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ON THE NON-CENTRAL DISTRIBUTIONS OF TWO TEST CRITERIA IN MULTIVARIATE ANALYSIS OF VARIANCE

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1. Introduction and summary. Let $\mathbf{X}$ be a $p \times f_2$ matrix variate ($p \leqq f_2$) and $\mathbf{Y}$ a $p \times f_1$ matrix variate ($p \leqq f_1$) and the columns be all independently normally distributed with covariance matrix $\mathbf{\Sigma}$, $E(\mathbf{X}) = \mathbf{M}$ and $E(\mathbf{Y}) = \mathbf{0}$. Let $0 < l_1 \leqq \cdots \leqq l_p < 1$ be the ordered characteristic roots of

$$(1.1) \qquad |\mathbf{XX}' - l(\mathbf{YY}' + \mathbf{XX}')| = 0$$

and $\omega_1, \cdots, \omega_p$ those of

$$|\mathbf{MM}' - \omega\mathbf{\Sigma}| = 0,$$

then the joint density function of $l_1, \cdots, l_p$ is given by Constantine [1], James [2] in the form

$$(1.2) \qquad \exp\left(-\tfrac{1}{2}\operatorname{tr}\mathbf{\Omega}\right) {}_1F_1(\tfrac{1}{2}\nu; \tfrac{1}{2}f_2; \tfrac{1}{2}\mathbf{\Omega}, \mathbf{L}) f_0(l_1, \cdots, l_p),$$

where

$$(1.3) \qquad f_0(l_1, \cdots, l_p) = C(p, f_1, f_2) \prod_{i=1}^{p} \{l_i^{\frac{1}{2}(f_2-p-1)}(1 - l_i)^{\frac{1}{2}(f_1-p-1)}\}\alpha_p(\mathbf{L}),$$
$$\mathbf{L} = \mathbf{X}'(\mathbf{YY}' + \mathbf{XX}')^{-1}\mathbf{X}, \qquad \mathbf{\Omega} = \mathbf{M}'\mathbf{\Sigma}^{-1}\mathbf{M}, \qquad \nu = f_1 + f_2,$$

${}_1F_1$ is the hypergeometric function of matrix argument defined in [2] given by

$$\sum_{k=0}^{\infty} \sum_{\kappa} (\tfrac{1}{2}\nu)_\kappa C_\kappa(\tfrac{1}{2}\mathbf{\Omega}) C_\kappa(\mathbf{L}) / (\tfrac{1}{2}f_2)_\kappa C_\kappa(\mathbf{I}_p) k\,!$$

and where

$$C(p, f_1, f_2) = \pi^{\frac{1}{2}p^2}\Gamma_p(\tfrac{1}{2}\nu)/\{\Gamma_p(\tfrac{1}{2}f_1)\Gamma_p(\tfrac{1}{2}f_2)\Gamma_p(\tfrac{1}{2}p)\}$$

and

$$\alpha_p(\mathbf{L}) = \prod_{i>j}(l_i - l_j).$$

In this paper the distribution of Pillai's $V^{(p)}$ criterion which is the trace of $\mathbf{L}$, [5], [6] and that of Roy's largest root criterion, l_p, [8], [10], have been obtained in series forms and certain constants involved in the series tabulated. In addition, the first four moments of $V^{(p)}$ are also obtained in the linear case, illustrating further use of some of the tabulations.

2. Distribution of $V^{(p)}$. First a lemma is proved which will be used later in the section.

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LEMMA 1. *If A_κ denotes the integral*

$$\int_{\mathfrak{D}} |\mathbf{Z}|^{\frac{1}{2}(f_2-p-1)}\alpha_p(\mathbf{Z})C_\kappa(\mathbf{Z})\, dz_1, \cdots, dz_{p-1}, \tag{2.1}$$

where

$$\mathbf{Z} = \operatorname{diag}(z_1, \cdots, z_p), \qquad z_p = 1 - z_1 - \cdots - z_{p-1}$$

and $\mathfrak{D}$ *is given by*

$$\mathfrak{D}: \{0 < z_1 < z_2 < \cdots < z_{p-2} < z_{p-1} < z_p = 1 - z_1 - \cdots - z_{p-2} - z_{p-1}\}, \tag{2.2}$$

then

$$A_\kappa = \{(\tfrac{1}{2}f_2)_\kappa\Gamma_p(\tfrac{1}{2}f_2)\Gamma_p(\tfrac{1}{2}p)C_\kappa(\mathbf{I}_p)\}/\{\pi^{\frac{1}{2}p^2}(\tfrac{1}{2}f_2p)_k\Gamma(\tfrac{1}{2}f_2p)\}. \tag{2.3}$$

PROOF. Consider the null distribution of $T = \sum_{i=1}^{p} c_i$, $c_i = l_i/(1 - l_i)$, $i = 1, \cdots, p$, given by Constantine [1] in the form

$$\Gamma_p(\tfrac{1}{2}\nu)\{\Gamma(\tfrac{1}{2}f_2p)\Gamma_p(\tfrac{1}{2}f_1)\}^{-1}T^{\frac{1}{2}f_2p-1}\sum_{k=0}^{\infty}(-T)^k\{k!\,(\tfrac{1}{2}f_2p)_k\}^{-1} \cdot \sum_\kappa (\tfrac{1}{2}\nu)_\kappa(\tfrac{1}{2}f_2)_\kappa C_\kappa(\mathbf{I}_p), \tag{2.4}$$

for any $|T| < 1$. Now, from $f_0(l_1, \cdots, l_p)$ in (1.3), the joint density of $c_1, \cdots, c_p$ can be obtained as

$$C(p, f_1, f_2)\prod_{i=1}^{p}\{c_i^{\frac{1}{2}(f_2-p-1)}(1 + c_i)^{-\frac{1}{2}\nu}\}\alpha_p(\mathbf{C}), \quad 0 < c_1 \leqq \cdots \leqq c_p < \infty, \tag{2.5}$$

where $\mathbf{C} = \operatorname{diag}(c_1, \cdots, c_p)$. Expanding

$$\prod_{i=1}^{p}(1 + c_i)^{-\frac{1}{2}\nu} = \sum_{k=0}^{\infty}(-1)^k\sum_\kappa(\tfrac{1}{2}\nu)_\kappa C_\kappa(\mathbf{C})/k!, \tag{2.6}$$

transforming $z_i = c_i/T$ and integrating z_i's in the region $\mathfrak{D}$ given in (2.2) we get

$$C(p, f_1, f_2)T^{\frac{1}{2}f_2p-1}\sum_{k=0}^{\infty}(-T)^k(k!)^{-1}\sum_\kappa(\tfrac{1}{2}\nu)_\kappa A_\kappa, \tag{2.7}$$

where A_κ is given in (2.1). Hence equating the coefficients of $(\frac{1}{2}\nu)_\kappa$ in (2.4) and (2.7) we get (2.3). Hence the lemma.

Now consider the distribution of $l_1, \cdots, l_p$ in (1.2). First note that

$$\begin{aligned}|\mathbf{I} - \mathbf{L}|^{\frac{1}{2}(f_1-p-1)}C_\kappa(\mathbf{L}) &= \sum_{n=0}^{\infty}\sum_\eta(\tfrac{1}{2}(p + 1 - f_1))_\eta C_\eta(L)\mathbf{C}_\kappa(\mathbf{L})/n! \\ &= \sum_{n=0}^{\infty}\sum_\eta\sum_\delta(\tfrac{1}{2}(p + 1 - f_1))_\eta g_{\kappa,\eta}^{\delta}C_\delta(\mathbf{L})/n!\end{aligned} \tag{2.8}$$

where $g_{\kappa,\eta}^{\delta}$ is the coefficient of $C_\delta(\mathbf{L})$ in the product $C_\kappa(\mathbf{L})C_\eta(\mathbf{L})$, $\delta = (\delta_1, \cdots, \delta_p)$, $\delta_1 \geqq \cdots \geqq \delta_p \geqq 0$ and $\sum_{i=1}^{p}\delta_i = n + k = d$. Using (2.8) in (1.2) and integrating $l_1, \cdots, l_p$ over the surface $\sum_{i=1}^{p} l_i = V^{(p)}$, we get the density of $V^{(p)}$ as

$$C(p, f_1, f_2)\exp(-\tfrac{1}{2}\operatorname{tr}\mathbf{\Omega})\sum_{k=0}^{\infty}\sum_\kappa\sum_{\eta=0}^{\infty}\sum_\eta\sum_\delta(\tfrac{1}{2}\nu)_\kappa(\tfrac{1}{2}(p + 1 - f_1))_\eta \cdot C_\kappa(\tfrac{1}{2}\mathbf{\Omega})g_{\kappa,\eta}^{\delta}A_\delta\{V^{(p)}\}^{\frac{1}{2}f_2p+d-1}/\{(\tfrac{1}{2}f_2)_\kappa C_\kappa(\mathbf{I}_p)k!n!\}, \tag{2.9}$$

where A_δ is defined by (2.1) whose value is obtained from (2.3) by putting $\kappa = \delta$. The series (2.9) is convergent for $0 < V^{(p)} < 1$. Tabulations of the g-coefficients are presented in Table 1 for various values of the three arguments.

3. The moments of $V^{(p)}$ in the linear case. Consider now the evaluation of the moments of $V^{(p)}$ in the linear case, i.e. when there is only one non-zero population root, say, ω. This will also serve to illustrate further uses of Table 1. For this, as the authors have shown in a previous paper, [3], we may write

$$(3.1)\qquad \begin{aligned} V^{(p)} - 1 &= \operatorname{tr} \mathbf{L}_{22} - (1 - l_{11})[(1 - \mathbf{u}'\mathbf{u}) + \mathbf{u}'\mathbf{L}_{22}\mathbf{u}] \\ &= \textstyle\sum_{i=1}^{p-1} c_i' - (1 - l_{11})[(1 - \mathbf{u}'\mathbf{u}) + \sum_{i=1}^{p-1} c_i' u_i^2], \end{aligned}$$

where l_{11}, $\mathbf{u}:(p - 1 \times 1)$ and $\mathbf{L}_{22}:(p - 1) \times (p - 1)$ are independently distributed and whose distributions are given in [3], [4], and c_i' $(i = 1, \cdots, p - 1)$ are the characteristic roots of $\mathbf{L}_{22}$. It may be pointed out that the density of l_{11} alone involves ω. Now let $l_{11,0}$ be a variate whose distribution is the same as that of l_{11} when $\omega = 0$ and $V_0^{(p)}$ be the corresponding $V^{(p)}$ statistic. We may note that

$$(3.2)\qquad x_1 = E(1 - l_{11,0}) - E(1 - l_{11}) = f_1\delta(\nu),$$

$$(3.3)\qquad x_2 = E(1 - l_{11,0})^2 - E(1 - l_{11})^2 = \tfrac{1}{2}f_1(f_1 + 2)\Delta_1,$$

$$(3.4)\qquad x_3 = E(1 - l_{11,0})^3 - E(1 - l_{11})^3 = \tfrac{1}{8}f_1(f_1 + 2)(f_1 + 4)\Delta_2,$$

and

$$(3.5)\qquad x_4 = E(1 - l_{11,0})^4 - E(1 - l_{11})^4 = \tfrac{1}{48}f_1(f_1 + 2)(f_1 + 4)(f_1 + 6)\Delta_3,$$

where

$$(3.6)\qquad \delta(\nu) = [\omega/2\nu] \exp(-\tfrac{1}{2}\omega) \textstyle\sum_{i=0}^{\infty} \{(\tfrac{1}{2}\omega)^i/[i!(\tfrac{1}{2}\nu + i + 1)]\},$$

$$(3.7)\qquad \begin{aligned} &\Delta_1 = \delta(\nu) - \delta(\nu + 2), \quad \Delta_2 = \delta(\nu) - 2\delta(\nu + 2) + \delta(\nu + 4) \\ &\text{and} \quad \Delta_3 = \delta(\nu) - 3\delta(\nu + 2) + 3\delta(\nu + 4) - \delta(\nu + 6). \end{aligned}$$

Now from (3.1) we can write

$$(3.8)\qquad E(V^{(p)} - 1) = E(V_0^{(p)} - 1) + x_1E(\beta),$$

$$(3.9)\qquad E(V^{(p)} - 1)^2 = E(V_0^{(p)} - 1)^2 - x_2E(\beta^2) + 2x_1E(\beta\alpha),$$

$$(3.10)\qquad E(V^{(p)} - 1)^3 = E(V_0^{(p)} - 1)^3 + \textstyle\sum_{i=1}^{3} \binom{3}{i}(-1)^{i-1}E(\beta^i\alpha^{3-i})x_i,$$

$$(3.11)\qquad E(V^{(p)} - 1)^4 = E(V_0^{(p)} - 1)^4 + \textstyle\sum_{i=1}^{4} \binom{4}{i}(-1)^{i-1}E(\beta^i\alpha^{4-i})x_i$$

where

$$(3.12)\qquad \alpha = \operatorname{tr} \mathbf{L}_{22} \quad \text{and} \quad \beta = 1 - \mathbf{u}'\mathbf{u} + \mathbf{u}'\mathbf{L}_{22}\mathbf{u}.$$

Now, the expected values on the right side of (3.8)–(3.11) which are the coefficients of the x's have been obtained in terms of functions of $\operatorname{tr}_i \mathbf{L}_{22}$ i.e. the ith elementary symmetric function in the characteristic roots, $c_1', \cdots, c_{p-1}'$,

TABLE 1

Values of $g^{\kappa}_{\alpha,\beta}$

$C_\alpha(\cdot)$	$C_\beta(\cdot)$	$k=2$		$k=3$			$k=4$				
		$C_{(2)}$	$C_{(1^2)}$	$C_{(3)}$	$C_{(2\,1)}$	$C_{(1^3)}$	$C_{(4)}$	$C_{(3\,1)}$	$C_{(2^2)}$	$C_{(2\,1^2)}$	$C_{(1^4)}$
$C_{(1)}$	$C_{(1)}$	1	1								
$C_{(1)}$	$C_{(2)}$			1	4/9						
$C_{(1)}$	$C_{(1^2)}$				5/9	1					
$C_{(1)}$	$C_{(3)}$						1	3/10			
$C_{(1)}$	$C_{(2\,1)}$							7/10	1	5/8	
$C_{(1)}$	$C_{(1^3)}$									3/8	1
$C_{(2)}$	$C_{(2)}$						1	2/9	4/9		
$C_{(2)}$	$C_{(1^2)}$							7/18		5/18	
$C_{(1^2)}$	$C_{(1^2)}$								5/9	4/9	1

$C_\alpha(\cdot)$	$C_\beta(\cdot)$	$k=5$						
		$C_{(5)}$	$C_{(4\,1)}$	$C_{(3\,2)}$	$C_{(3\,1^2)}$	$C_{(2^2\,1)}$	$C_{(2\,1^3)}$	$C_{(1^5)}$
$C_{(1)}$	$C_{(4)}$	1	8/35					
$C_{(1)}$	$C_{(3\,1)}$		27/35	8/15	7/15			
$C_{(1)}$	$C_{(2^2)}$			7/15	1/3			
$C_{(1)}$	$C_{(2\,1^2)}$				8/15	2/3	18/25	
$C_{(1)}$	$C_{(1^4)}$						7/25	1
$C_{(2)}$	$C_{(3)}$	1	4/25	4/25				
$C_{(2)}$	$C_{(2\,1)}$		27/50	28/75	7/30	1/3		
$C_{(2)}$	$C_{(1^3)}$				1/5		1/5	
$C_{(1^2)}$	$C_{(3)}$		3/10		7/50			
$C_{(1^2)}$	$C_{(2\,1)}$			7/15	32/75	5/12	9/20	
$C_{(1^2)}$	$C_{(1^3)}$					1/4	7/20	1

$C_\alpha(\cdot)$	$C_\beta(\cdot)$	$k=6$										
		$C_{(6)}$	$C_{(5\,1)}$	$C_{(4\,2)}$	$C_{(4\,1^2)}$	$C_{(3^2)}$	$C_{(3\,2\,1)}$	$C_{(3\,1^3)}$	$C_{(2^3)}$	$C_{(2^2\,1^2)}$	$C_{(2\,1^4)}$	$C_{(1^6)}$
$C_{(1)}$	$C_{(5)}$	1	1/54									
$C_{(1)}$	$C_{(4\,1)}$		22/27	2/5	3/8							
$C_{(1)}$	$C_{(3\,2)}$			3/5		1	7/27					
$C_{(1)}$	$C_{(3\,1^2)}$				5/8		10/27	4/7				
$C_{(1)}$	$C_{(2^2\,1)}$						10/27		1	1/2		
$C_{(1)}$	$C_{(2\,1^3)}$							3/7		1/2	7/9	
$C_{(1)}$	$C_{(1^5)}$										2/9	1
$C_{(2)}$	$C_{(4)}$	1	8/63	16/175								
$C_{(2)}$	$C_{(3\,1)}$		22/35	8/35	1/5	8/15	56/405					
$C_{(2)}$	$C_{(2^2)}$			7/25			7/81		1/3			
$C_{(2)}$	$C_{(2\,1^2)}$				1/3		16/81	16/75		4/15		
$C_{(2)}$	$C_{(1^4)}$							3/25			7/45	
$C_{(1^2)}$	$C_{(4)}$		11/45		3/35							
$C_{(1^2)}$	$C_{(3\,1)}$			2/5	8/21		14/81	4/15				
$C_{(1^2)}$	$C_{(2^2)}$					7/15	64/405			1/6		
$C_{(1^2)}$	$C_{(2\,1^2)}$						20/81	2/5	2/3	32/75	14/25	
$C_{(1^2)}$	$C_{(1^4)}$									7/50	64/225	1
$C_{(3)}$	$C_{(3)}$	1	3/25	16/25		4/25						
$C_{(3)}$	$C_{(2\,1)}$		11/25	27/125	27/200		7/75					
$C_{(3)}$	$C_{(1^3)}$				1/8			2/25				
$C_{(2\,1)}$	$C_{(2\,1)}$			63/125	12/25	21/25	12/25	3/10	3/4	3/8		
$C_{(2\,1)}$	$C_{(1^3)}$						1/6	27/100		9/40	7/20	
$C_{(1^3)}$	$C_{(1^3)}$								1/4	7/40	3/10	1

TABLE 1 (*Cont'd*)

$C_\alpha(\cdot)$	$C_\beta(\cdot)$	$k=7$														
		$C_{(7)}$	$C_{(6\,1)}$	$C_{(5\,2)}$	$C_{(5\,1^2)}$	$C_{(4\,3)}$	$C_{(4\,2\,1)}$	$C_{(4\,1^3)}$	$C_{(3^2\,1)}$	$C_{(3\,2^2)}$	$C_{(3\,2\,1^2)}$	$C_{(3\,1^4)}$	$C_{(2^3\,1)}$	$C_{(2^2\,1^3)}$	$C_{(2\,1^5)}$	$C_{(1^7)}$
$C_{(1)}$	$C_{(6)}$	1	12/77													
$C_{(1)}$	$C_{(5\,1)}$		65/77	16/49	11/35											
$C_{(1)}$	$C_{(4\,2)}$			33/49		4/7	3/14									
$C_{(1)}$	$C_{(4\,1^2)}$				24/35		2/7	10/21								
$C_{(1)}$	$C_{(3^2)}$					3/7			1/5							
$C_{(1)}$	$C_{(3\,2\,1)}$						1/2		4/5	5/7	20/49					
$C_{(1)}$	$C_{(3\,1^3)}$							11/21			2/7	9/14				
$C_{(1)}$	$C_{(2^3)}$									2/7			2/7			
$C_{(1)}$	$C_{(2^2\,1^2)}$										15/49		5/7	3/5		
$C_{(1)}$	$C_{(2\,1^4)}$											5/14		2/5	40/49	
$C_{(1)}$	$C_{(1^6)}$														9/49	1
$C_{(2)}$	$C_{(5)}$	1	20/189	80/1323												
$C_{(2)}$	$C_{(4\,1)}$		130/189	176/945	11/63	8/35	3/35									
$C_{(2)}$	$C_{(3\,2)}$			99/245		12/35	4/45		4/27	5/27						
$C_{(2)}$	$C_{(3\,1^2)}$				3/7		25/189	10/49	8/27		160/1323					
$C_{(2)}$	$C_{(2^2\,1)}$						5/27			40/189	20/189		2/7			
$C_{(2)}$	$C_{(2\,1^3)}$							11/49			6/49	4/21		2/9		
$C_{(2)}$	$C_{(1^5)}$											5/63			8/63	
$C_{(1^2)}$	$C_{(5)}$		13/63		11/189											
$C_{(1^2)}$	$C_{(4\,1)}$			22/63	64/189		3/28	5/28								
$C_{(1^2)}$	$C_{(3\,2)}$					3/7	32/189		7/27		20/189					
$C_{(1^2)}$	$C_{(3\,1^2)}$						25/108	77/196		50/189	256/1323	18/49				
$C_{(1^2)}$	$C_{(2^2\,1)}$								8/27	64/189	25/126		5/14	3/10		
$C_{(1^2)}$	$C_{(2\,1^3)}$										15/98	160/441	5/14	7/18	40/63	
$C_{(1^2)}$	$C_{(1^5)}$													4/45	5/21	1
$C_{(3)}$	$C_{(4)}$	1	24/245	48/1225		64/1225										
$C_{(3)}$	$C_{(3\,1)}$		26/49	176/1225	22/175	216/1225	8/175		8/75							

$C_{(3)}$	$C_{(2^2)}$			33/175			1/25			1/15						
$C_{(3)}$	$C_{(2\,1^2)}$				8/35		2/21	18/175			32/525					
$C_{(3)}$	$C_{(1^4)}$							11/175				9/175				
$C_{(2\,1)}$	$C_{(4)}$		13/35	176/1225	22/245		54/1225									
$C_{(2\,1)}$	$C_{(3\,1)}$			594/1225	576/1225	72/175	6427/22050	3/14	56/225	2/9	8/63					
$C_{(2\,1)}$	$C_{(2^2)}$					9/25	32/225		7/45	64/315	5/63		3/14			
$C_{(2\,1)}$	$C_{(2\,1^2)}$						2/9	66/175	16/45	20/63	23/63	54/175	3/7	9/25		
$C_{(2\,1)}$	$C_{(1^4)}$										27/350	32/175		7/50	2/7	
$C_{(1^3)}$	$C_{(4)}$				6/70			2/49								
$C_{(1^3)}$	$C_{(3\,1)}$						5/42	99/490			8/105	6/35				
$C_{(1^3)}$	$C_{(2^2)}$								2/15		1/14			1/10		
$C_{(1^3)}$	$C_{(2\,1^2)}$									4/21	1/7	2/7	9/35	7/25	16/35	
$C_{(1^3)}$	$C_{(1^4)}$												1/10	3/25	9/35	1

of $\mathbf{L}_{22}$ [4]. Thus (3.10) reduces to the form [4]

$$
\begin{aligned}
E(V^{(p)}-1)^3 = {} & E(V_0^{(p)}-1)^3 + \tfrac{1}{8}\Delta_2 f^{(3)} \\
& + \tfrac{3}{8}[\{-3\delta(\nu)+2\delta(\nu+2)+\delta(\nu+4)\}f^{(2)}E(\operatorname{tr}\mathbf{L}_{22}) \\
& + \{3\delta(\nu)+2\delta(\nu+2)+3\delta(\nu+4)\}f^{(1)}E(\operatorname{tr}\mathbf{L}_{22})^2 \\
& + \{\delta(\nu)+2\delta(\nu+2)+5\delta(\nu+4)\}E(\operatorname{tr}\mathbf{L}_{22})^3 \\
& + 4\{\delta(\nu)+2\delta(\nu+2)-3\delta(\nu+4)\}E(\operatorname{tr}\mathbf{L}_{22}\operatorname{tr}_2\mathbf{L}_{22}) \\
& + 4\Delta_2\{-f^{(1)}E(\operatorname{tr}_2\mathbf{L}_{22})+2E(\operatorname{tr}_3\mathbf{L}_{22})\}],
\end{aligned}
\tag{3.13}
$$

where $f^{(i)} = \prod_{j=1}^{i}(f_1-p-1+2j)$,

$$
E(\operatorname{tr}_i\mathbf{L}_{22}) = \binom{p-1}{i}\prod_{j=1}^{i}[(f_2-j)/(\nu-j)] \qquad (i=1,2,\cdots,p-1;\ \operatorname{tr}_1\mathbf{L}_{22}=\operatorname{tr}\mathbf{L}_{22}),
\tag{3.14}
$$

$$
\begin{aligned}
E(\operatorname{tr}\mathbf{L}_{22}\operatorname{tr}_i\mathbf{L}_{22}) = {} & [E(\operatorname{tr}_i L_{22})/\{(\nu+1)(\nu-i-1)\}] \\
& \cdot[(f_1-p)\{(f_2-1)(p-1)+2i\} \\
& + (p-1)(f_2-p-1)(f_2+2p-i+1) \\
& + 2p^3-ip^2+(i-2)p+2i]
\end{aligned}
\tag{3.15}
$$

and

$$
\begin{aligned}
E(\operatorname{tr}\mathbf{L}_{22})^3 = {} & E(\operatorname{tr}\mathbf{L}_{22})[2^3(f_1-f_2+1)f_1(\nu-2p+1)(\nu-p) \\
& \cdot\{(\nu-3)(\nu-2)(\nu-1)^2(\nu+1)(\nu+3)\}^{-1}+3E(\operatorname{tr}\mathbf{L}_{22})^2 \\
& - 2\{E(\operatorname{tr}\mathbf{L}_{22})\}^2].
\end{aligned}
\tag{3.16}
$$

$$
\begin{aligned}
E(V^{(p)}-1)^4 = {} & E(V_0^{(p)}-1)^4 - \tfrac{1}{48}\Delta_3 f^{(4)} \\
& + \tfrac{1}{12}\{5\delta(\nu)-9\delta(\nu+2)+3\delta(\nu+4)+\delta(\nu+6)\}f^{(3)} \\
& \quad\cdot E(\operatorname{tr}\mathbf{L}_{22}) \\
& + \tfrac{3}{8}\{-5\delta(\nu)+3\delta(\nu+2)+\delta(\nu+4)+\delta(\nu+6)\}f^{(2)} \\
& \quad\cdot E(\operatorname{tr}\mathbf{L}_{22})^2 \\
& + \tfrac{1}{4}\{5\delta(\nu)+3\delta(\nu+2)+3\delta(\nu+4)+5\delta(\nu+6)\}f^{(1)} \\
& \quad\cdot E(\operatorname{tr}\mathbf{L}_{22})^3 \\
& + \tfrac{1}{16}\{5\delta(\nu)+9\delta(\nu+2)+15\delta(\nu+4)+35\delta(\nu+6)\} \\
& \quad\cdot E(\operatorname{tr}\mathbf{L}_{22})^4 \\
& + \tfrac{3}{2}\{\delta(\nu)+\delta(\nu+2)+3\delta(\nu+4)-5\delta(\nu+6)\} \\
& \quad\cdot E[(\operatorname{tr}\mathbf{L}_{22})^2\operatorname{tr}_2\mathbf{L}_{22}] \\
& + 3\Delta\{-f^{(1)}E(\operatorname{tr}\mathbf{L}_{22}\operatorname{tr}_2\mathbf{L}_{22})+2E(\operatorname{tr}\mathbf{L}_{22}\operatorname{tr}_3\mathbf{L}_{22})\} \\
& + \Delta_3\{(f^{(2)}/2)E(\operatorname{tr}_2\mathbf{L}_{22})-2f^{(1)}E(\operatorname{tr}_3\mathbf{L}_{22})+4E(\operatorname{tr}_4\mathbf{L}_{22}) \\
& - 3E(\operatorname{tr}_2\mathbf{L}_{22})^2\},
\end{aligned}
\tag{3.17}
$$

where

$$\Delta = \delta(\nu) - \delta(\nu + 2) - \delta(\nu + 4) + \delta(\nu + 6).$$

Now using Table 1,

$$(3.18) \quad E[(\mathrm{tr}\ \mathbf{L}_{22})^2\ \mathrm{tr}_2\ \mathbf{L}_{22}] = \tfrac{1}{24}E[7C_{(31)}(\mathbf{L}_{22}) + 10C_{(2^2)}(\mathbf{L}_{22}) + 13C_{(21^2)}(\mathbf{L}_{22}) + 18C_{(1^4)}(\mathbf{L}_{22})].$$

Now using a theorem of Constantine (See Theorem 3 of [1]) the right side of of (3.18) equals

$$(3.19) \quad \tfrac{1}{24}\Gamma_{p-1}(\tfrac{1}{2}f_1)[7b_1 + 10b_2 + 13b_3 + 18b_4],$$

where

$$b_1 = \Gamma_{p-1}(t, (31))C_{(31)}(\mathbf{I})/\Gamma_{p-1}(v, (31)),$$
$$b_2 = \Gamma_{p-1}(t, (2^2))C_{(2^2)}(\mathbf{I})/\Gamma_{p-1}(v, (2^2)),$$
$$b_3 = \Gamma_{p-1}(t, (21^2))C_{(21^2)}(\mathbf{I})/\Gamma_{p-1}(v, (21^2)),$$
$$b_4 = \Gamma_{p-1}(t, (1^4))C_{(1^4)}(\mathbf{I})/\Gamma_{p-1}(v, (1^4)),$$

where

$$t = \tfrac{1}{2}(f_2 - 1) \quad \text{and} \quad v = \tfrac{1}{2}(\nu - 1).$$

Similarly from Table 1,

$$(3.20) \quad E(\mathrm{tr}_2\ \mathbf{L}_{22})^2 = \tfrac{1}{16}\Gamma_{p-1}(\tfrac{1}{2}f_1)[5b_2 + 4b_3 + 9b_4]$$

and

$$(3.21) \quad E(\mathrm{tr}_1\ \mathbf{L}_{22})^4 = \Gamma_{p-1}(\tfrac{1}{2}f_1) \textstyle\sum_{i=0}^{4} b_i,$$

where

$$b_0 = \Gamma_{p-1}(t, (4))C_{(4)}(\mathbf{I})/\Gamma_{p-1}(v, (4)).$$

It may be pointed out that alternate expressions for $E[(\mathrm{tr}\ \mathbf{L}_{22})^2\mathrm{tr}_2\ \mathbf{L}_{22}]$, $E(\mathrm{tr}_2\ \mathbf{L}_{22})$ and $E(\mathrm{tr}_1\ \mathbf{L}_{22})^4$ are available in [4]. The latter two can further be obtained from [7], [8], [9]. In addition, the first four moments of $V_0^{(p)}$ are available in Pillai [5], [7], [8] and the first two moments of $V^{(p)}$ in Khatri and Pillai [3] and hence (3.8) and (3.9) are not treated here separately.

4. Distribution of the largest root, l_p. In the density function of $l_1, \cdots, l_p$ in (1.2) let us make the following transformation: $l_i/l_p = g_i$, $i = 1, 2, \cdots, p-1$, $l_p = l_p$. Then the joint distribution of $(g_1, \cdots, g_{p-1})$ and l_p is given by

$$(4.1) \quad \begin{aligned} &C(p, f_1, f_2) \exp(-\tfrac{1}{2}\ \mathrm{tr}\ \Omega)l_p^{\frac{1}{2}pf_2-1}(1 - l_p)^{\frac{1}{2}(f_1-p-1)}\alpha_{p-1}(\mathbf{G})|\mathbf{G}|^{\frac{1}{2}(f_2-p-1)} \\ &\cdot|\mathbf{I}_{p-1} - l_p\mathbf{G}|^{\frac{1}{2}(f_1-p-1)}|\mathbf{I}_{p-1} - \mathbf{G}| \textstyle\sum_{k=0}^{\infty}\sum_{\kappa}(\tfrac{1}{2}\nu)_{\kappa}C_{\kappa}(\tfrac{1}{2}\Omega)C_{\kappa}(\mathbf{G}_1)l_p^{\kappa}/ \\ &\{(\tfrac{1}{2}f_2)_{\kappa}C_{\kappa}(\mathbf{I}_p)k\,!\},\end{aligned}$$

TABLE 2

Values of $b_{\kappa,\eta}$

n	η	$k=1$	$k=2$		$k=3$		
		κ					
		[1]	[2]	$[1^2]$	[3]	[2 1]	$[1^3]$
0		1	1		1		
1	[1]	1	2/3	4/3	3/5	12/5	
2	[2]		1		3/5	12/5	
	$[1^2]$			1		3/2	3/2
3	[3]				1		
	[2 1]					1	
	$[1^3]$						1

n	η	$k=4$				
		κ				
		[4]	[3 1]	$[2^2]$	$[2\ 1^2]$	$[1^4]$
0		1				
1	[1]	4/7	24/7			
2	[2]	18/35	16/7	16/5		
	$[1^2]$		2		4	
3	[3]	4/7	24/7			
	[2 1]		8/9	8/9	20/9	
	$[1^3]$				12/5	8/5
4	[4]	1				
	[3 1]		1			
	$[2^2]$			1		
	$[2\ 1^2]$				1	
	$[1^4]$					1

n	η	$k=5$						
		κ						
		[5]	[4 1]	[3 2]	$[3\ 1^2]$	$[2^2\ 1]$	$[2\ 1^3]$	$[1^5]$
0		1						
1	[1]	5/9	40/9					
2	[2]	10/21	8/3	48/7				
	$[1^2]$		5/2		15/2			
3	[3]	10/21	8/3	48/7				
	[2 1]		1	16/9	25/9	40/9		
	$[1^3]$				30/7		40/7	
4	[4]	5/9	40/9					
	[3 1]		3/4	4/3	35/12			
	$[2^2]$			5/3		10/3		
	$[2\ 1^2]$				25/21	5/3	15/7	
	$[1^4]$						10/3	5/3
5	[5]	1						
	[4 1]		1					
	[3 2]			1				
	$[3\ 1^2]$				1			
	$[2^2\ 1]$					1		
	$[2\ 1^3]$						1	
	$[1^5]$							1

TABLE 2 (*Cont'd.*)

n	η	$k=6$										
		κ										
		[6]	[5 1]	[4 2]	[4 1^2]	[3^2]	[3 2 1]	[3 1^3]	[2^3]	[2^2 1^2]	[2 1^4]	[1^6]
0		1										
1	[1]	6/11	60/11									
2	[2]	5/11	240/77	80/7								
	[1^2]		3		12							
3	[3]	100/231	216/77	160/21		64/7						
	[2 1]		8/7	20/7	4		12					
	[1^3]				20/3			40/3				
4	[4]	5/11	240/77	80/7								
	[3 1]		27/35	10/7	14/5	8/5	42/5					
	[2^2]			5/2			15/2		5			
	[2 1^2]				5/3		30/7	10/3		40/7		
	[1^4]							15/2			15/2	
5	[5]	6/11	60/11									
	[4 1]		24/35	12/7	18/5							
	[3 2]			1		4/5	21/5					
	[3 1^2]				14/15		12/5	8/3				
	[2^2 1]						15/7		1	20/7		
	[2 1^3]							3/2		12/5	21/10	
	[1^5]										30/7	12/7
6	[6]	1										
	[5 1]		1									
	[4 2]			1								
	[4 1^2]				1							
	[3^2]					1						
	[3 2 1]						1					
	[3 1^3]							1				
	[2^3]								1			
	[2^2 1^2]									1		
	[2 1^4]										1	
	[1^6]											1

where $\mathbf{G} = \operatorname{diag}(g_1, \cdots, g_{p-1})$ and $\mathbf{G}_1 = \operatorname{diag}(\mathbf{G}, 1)$. Now the distribution of l_p is obtained from (4.1) by integrating $g_1, \cdots, g_{p-1}$ i.e. by evaluating the integral

$$(4.2)\quad B_k = \int \int_{0 \leqq g_1 \leqq \cdots \leqq g_{p-1} \leqq 1} |\mathbf{G}|^{\frac{1}{2}(f_2-p-1)} \cdot |\mathbf{I}_{p-1} - \mathbf{G}| \alpha_{p-1}(\mathbf{G}) |\mathbf{I}_{p-1} - l_p \mathbf{G}|^{\frac{1}{2}(f_1-p-1)} C_\kappa(\mathbf{G}_1)\, d\mathbf{G}.$$

Now let us note the following results:

$$(4.3)\quad C_\kappa(\mathbf{G}_1) = \sum_{n=0}^{k} \sum_{\eta} b_{\kappa,\eta} C_\eta(\mathbf{G}),$$

where $b_{\kappa,\eta}$'s are constants depending on κ and η,

$$(4.4)\quad |\mathbf{I}_p - l_p \mathbf{G}|^{\frac{1}{2}(f_1-p-1)} = \sum_{d=0}^{\infty} \sum_{\tau} (\tfrac{1}{2}(p+1-f_1))_\tau l_p{}^d C_\tau(\mathbf{G})/d\,!.$$

As in (2.8)

$$C_\eta(\mathbf{G})C_\tau(\mathbf{G}) = \sum_\delta g_{\eta,\tau}^\delta C_\delta(\mathbf{G}). \tag{4.5}$$

Using (4.3)–(4.5) in (4.2) we get

$$B_\kappa = \sum_{n=0}\sum_\eta \sum_{d=0}\sum_\tau \sum_\delta b_{\kappa,\eta} g_{\eta,\tau}^\delta (\tfrac{1}{2}(p + 1 - f_1))_\tau l_p{}^d D_\delta / d\,!, \tag{4.6}$$

where

$$\begin{aligned} D_\delta &= \int \int_{0 \leqq g_1 \leqq \cdots \leqq g_{p-1} \leqq 1} |\mathbf{G}|^{\frac{1}{2}(f_2-p-1)} |\mathbf{I}_{p-1} - \mathbf{G}| \alpha_{p-1}(\mathbf{G}) C_\delta(\mathbf{G})\, d\mathbf{G} \\ &= \Gamma_{p-1}(\tfrac{1}{2}(p-1))\Gamma_{p-1}(\tfrac{1}{2}(p+1))\Gamma_{p-1}(\tfrac{1}{2}(f_2-1), \delta) C_\delta(\mathbf{I}_{p-1}) \\ &\quad \cdot [\pi^{\frac{1}{2}(p-1)^2} \Gamma_{p-1}(\tfrac{1}{2}(f_2+p), \delta)]^{-1}. \end{aligned} \tag{4.7}$$

Using (4.6) in (4.1) we get the distribution of l_p. Table 2 gives the values of $b_{\kappa,\eta}$ for various values of κ and η. If a_i denotes the ith elementary symmetric function of the roots of $\mathbf{G}_1$ i.e. $a_i = \mathrm{tr}_i\, \mathbf{G}_1$, $i = 1, 2, \cdots$, the $b_{\kappa,\eta}$ coefficients were obtained by using the zonal polynomials tabulated by James [2] in terms of the a_i's and using the relation $\mathrm{tr}_i\, \mathbf{G}_1 = \mathrm{tr}_i\, \mathbf{G} + \mathrm{tr}_{i-1}\, \mathbf{G}$.

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