xi_i)^2$ *is minimized subject to* $\bar{H}_\xi^*$, *then*

$$\text{(i)} \qquad \hat{\xi}_i = \hat{\xi}_i' \ (i = 1, \cdots, k) \qquad \textit{if } \hat{\xi}_1' \leqq \hat{\xi}_2'$$

$$\text{(ii)} \qquad \hat{\xi}_1 = \hat{\xi}_2 \qquad \textit{if } \hat{\xi}_1' > \hat{\xi}_2'.$$

The effect of this theorem is to locate the minimum either in H_ξ^* or on a boundary of H_ξ^* defined by inequalities among the ξ_i's. The problem is then solved by obtaining the unrestricted minimum of $\sum_{k=1}^{i} n_i(\bar{Z}_i - \xi_i)^2$ on that boundary. This result shows that the minimizing $\hat{\xi}_i$'s are of the form $(n_{i_1}\bar{Z}_{i_1} + \cdots + n_{i_1}\bar{Z}_{i_s})/(n_{i_1} + \cdots + n_{i_s})$ for appropriate choices of $s, i_1, \cdots, i_s$ it also allows us to use a step by step method which stops as soon as the resulting averages satisfy certain inequalities; and note that the result of this process is independent of the order of the steps.

THEOREM 2. *Under the hypothesis of Theorem* 1, *both Case* 0 *and Case* 1, $(Q - SGAS_\xi)/\sigma^2$ *has the* $\bar{D}^*[n_1, \cdots, n_k; k]$ *distribution and* $(Q - SGAS_\xi)/(T + Q)$ *has the* $\bar{B}^*[n_1, \cdots, n_k; k, \nu + k]$ *distribution.*

PROOF. We now let $\mathfrak{X}(t_1, \cdots, t_m)$ denote the region in the sample space where the generalized amalgation process yields

$$\hat{\xi}_{i_{\tau_j}+1} = \cdots = \hat{\xi}_{i_{\tau_{j+1}}} = (n_{i_{\tau_j}+1}\bar{Z}_{i_{\tau_j}+1} + \cdots + n_{i_{\tau_{j+1}}}\bar{Z}_{i_{\tau_{j+1}}})/$$
$$(n_{i_{\tau_j}+1} + \cdots + n_{i_{\tau_{j+1}}}),$$

$j = 0, 1, \cdots, m - 1$, for some permutation $(i_1, \cdots, i_k)$ of $(1, \cdots, k)$; and let $SS(t_1, \cdots, t_m)$ be the corresponding fixed sum of squares. The proof of Theorem 1 may be recopied once we show that the indicator function of the region $\mathfrak{X}(t_1, \cdots, t_m)$ and the function $Q - SS(t_1, \cdots, t_m)$ are independent random variables in Case 0 and 1 and that the $p_{m,k}(n_1, \cdots, n_k)$'s have identical values for Cases 0 and 1 in the present situation. We use the van Eeden theorem of this section and induction on the number s of essential restrictions to establish this independence and equality of probabilities. For $s = 0$ $\mathfrak{X}(t_1, \cdots, t_m)$ is the whole sample space and so independence is trivial and $p_{k,k}(n_1, \cdots, n_k) = 1$ in both Cases 0 and 1. We now assume that the independence and equality of probabilities hold for the case of $s - 1$ or fewer essential restrictions and proceed to infer that they hold for s essential restrictions. We are still taking R_s as $\xi_1 \leqq \xi_2$. Now either

(i) $\hat{\xi}_1' \leqq \hat{\xi}_2'$ or

(ii) $\hat{\xi}_1' > \hat{\xi}_2'$;

where these still denote the minimizing values under $\bar{H}_\xi^*$. Each $\mathfrak{X}(t_1, \cdots, t_m)$ representing case (i) is defined by a set of inequalities among $\bar{Z}_1, \cdots, \bar{Z}_k$ which are appropriate for a problem with $s - 1$ essential restrictions (and which just happen to imply $\hat{\xi}_1' \leqq \hat{\xi}_2'$) and to this case the inductive hypothesis applies directly yielding the desired results. Now consider an $\mathfrak{X}(t_1, \cdots, t_n)$ representing case (ii). In this case the van Eeden theorem tells us $\hat{\xi}_1 = \hat{\xi}_2$. Also

$$n_1(\bar{Z}_1 - \xi_1)^2 + n_2(\bar{Z}_2 - \xi_2)^2 = (n_1 + n_2)(\bar{Z}_{[1,2]} - \xi)^2 + \text{terms not involving } \xi$$

where $\xi = \xi_1 = \xi_2$. Thus $\mathfrak{X}(t_1, \cdots, t_m)$ can be described by a set of inequalities among $\bar{Z}_{[1,2]}, \bar{Z}_3, \cdots, Z_k$ that correspond to a problem having at most $s - 1$ essential restrictions. (The example at the beginning of this section shows why we say at most $s - 1$ instead of simply $s - 1$; since letting $\xi_1 = \xi_2$ cause two essential restrictions to be removed in that example.) When case α, α equal 0 or 1, is true, the covariance matrix of $\bar{Z}_{[1,2]}, \bar{Z}_3, \cdots, \bar{Z}_k$ is a $(k - 1)$ by $(k - 1)$ matrix of the type considered in case α. Applying the inductive hypothesis gives the desired results. The proof is completed. We acknowledge that it imitates the proof of Bartholomew (1961a). It is rephrased so as to be in terms of the regions $\mathfrak{X}(t_1, \cdots, t_m)$; and it proves three facts by induction (independence in two cases and an equality of probabilities) rather than one (an independence).

Duplicating the work of earlier sections shows that the LR-test of $H_{0,\alpha}$ against H_α^* under the normal theory models rejects H_α^* if

$$\bar{B}^* = (SS_\alpha - SGAS_\alpha)/(SS_E + SS_\alpha)$$

is too large. Let $\bar{H}^*$ and $\bar{\chi}_r^{2*}$ denote the obvious rank analogs of these test statistics in the one and two way layants. Then by Theorem 2, $\bar{B}^*$ has the $\bar{B}^*[n_1, \cdots, n_k; I, \nu + I]$ distribution, where ν is the number of degrees of freedom of SS_E; while $\bar{H}^*$ and $\bar{\chi}_r^{2*}$ are asymptotically distributed as $\bar{D}^*[J_1, \cdots, J_I; I]$ and $\bar{D}^*[I]$ respectively. Pitman are's of these rank tests with respect to the appropriate $\bar{B}^*$-test are the usual values. The distribution theory for all these statistics hinges on determination of the $p_{m,I}^*(n_1, \cdots, n_I)$'s. This problem is discussed in Bartholomew (1961a) where a partial solution is given. Finally, we remark that tests of $H_{0,\alpha}$ against H_α^* and of $H_{0,\alpha} \cap H_{0,\beta}$ against $H_{0,\alpha} \cap H_\beta^*$ are independent.

11. Asymptotically nonparametric testing against ordered alternatives. Follow Lehmann's (1963) approach to find nonparametric estimates of the components of $\boldsymbol{\beta}$ (see the model of our Section 3) based on the Mann-Whitney test statistic. Instead of replacing the normal theory estimates of the components of $\boldsymbol{\beta}$ in the classic normal theory statistic by the new estimates as Lehmann does, first apply the amalgamation process to the appropriate components of $\boldsymbol{\beta}$ and then use these to replace the corresponding quantities in the statistics of our Sections 4–7. Just as Lehmann's test yields a Pitman asymptotic relative efficiency of $12\sigma^2[\int f^2]^2$ with respect to the classical normal theory test, so the test just proposed should have Pitman are of $12\sigma^2[\int f^2]^2$ with respect to the appropriate test of Sections 4–7.

12. Relation to a paper of Nuesch. A knowledge of Nuesch (1966) is assumed in this section. Let $\hat{\mathbf{u}}(C)$ denote that unique vector satisfying $(\hat{\mathbf{u}}(C) - \bar{\mathbf{X}})'C \geqq 0$ and $\min(\hat{\mathbf{u}}(C), (\hat{\mathbf{u}}(C) - \bar{\mathbf{X}})'C) = 0$; see Lemma 1.1 and Equation (2.5) of Nuesch. The quantity $\hat{\mathbf{u}}$ of Nuesch's equation (3.2) is really $\hat{\mathbf{u}}(C)$; i.e. it is a function of C. Thus Nuesch's (3.4) is false; which renders his Theorem 3.1 false. Since he cannot compute the exact distribution of his statistic (3.5) in which $\hat{\mathbf{u}}$ in reality denotes $\hat{\mathbf{u}}(\Sigma^{-1})$, he derives the distribution of $N\hat{\mathbf{u}}(\Sigma^{-1})'\cdot A^{-1}\hat{\mathbf{u}}(\Sigma^{-1})$ in his Theorem 3.3. However, this quantity is unobservable. One is really interested in the statistic $N\hat{\mathbf{u}}(A^{-1})'A^{-1}\hat{\mathbf{u}}(A^{-1})$; and its distribution theory is still unknown.[1]

However, if one assumes $\Sigma = \sigma^2\Sigma_0$ with Σ_0 known and σ^2 unknown one may use Nuesch's technique to prove the following.

THEOREM 3. *The LR test for testing H vs. K_0 rejects H if*

$$\bar{B} = N\hat{\mathbf{u}}(\Sigma_0^{-1})'\Sigma_0^{-1}\hat{\mathbf{u}}(\Sigma_0^{-1})/\textstyle\sum_{\alpha=1}^{N}\mathbf{X}_\alpha'\Sigma_0^{-1}\mathbf{X}_\alpha \geqq c^2.$$

Moreover,

$$P(\bar{B} \geqq c^2) = \textstyle\sum_{k=1}^{p}\omega(p, k)P[B(k/2, (Np-k)/2) \geqq c^2]$$

where the weights $\omega(p, k)$ are computed for Σ_0.

PROOF. In Nuesch's notation

$$\begin{aligned} P(S^c \mid \mathfrak{X}_k) &= P(N\hat{\mathbf{u}}(\Sigma_0^{-1})'\Sigma_0^{-1}\mathbf{u}(\Sigma_0^{-1})/(\operatorname{trace}(\Sigma_0^{-1}\textstyle\sum_{\alpha=1}^{N}\mathbf{X}_\alpha\mathbf{X}_\alpha')) \geqq c^2 \mid \mathfrak{X}_k) \\ &= P(B(k/2, (Np-k)/2) \geqq c^2). \end{aligned}$$

Theorem 3 above and Nuesch's Theorem 2.1 overlap our Theorem 1. However, they do not overlap Case 1 of Theorem 1 since the $\omega(p, k)$ have not previously been evaluated for this case, they do not apply in the case of unequal sample sizes, and they unnecessarily obscure the simple nature of the derivation of the LR tests in the case of greatest interest when $\Sigma = \sigma^2 I$.

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