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LOCALLY AND ASYMPTOTICALLY MINIMAX TESTS OF A MULTIVARIATE PROBLEM

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1. Introduction and summary. Let $X_1, \cdots, X_N$ be independent normal p-vectors with common mean vector $\xi = (\xi_1, \cdots, \xi_p)'$ and common non-singular covariance matrix Σ. Write $N\bar{X} = \sum_1^N X_i$, $S = \sum_1^N (X_i - \bar{X})(X_i - \bar{X})'$, $b[i]$ for the i-vector consisting of the first i components of a p-vector b and $C[i]$ for the upper left-hand $i \times i$ sub-matrix of a $p \times p$ matrix C. Let $\delta = N\xi'\Sigma^{-1}\xi$ $(\geqq 0)$. We will consider here the problem of testing the hypothesis

$$H_0: \xi_1 = \cdots = \xi_p = 0$$

against the alternative

$$H_\lambda: \xi_1 = \cdots = \xi_q = 0, \qquad \delta = \lambda,$$

where ξ, Σ are unknown, $q < p$ and $\lambda > 0$ is given.

The problem of testing H_0 against H_λ remains invariant under the group G of $p \times p$ non-singular matrices

$$g = \begin{pmatrix} g_{11} & 0 \\ g_{21} & g_{22} \end{pmatrix} \tag{1.0}$$

where g_{11} is a $q \times q$ sub-matrix of g. A maximal invariant in the space of $(\bar{X}, S)$ under G is $\bar{R} = (\bar{R}_1, \bar{R}_2)$ where

$$\begin{aligned} \bar{R}_1 + \bar{R}_2 &= N\bar{X}'(S + N\bar{X}\bar{X}')^{-1}\bar{X}, \\ \bar{R}_1 &= N\bar{X}'_{[q]}(S_{[q]} + N\bar{X}_{[q]}\bar{X}'_{[q]})^{-1}\bar{X}_{[q]}\,; \end{aligned} \tag{1.1}$$

and a corresponding maximal invariant in the parametric space of (ξ, Σ) under G is $\bar{\Delta} = (\bar{\delta}_1, \bar{\delta}_2)$, where

$$\begin{aligned} \bar{\delta}_1 + \bar{\delta}_2 &= N\xi'\Sigma^{-1}\xi, \\ \bar{\delta}_1 &= N\xi'_{[q]}\Sigma^{-1}_{[q]}\xi_{[q]} \end{aligned} \tag{1.2}$$

(see Giri (1961)). Giri (1962) has shown that for testing H_0 against the alternatives

$$H_1': \xi_1 = \cdots = \xi_q = 0, \qquad \delta > 0, \tag{1.3}$$

the likelihood ratio test of significance level α is given by

$$\begin{aligned} \phi(X_1, \cdots, X_N) &= 1, \quad \text{if} \quad Z = (1 - \bar{R}_1 - \bar{R}_2)/(1 - \bar{R}_1) \leqq C, \\ &= 0, \quad \text{otherwise}; \end{aligned} \tag{1.4}$$

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where the constant C is chosen in such a way that $E_{H_0}\phi = \alpha$ and under H_0, Z has central beta distribution with parameters $(N - p)/2$, $(p - q)/2$. It follows from Giri (1961) (also from (2.3) in Section 2) that with respect to G the likelihood ratio test for testing H_0 against H_1' is not uniformly most powerful invariant and that there is no uniformly most powerful invariant test for this problem. However, for fixed p, this test is nearly optimum as the sample size becomes large (Wald (1943)). Thus if p is not large, it seems likely that the sample size occurring in practice will usually be large enough for this result to be relevant. However, if the dimension p of the basic multivariate distribution is large, it may be that the sample size must be extremely large for this large sample result to apply, for example, there are cases where N/p^3 is large. The only satisfactory property of this test procedure known to us at this writing is that the difference of the powers of this test and the best invariant test with respect to G is $O(N^{-1})$ when p, q are both $O(N)$, δ is $O(N^{\frac{1}{2}})$ and N becomes large (Giri (1967)).

In this paper it will be shown that the likelihood ratio test for this problem is neither locally minimax as $\lambda \to 0$ nor asymptotically minimax as $\lambda \to \infty$. Thus it will be established here that it can not be minimax for every λ for this problem. Attempts are also made to find test procedures based on $\bar{R}$ which are locally and asymptotically minimax. It is easy to see that the power function of any test procedure based on $\bar{R}$ is constant on each contour $\delta = \lambda$. It will be shown in Sections 2 and 3 that the test (hereafter called ϕ^*) which rejects H_0 if $\bar{R}_1 + (N - q)\bar{R}_2/(p - q) \geqq C_\alpha$, where C_α is chosen so that $E_{H_0}\phi^* = \alpha$, is locally minimax as $\lambda \to 0$ but not asymptotically logarithmically (sometimes will be called asymptotically only) minimax as $\lambda \to \infty$ and the test (hereafter called Hotelling's test) which rejects H_0 if $\bar{R}_1 + \bar{R}_2 \geqq C_\alpha''$ (C_α'' depends on the size α of the test) is asymptotically minimax as $\lambda \to \infty$ but not locally minimax as $\lambda \to 0$. Furthermore for this problem no invariant test (under G) is minimax for every λ. This includes the likelihood ratio test, Hotelling's test and ϕ^*. It may be pointed out here that none of these last two test procedures is derived following after any well known statistical theory. They are derived so as to possess the required local and asymptotic property stated above.

2. Locally minimax tests. We may restrict attention to the space of minimal sufficient statistic $(\bar{X}, S)$ of (ξ, Σ). It is well known that the Hunt-Stein theorem can not be applied to the group G which leaves the present problem invariant, operating as $(\bar{X}, S; \xi, \Sigma) \to (g\bar{X}, gSg'; g\xi, g\Sigma g')$ for $g \varepsilon G$. For a discussion of the Hunt-Stein theorem in this context the reader is referred to Giri, Kiefer and Stein (1963). However the theorem does apply to the smaller group G_T of $p \times p$ non-singular lower triangular matrices (zero above the main diagonal) which is almost invariant under G_T and hence, in the present problem, there is such a test which is invariant (see Lehmann (1959), p. 225) and which maximizes, among all level α tests, the minimum power over H_λ. Whereas $\bar{R}$ was a maximal invariant under G with a single distribution under H_0 and H_λ respectively, the maximal invariant under G_T is a p-dimensional statistic $R = (R_1, \cdots, R_p)$ ($R_i \geqq 0$, $i = 1, \cdots, p$) defined by (Giri, Kiefer and Stein (1963)).

$$(2.0)\qquad \textstyle\sum_1^i R_j = N\bar{X}'_{[i]}(S_{[i]} + N\bar{X}_{[i]}\bar{X}'_{[i]})^{-1}\bar{X}_{[i]}, \qquad i = 1, \cdots, p;$$
$$\textstyle\sum_1^q R_j = \bar{R}_1, \qquad \sum_1^p R_j = \bar{R}_1 + \bar{R}_2,$$

with a single distribution under H_0 and with a distribution which depends continuously on the $(p - q - 1)$-dimensional parameter $(\delta_1 = 0, \cdots, \delta_q = 0, \delta_{q+1}, \cdots \delta_p)$, $\delta_i \geqq 0, i = q + 1, \cdots, p, \sum \delta_j = \delta$, under H_1'. Let

$$(2.1)\qquad \Delta = \{(\delta_1 = 0, \cdots, \delta_q = 0, \delta_{q+1}, \cdots, \delta_p); \delta_i \geqq 0, i = q + 1, \cdots, p;$$
$$\textstyle\sum_{q+1}^p \delta_j = \delta = \lambda \text{ under } H_\lambda\}$$

where

$$(2.2)\qquad \textstyle\sum_1^i \delta_j = N\xi'_{[i]}\ \Sigma_{[i]}^{-1}\ \xi_{[i]}, \qquad i = 1, \cdots, p$$
$$\textstyle\sum_1^q \delta_j = \bar{\delta}_1, \qquad \sum_1^p \delta_j = \bar{\delta}_1 + \bar{\delta}_2.$$

From Giri and Kiefer (1964), the Lebesgue density of R on $H = \{r : r_i > 0, 1 \leqq i \leqq p, \sum_i^p r_i < 1\}$ under H_1' is the function f_Δ^*, given by

$$(2.3)\qquad f_\Delta^*(r) = \pi^{-p/2}\Gamma(N/2)(1 - \textstyle\sum_1^p r_i)^{(N-p-2)/2}/\Gamma((N - p)/2) \prod_1^p r_i^{\frac{1}{2}}$$
$$\exp\{(\delta/2)(-1 + \textstyle\sum_1^q r_j + \sum_{q+1}^p r_j \sum_{i>j}(\delta_i/\delta)\}$$
$$\cdot \textstyle\prod_{q+1}^p \phi((N - i + 1)/2, \tfrac{1}{2}, r_i\delta_i/2)$$

where ϕ is the confluent hypergeometric series (sometimes denoted by ${}_1F_1$)

$$(2.4)\qquad \phi(a, b, x) = 1 + \textstyle\sum_1^\infty \Gamma(a + j)\Gamma(b)x^j/\Gamma(b + j)\Gamma(a)j!$$

From Giri (1961), the Lebesgue density of $\bar{R}$ on the set $\{\bar{r} : \bar{r}_i > 0, i = 1, 2, \sum_1^2 \bar{r}_i < 1\}$ under H_1' is the function f_Δ, given by

$$f_\Delta(\bar{r}) = [\Gamma(N/2)/\Gamma((N - p)/2)\Gamma((p - q)/2)\Gamma(q/2)]$$
$$(2.5)\qquad \cdot(1 - \bar{r}_1 - \bar{r}_2)^{(N-p-2)/2}\bar{r}_1^{(q-2)/2}\bar{r}_2^{(p-q-2)/2}$$
$$\cdot\exp\{(\bar{\delta}_2/2)(-1 + \bar{r}_1)\}\ \phi((N - q)/2, (p - q)/2, \bar{r}_2\bar{\delta}_2/2).$$

Because of the compactness of the reduced parameter spaces $\{0\}$ and $\Gamma_\lambda = \{(0, \cdots, 0, \delta_{q+1}, \cdots, \delta_p); \delta_i \geqq 0, \sum_{q+1}^p \delta_j = \lambda\}$ and the continuity of f_Δ^* in Δ (see Wald (1950)) we conclude that every minimax test for the reduced problem in terms of R is Bayes. Furthermore, every test based on R with constant power on Γ_λ is minimax for testing H_0 against H_λ if and only if it is Bayes. In this section we will show that whereas the likelihood ratio test and Hotelling's test are not locally minimax, the test ϕ^* is so as $\lambda \to 0$.

The theory of locally minimax tests has been developed by Giri and Kiefer (1964). We will outline here only the basic steps needed for this development. For each point (δ, η) in the parameter set Ω (where $\delta \geqq 0$), let $p(\cdot; \delta, \eta)$ be the probability density of X with respect to some σ-finite measure μ (the range of η may depend on δ). Throughout this section such expression as $O(1)$, $O(h(\lambda))$ etc. are to be interpreted as $\lambda \to 0$. For each point α, $0 < \alpha < 1$, we will consider critical region of the form $R = \{X : U(X) \geqq C_\alpha\}$ where $U(x)$ is bounded, positive and has continuous distribution function for each (δ, η), equi-continuous for $\delta <$ some δ_0 and where

$$(2.6)\qquad P_{0,\eta}(R) = \alpha, \qquad P_{\lambda,\eta}(R) = \alpha + h(\lambda) + q(\lambda, \eta),$$

where $q(\lambda, \eta) = O(h(\lambda))$ uniformly in η with $h(\lambda) > 0$ for $\lambda > 0$ and $h(\lambda) = O(1)$. We shall also be concerned with probability measures $\xi_{0,\lambda}$, $\xi_{1,\lambda}$ on the sets $\delta = 0$ and $\delta = \lambda$ respectively, for which

$$(2.7)\qquad \int p(x; \lambda, \eta)\xi_{1,\lambda}(d\eta)/\int p(x; 0, \eta)\xi_{0,\lambda}(d\eta) = 1 + h(\lambda)[g(\lambda) + r(\lambda)U(x)] + B(x, \lambda)$$

where $0 < C_1 < r(\lambda) < C_2 < \infty$ for λ sufficiently small and where $g(\lambda) = O(1)$ and $B(x, \lambda) = O(h(\lambda))$ uniformly in x. The following theorem is proved in Giri and Kiefer (1964).

THEOREM 2.1. *If U satisfies* (2.6) *and for sufficiently small λ there exists $\xi_{0,\lambda}$ and $\xi_{1,\lambda}$ satisfying* (2.7), *then U is locally minimax of level α for testing $H_0{:}\delta = 0$ against the alternative $H_\lambda{:}\delta = \lambda$ as $\lambda \to 0$, that is,*

$$(2.8)\qquad \lim_{\lambda\to 0}\,[(\inf_\eta P_{\lambda,\eta}(R) - \alpha)/(\sup_{\phi\varepsilon Q_\alpha} \inf_\eta P_{\lambda,\eta}\{\phi \text{ rejects } H_0\} - \alpha)] = 1$$

where Q_α is the class of tests of level α.

REMARK 1. It is easy to conclude that if the test $U(x) \geqq C_\alpha$ satisfies (2.6) but does not satisfy (2.7) for some $\xi_{0,\lambda}$ and $\xi_{1,\lambda}$, then U is not locally minimax in the above sense.

Writing

$$(2.9)\qquad \eta_i = \delta_i/\delta, \quad i = 1, \cdots, p; \qquad \textstyle\sum_{q+1}^p \eta_i = 1$$

we get from (2.1) as $\delta = \lambda \to 0$

$$(2.10)\qquad f^*_{\lambda,\eta}(r)/f^*_{0,\eta}(r) = 1 + (\lambda/2)\{-1 + \textstyle\sum_1^q r_j + \sum_{q+1}^p r_j[\sum_{i>j}\eta_i + (N - j + 1)\eta_j]\} + B(r, \eta, \lambda)$$

where $B(r, \eta, \lambda) = O(\lambda)$ uniformly in r and η. Also from (2.5) when $\bar\delta_2 = \delta = \lambda \to 0$

$$(2.11)\qquad f_{H_\lambda}(\bar r) = f_{H_0}(\bar r)[1 + (\lambda/2)(-1 + \bar r_1 + (N - q)\bar r_2/(p - q)) + B(\bar r, \lambda)]$$

where $B(\bar r, \lambda) = O(\lambda)$ uniformly in $\bar r$. The set $\delta = 0$ is a single point $\eta = 0$. So $\xi_{0,\lambda}$ assigns measure 1 to the point $\eta = 0$. The set $\delta = \lambda$ is a convex $(p - q)$-dimensional Euclidean set wherein each component η_i is $O(h(\lambda))$. Any probability measure can be replaced by the degenerate measure $\xi_{1,\lambda}$ which assigns measure one to the mean of $\xi_{1,\lambda}$ (see Remark 1 of Giri and Kiefer (1964)). Hence

$$\begin{aligned}
&\int f^*_{\lambda,\eta}(r)\xi_{1,\lambda}(d\eta)/\int f^*_{0,0}(r)\xi_{0,\lambda}(d\eta)\\
&\quad= \int \{1 + (\lambda/2)[-1 + \bar r_1 + \textstyle\sum_{q+1}^p r_j(\sum_{i>j}\eta_i + (N - j + 1)\eta_j)]\\
(2.12)\qquad &\qquad + B(r, \lambda, \eta)\}\,\xi_{1,\lambda}(d\eta)\\
&\quad= 1 + (\lambda/2)\{-1 + \bar r_1 + \textstyle\sum_{q+1}^p r_j[\sum_{i>j}\eta_i^0 + (N - j + 1)\eta_j^0]\}\\
&\qquad + B(r, \lambda)
\end{aligned}$$

where $B(r, \lambda) = O(h(\lambda))$ uniformly in r and $\xi_{1,\lambda}$ assigns measure one to $(\eta_{q+1}^0, \cdots, \eta_p^0)$. Let us now consider the rejection region R_K given by

$$R_K = \{X: U(X) = \bar{R}_1 + K\bar{R}_2 \geqq C_\alpha\} \tag{2.13}$$

where K is chosen such that (2.12) is reduced to yield (2.7) and C_α depends on the level of significance α of the test for the chosen K.

Now choosing

$$\begin{aligned} \eta_j^0 &= [(N-j-1)\cdots(N-p)/(N-j+1)\cdots(N-p+2) \\ &\quad \cdot[(N-q)/(N-p+1)(p-q)], \quad j = q+1, \cdots, (p-1), \\ \eta_p^0 &= (N-q)/(N-p+1)(p-q), \end{aligned} \tag{2.14}$$

so that $\sum_{i>j} \eta_i^0 + (N-j+1)\eta_j^0 = (N-q)/(p-q)$ for $j = q+1, \cdots, p$. We see that the test ϕ^* with the rejection region R'

$$R' = \{X: U(X) = \bar{r}_1 + (N-q)\bar{r}_2/(p-q) \geqq C_\alpha\} \tag{2.15}$$

with $P_{0,\lambda}(R') = \alpha$ satisfies (2.7) as $\lambda \to 0$. Furthermore, it is easy to check that any region R_K of structure (2.13) must have $K = (N-q)/(p-q)$ to satisfy (2.7) for some $\xi_{1,\lambda}$. For the invariant region R', $P_{\lambda,\eta}(R')$ depends only on λ and hence from (2.6) $q(\lambda, \eta) = 0$. From (2.5) it is easy to conclude that the test ϕ^* is locally most powerful invariant as $\lambda \to 0$ (see Lehmann (1959), p. 342). Hotelling's test does not coincide with ϕ^* and hence it is locally worse. It is well known that the power function of Hotelling's test, which depends only on δ, has positive derivative everywhere, in particular, at $\delta = 0$. Hence the test ϕ^*, being locally most powerful, satisfies the same condition at $\lambda = 0$. Thus from (2.6), with $R = R'$, $h(\lambda) > 0$. Hence we have

THEOREM 2.2. *For testing H_0 against H_1' the test given by the test function ϕ^*, which is easily shown to be locally most powerful invariant under G, is locally minimax.*

Hotelling's test and the likelihood ratio test are also invariant under G and therefore their power functions are functions of δ only. However it follows from above theorem that neither of these tests maximises the derivative of power function at $\delta = 0$. So Hotelling's test and the likelihood ratio test are not locally minimax for this problem.

3. Asymptotic minimax tests. In this section we will treat the setting of Section 2 as $\lambda \to \infty$ and expressions such as $O(1)$, $O(H(\lambda))$ are to be interpreted in this light and will show that when $\lambda \to \infty$ the likelihood ratio test and the test ϕ^* are not asymptotically minimax but Hotelling's test is so. We are here interested in minimaxing a probability error which is going to zero. This notion has been developed in Giri and Kiefer (1964). In short, suppose that the region $R = \{X: U(X) \geqq C_\alpha\}$ satisfies (in place of (2.6)),

$$P_{0,\eta}\{R\} = \alpha, \qquad P_{\lambda,\eta}\{R\} = 1 - \exp\{-H(\lambda)[1 + O(1)]\}, \tag{3.1}$$

where $H(\lambda) \to \infty$ with λ and the $O(1)$ is uniform in η. Suppose, replacing (2.7), that

$$(3.2)\quad \int p(x, \lambda, \eta)\xi_{1,\lambda}(d\eta)/\int p(x, 0, \eta)\xi_{0,\lambda}(d\eta) = \exp\{H(\lambda)[G(\lambda) + R(\lambda)U(x)] + B(x, \lambda)\}$$

where $\sup_x |B(x, \lambda)| = O(H(\lambda))$ and $0 < C_1 < R(\lambda) < C_2 < \infty$.

One other regularity assumption is that C_α is a point of increase from the left of the distribution function of U when $\delta = 0$ uniformly in η, that is,

$$(3.3)\quad \inf_\eta P_{0,\eta}\{U(x) \geqq C_\alpha - \epsilon\} > \alpha$$

for every $\epsilon > 0$. The following theorem is proved in Giri and Kiefer (1964).

THEOREM 3.1. *If U satisfies (3.1) and (3.3) and if for sufficiently large λ there exist $\xi_{0,\lambda}$, $\xi_{1,\lambda}$ satisfying (3.2) then U is asymptotically logarithmically minimax of level α for testing $H_0:\delta = 0$, against $H_\lambda:\delta = \lambda$ as $\lambda \to \infty$, that is*

$$(3.4)\quad \lim_{\lambda\to\infty} [\inf_\eta \{-\log(1 - P_{\lambda,\eta}(R))\})^{-1} (\sup_{\phi\epsilon Q_\alpha} \inf_\eta \{-\log[1 - P_{\lambda,\eta}\{\phi \text{ rejects } H_0\}]\})^{-1}] = 1.$$

REMARK 2. If U satisfies (3.1) and (3.3) but does not satisfy (3.2) then U is not asymptotically logarithmically minimax.

Since $\phi(a, b, x) = \exp\{x(1 + O(1))\}$ as $x \to \infty$ we have from (2.3)

$$(3.5)\quad f^*_{\lambda,\eta}(r)/f^*_{0,0}(r) = \exp\{(\lambda/2)[-1 + \bar{r}_1 + \textstyle\sum_{q+1}^p r_j \sum_{i\geqq j} \eta_i][1 + B(r, \eta, \lambda)]\}$$

with $\sup_{r,\eta} B(r, \eta, \lambda) = O(1)$ as $\lambda \to \infty$. Also from (2.5) when $\delta_2 = \delta = \lambda \to \infty$,

$$(3.6)\quad f_{H_1}(\bar{r})/f_{H_0}(\bar{r}) = \exp\{(\lambda/2)[-1 + \bar{r}_1 + \bar{r}_2](1 + B(\bar{r}, \lambda))\}$$

where $\sup_{\bar{r}}|B(r, \lambda)| = O(1)$ as $\lambda \to \infty$. Since $\xi_{0,\lambda}$ assigns measure one to the single point $\eta = 0$, from (3.5),

$$\begin{aligned} &\int f^*_{\lambda,\eta}(r)\xi_{1,\lambda}(d\eta)/\int f^*_{0,0}(r)\xi_{0,\lambda}(d\eta) \\ (3.7)\quad &= \int \exp\{(\lambda/2)[-1 + \bar{r}_1 + \textstyle\sum_{q+1}^p r_j \sum_{i\geqq j} \eta_i](1 + B(r, \eta, \lambda))\}\xi_{1,\lambda}(d\eta) \\ &= \int \exp\{\lambda/2[-1 + \bar{r}_1 + r_{q+1} + \textstyle\sum_{q+2}^p r_j \sum_{i\geqq j} \eta_i] \\ &\quad (1 + B(r, \eta, \lambda))\}\xi_{1,\lambda}(d\eta). \end{aligned}$$

As in Section 2, we will consider test procedures of level α based on $\bar{R}$ with rejection region of the form R_K where K is chosen in such a way that (3.7) is reduced to yield (3.2) and R_K, for this chosen K, satisfies (3.1) and (3.3). Letting $\xi_{1,\lambda}$ assigning measure one to the point $\eta_{q+1} = \cdots = \eta_{p-1} = 0$, $\eta_p = 1$, we see that (3.7) is reduced to (3.2) with $U(x) = \bar{R}_1 + \bar{R}_2$. From (3.6) we get

$$(3.8)\quad P_{\lambda,\eta}(\bar{R}_1 + \bar{R}_2 < C_\alpha'') = \exp\{(\lambda/2)(C_\alpha'' - 1)(1 + O(1))\}.$$

Hence Hotelling's test region satisfies (3.1) with $H(\lambda) = (\lambda/2)(1 - C_\alpha'')$. It is easy to check that Hotelling's test satisfies (3.3). Furthermore, since the coefficient of r_{q+1} in the expression inside the bracket in the exponent of (3.7) is one, any rejection region R_K must have $K = 1$ to satisfy (3.2) for some $\xi_{1,\lambda}$. Thus, from Remark 2, a R_K with $K \neq 1$ and which satisfies (3.1) and (3.3) can not be asymptotically minimax as $\lambda \to \infty$. The fact that Hotelling's test is asymptotically most powerful invariant under G follows trivially from (2.5). Hence we have

THEOREM 3.2 *For testing H_0 against H_1' Hotelling's test which is asymptotically most powerful invariant under G, is asymptotically minimax.*

REMARK 3. It has been shown in Giri and Kiefer (1964) that there are other asymptotically optimum tests, not of the form R_K for this problem.

After some standard calculations, it is easy to see that

$$\begin{aligned} &P_{\lambda,\eta}(\bar{R}_1 + \bar{R}_2/(1 - C) \geqq 1) \\ &\qquad = 1 - \exp\{-(\log \lambda^2 - \log((1 - \grave{C})(1 + O(1)))\}, \\ &P_{\lambda,\eta}(\bar{R}_1 + (N - q)\bar{R}_2/(p - q) \geqq C_\alpha) \\ &\qquad = 1 - \exp\{-(\lambda/2)(1 - C_\alpha)(1 + O(1))\}. \end{aligned} \tag{3.9}$$

Thus the likelihood ratio test and the locally minimax test ϕ^* satisfy (3.3). (3.1) in both cases is trivial. From Theorem 3.2 it follows that the likelihood ratio test and ϕ^* are not asymptotically minimax for this problem.

4. Lack of minimax property of any invariant test under G for every choice of λ.

We will investigate the minimax property of any invariant test under G for every fixed value of δ, and show that no invariant test of H_0 against H_λ is minimax for every choice of λ. This includes Hotelling's test, the likelihood ratio test and ϕ^*.

In order for an invariant test of H_0 against H_λ to be minimax, it has to be minimax among all invariant tests. However, since for an invariant test the power function is constant on the contour $\delta = \lambda$, "minimax" simply means "most powerful". The rejection region of the most powerful invariant test is obtained from (2.5), by setting ratio of the density of $\bar{r}$ under H_λ to that under H_0 greater than a constant, (depending on the size of the test). But the ratio

$$f_{H_\lambda}(\bar{r})/f_{H_0}(\bar{r}) = \exp\{(\lambda/2)(-1 + \bar{r}_1)\phi((N - q)/2, (p - q)/2, \bar{r}_2\lambda/2)\} \tag{4.1}$$

depends non-trivially on λ so that no test can be most powerful for every value of λ.

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