The Annals of Mathematical Statistics
1968, Vol. 39, No. 1, 261–262

A NOTE ON THE ABSENCE OF TANGENCIES IN GAUSSIAN SAMPLE PATHS[1]

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Let $Y(t)$, $t \varepsilon [0, 1]$, be a Gaussian process having continuous sample paths with probability one, and let the mean and standard deviation of $Y(t)$ be denoted by $m(t)$ and $\sigma(t)$, respectively. In this Gaussian setting, quadratic mean continuity follows continuity with probability one and hence the functions m and σ, as specified above, are continuous on [0, 1]. It is assumed that $\min_{0 \leqq t \leqq 1} \sigma(t) > 0$.

Suppose u is a fixed continuous function on [0, 1]. We will say Y is somewhere tangent to u if there is an open interval $I \subset [0, 1]$ on which $Y - u$ has a constant sign and a $t \varepsilon I$ for which $Y(t) - u(t) = 0$ (intervals of the form $[0, a)$ and $(b, 1]$ are taken as open). It will be convenient to think of tangencies as being from above or from below, from above being associated with $Y - u$ non-negative on I. The purpose of this note is to show that Y is somewhere tangent to u with probability zero. This is known for stationary Gaussian processes with $m \equiv 0$, $u \equiv 0$ and $\sigma \not\equiv 0$, [2]. A simple modification of the proof in [2] shows it true for $u - m$ and σ constant functions, $\sigma \not\equiv 0$. The result is also known when, essentially, Y has a quadratic mean derivative at each $t \varepsilon [0, 1]$, $\min_{0 \leqq t \leqq 1} \sigma(t) > 0$, and u is continuously differentiable [1], pg. 289. After a reduction of the problem, much reliance is placed on the method of proof used in [2].

Consider a new process defined by

$$X(t) = [Y(t) - u(t)]/\sigma(t) + \lambda, \qquad t \varepsilon [0, 1],$$

with λ a constant to be fixed. $X(t)$, $t \varepsilon [0, 1]$, is a Gaussian process having continuous sample paths with probability one and

$$\bar{m}(t) = EX(t) = [m(t) - u(t)]/\sigma(t) + \lambda, \qquad \bar{\sigma}(t) = [E(X(t) - \bar{m}(t))^2]^{\frac{1}{2}} \equiv 1.$$

Let λ be chosen so large that $\bar{m}$ is non-negative on [0, 1]. Evidently tangencies of Y to u from above (below) are reflected in tangencies of X to the constant function λ from above (below).

We show X is somewhere tangent to λ from below with probability zero. This follows by first noting that some I in the definition of a tangency may be replaced by some I from among a fixed countable collection of intervals (see the remark above Lemma 1 of [2]). Second, for any given I, X is tangent to λ from below on I with probability zero. For, such an event implies that $\sup_I X(\cdot) = \lambda$ and we have the

LEMMA. $\sup_I X(\cdot)$ *has a continuous distribution for any open interval* $I \subset [0, 1]$.

PROOF. It is first claimed that if $T = \{t_1, \cdots, t_n\}$ is a finite set drawn from

Received 25 May 1967.

[1] Research sponsored by NSF Grant G-25211.

[0, 1], then $\max_T X(\cdot)$ has a density of the form φG_T where φ is the standard normal density and G_T is a non-decreasing function. For, we may as well assume that no two of these n variables are equal with probability one and then, the density of $\max_T X(\cdot)$ may be written as

$$\sum_{j=1}^{n} (2\pi)^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}(x - \bar{m}(t_j))^2\} P[X(t_i) \leqq x \text{ for all } i \neq j \mid X(t_j) = x]$$
$$= \varphi(x) \sum_{j=1}^{n} \exp\{x\bar{m}(t_j) - \tfrac{1}{2}\bar{m}^2(t_j)\} G_{t_j}(x).$$

It may be verified that each G_{t_j} is a non-decreasing function of x (the details of such an argument are given in the proof of Lemma 1 of [2]). But, since $\bar{m}$ is a non-negative function, the density of $\max_T X(\cdot)$ has the form φG_T with G_T non-decreasing. If $\{T_n\}$ is a sequence of finite sets becoming dense in an open interval $I \subset [0, 1]$, the conclusion of the present lemma can be false only if $G_{T_n}(x_0)$ becomes unbounded for some x_0. However this cannot be since

$$\left(\int_{x_0}^{\infty} \varphi(x)\, dx\right) G_{T_n}(x_0) \leqq \int_{x_0}^{\infty} \varphi(x) G_{T_n}(x)\, dx \leqq 1.$$

In view of the preceding lemma and the remarks prior to it, we have the

THEOREM. *Let $Y(t)$, $t \in [0, 1]$, be a Gaussian process having continuous sample paths with probability one. If $E(Y(t) - EY(t))^2 > 0$ for all $t \in [0, 1]$, then for any fixed continuous function u on $[0, 1]$, $P\{Y$ is somewhere tangent to $u\} = 0$.*

The above result applies as well to Gaussian processes over more general domains, for example if t runs over a compact metric space.

REFERENCES

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[2] YLVISAKER, N. DONALD (1965). The expected numbersof zeros of a stationary Gaussian process. *Ann. Math. Statist.* **36** 1043–1046.