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CONVERGENCE RATES FOR PROBABILITIES OF MODERATE DEVIATIONS[1]

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1. Introduction. In [15] Rubin and Sethuraman consider sequences $\{X_n\}$ of independent random variables with partial sums $S_n = \sum_{k=1}^{n} X_k$. Under appropriate moment conditions on the X_k asymptotic forms are determined for $P[S_n > c(n \lg n)^{\frac{1}{2}}]$. As in this paper, they also study the two sided deviation problem, $P[|S_n| > c(n \lg n)^{\frac{1}{2}}]$, and term expressions of the above forms, probabilities of moderate deviations.

In this paper, the convergence rate problem is studied for both

$$P[|S_n| > c(n \lg n)^{\frac{1}{2}}] \quad \text{and} \quad P[\sup_{k \geqq n} |S_k(k \lg k)^{-\frac{1}{2}}| > c].$$

For random variables with mean zero and finite variance, it follows from the central limit theorem that $P[|S_n| > c(n \lg n)^{\frac{1}{2}}]$ tends to zero. Theorem 1 of this paper completely solves the convergence rate problem for probabilities of moderate deviations under the preceding moment conditions. Theorems 2 and 3 result from a study of the convergence rate problem when, as in [15], moments higher than the second are assumed finite.

In the last section some of the properties of moderate deviations are abstracted and analogous theorems presented in a somewhat more general setting. Here the proofs are merely outlined when similar to earlier arguments.

Throughout this paper sequences $\{X_n\}$ of independent identically distributed random variables with common distribution function F are considered. A median for the random variable X is denoted by $\mu(X)$, and $\Phi(x)$ represents the standard normal distribution function. $[x]$ stands for the largest integer less than or equal to x and $\lg x$ is the function defined by

$$\begin{aligned} \lg x &= \log_e x \quad \text{for} \quad x > 1 \\ &= 0 \qquad\quad \text{otherwise}. \end{aligned}$$

Also positive constants are written c, with or without subscripts.

2. Moderate deviations. This section deals exclusively with probabilities of the form $P[|S_n| > c(n \lg n)^{\frac{1}{2}}]$ and $P[\sup_{k \geqq n}|S_k(k \lg k)^{-\frac{1}{2}}| > c]$ for sequences of independent identically distributed random variables. In Theorem 1 characterizations are given for such sequences with $EX_1 = 0$, $EX_1^2 < \infty$ in terms of each of the above probabilities. Lemmas 1 and 2 are useful here and in later arguments.

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LEMMA 1. *Let X be a random variable and $\alpha > 0$. Then*

$$\sum n^{\alpha}(\lg n)^{\alpha+1}P[|X| > n \lg n] < \infty$$

if and only if $E|X|^{\alpha+1} < \infty$.

PROOF. Let $\psi(x \lg x) = x$ for $x > 1$. Then ψ is monotonic, hence has an inverse. Letting $Y = \psi(|X|)$, one may show by a standard argument that

$$\sum n^{\alpha}(\lg n)^{\alpha+1}P[Y > n] < \infty$$

if and only if $E|Y \lg Y|^{\alpha+1} < \infty$. By inverting Y this is seen to be equivalent to the conclusion.

LEMMA 2. *Let $\{A_n\}$ be a sequence of independent events. If $\sum P[A_n]$ is finite then*

$$P[\cup A_n] \geqq \sum P[A_n] - \sum P[A_n]\sum_{j=n+1}^{\infty} P[A_j].$$

PROOF. Proceed by induction when only finitely many terms are positive. The inequality is then preserved after taking limits.

THEOREM 1. *The following three statements are equivalent.*

1) $EX_1 = 0, EX_1^2 < \infty$.

2) $\sum_{n=1}^{\infty} \lg n(n^{-1})P[|S_n| > \epsilon(n \lg n)^{\frac{1}{2}}] < \infty$ *for all* $\epsilon > 0$.

3) $\sum_{n=1}^{\infty} n^{-1}P[\sup_{k \geqq n}|S_k(k \lg k)^{-\frac{1}{2}}| > \epsilon] < \infty$ *for all* $\epsilon > 0$.

PROOF. Note that according to Erdös' result [3], statement 1) is equivalent to $\sum_{n=1}^{\infty} P[|S_n| > n\epsilon] < \infty$ for all $\epsilon > 0$.

To show that 1) $\Rightarrow$ 2), let, for $1 \leqq k \leqq n$,

$$\begin{aligned} X_{kn} &= X_k \quad \text{if} \quad X_k < \epsilon(n \lg n)^{\frac{1}{2}} \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

and let

$$S_{nn} = \sum_{k=1}^{n} X_{kn}.$$

Then

$$\sum_{n=2}^{\infty} \lg n(n^{-1})P[|S_n| > \epsilon(n \lg n)^{\frac{1}{2}}]$$
$$\leqq \sum_{n=2}^{\infty} \lg nP[|X_1| > \epsilon(n \lg n)^{\frac{1}{2}}] + \lg n(n^{-1})P[|S_{nn}| > \epsilon(n \lg n)^{\frac{1}{2}}].$$

By Lemma 1,

$$EX_1^2 < \infty \Rightarrow \sum_{n=2}^{\infty} \lg nP[|X_1| > \epsilon(n \lg n)^{\frac{1}{2}}] < \infty.$$

Thus the series of first terms above converge, and one may deal only with the truncated random variables X_{kn}. Let X_{kn}^s be the symmetrized random variables, i.e., $X_{kn}^s = X_{kn} - X_{kn}'$ where X_{kn} and X_{kn}' are independent and identically distributed.

By Markov's inequality [13], p.158,

$$\begin{aligned} P[|S^s_{nn}| > \epsilon(n \lg n)^{\frac{1}{2}}] &\leqq E(S^s_{nn})^6\epsilon^{-6}n^{-3}(\lg n)^{-3} \\ &= [nE(X^s_{1n})^6 + 15n(n-1)E(X^s_{1n})^4E(X^s_{1n})^2 \\ &\qquad + 90n(n-1)(n-2)\{E(X^s_{1n})^2\}^3]\epsilon^{-6}(n \lg n)^{-3}. \end{aligned}$$

Here the equality follows by symmetry (odd moments vanish), independence, and identical distribution. Consider the first term in the expansion

$$\begin{aligned} &\textstyle\sum_{n=2}^{\infty} \lg n(n^{-1})E(X^s_{1n})^6 n^{-2}(\lg n)^{-3} \\ &\leqq \textstyle\sum_{n=2}^{\infty} n^{-3}(\lg n)^{-2}\sum_{k=2}^{n} (k \lg k)^3 \cdot P[(k-1)\lg(k-1) \leqq (X^s)^2 < k \lg k] \\ &= \textstyle\sum_{k=2}^{\infty} (k \lg k)^3 P[(k-1)\lg(k-1) \leqq (X^s)^2 < k \lg k]\sum_{n=k}^{\infty} n^{-3}(\lg n)^{-2} \\ &\leqq c\textstyle\sum_{k=2}^{\infty} (k \lg k)P[(k-1)\lg(k-1) \leqq (X^s)^2 < k \lg k] < \infty. \end{aligned}$$

Here the last inequality obtains by integration by parts, i.e.,

$$\textstyle\sum_{n=k}^{\infty} n^{-3}(\lg n)^{-2} = O(k^{-2}(\lg k)^{-2}).$$

The last series converges because $EX^2 < \infty \Rightarrow E(X^s)^2 < \infty$. Similar arguments hold for the other terms in the expansion. Thus since condition 1) on X_k implies 1) on X_k^s, 2) has been demonstrated for the symmetrized random variables.

By the weak symmetrization inequalities [13], p. 245,

$$\begin{aligned} &\textstyle\sum_{n=2}^{\infty} \lg n(n^{-1})P[|S_n^s| > \epsilon(n \lg n)^{\frac{1}{2}}] < \infty \\ &\qquad \Rightarrow \textstyle\sum_{n=2}^{\infty} \lg n(n^{-1})P[|S_n - \mu(S_n)| > \epsilon(n \lg n)^{\frac{1}{2}}] < \infty, \end{aligned}$$

where μ denotes the median. By [13], p. 244,

$$\text{(A)} \qquad |\mu(S_n) - ES_n| = |\mu(S_n)| \leqq (2\sigma^2(S_n))^{\frac{1}{2}} = (2nEX_1^2)^{\frac{1}{2}}.$$

Then we have

$$\begin{aligned} \infty &> \textstyle\sum_{n=2}^{\infty} \lg n(n^{-1})(P[|S_n - \mu(S_n)| > \epsilon(n \lg n)^{\frac{1}{2}}] \\ &\geqq \textstyle\sum_{n=2}^{\infty} \lg n(n^{-1})P[|S_n| > \epsilon(n \lg n)^{\frac{1}{2}} + (2nEX_1^2)^{\frac{1}{2}}] \\ &\geqq c\textstyle\sum_{n=2}^{\infty} \lg n(n^{-1})P[|S_n| > 2\epsilon(n \lg n)^{\frac{1}{2}}] \end{aligned}$$

and 1) $\Rightarrow$ 2) is proven. The converse, 2) $\Rightarrow$ 1) is demonstrated in the proof of Theorem 5.

Now it will be shown that together 1) and 2) imply 3). Let i be chosen such that $2^i \leqq n < 2^{i+1}$. Then

$$P[\sup_{k \geqq n} |S_k|(k \lg k)^{-\frac{1}{2}} > \epsilon] \leqq P[\sup_{2^i \leqq k} |S_k|(k \lg k)^{-\frac{1}{2}} > \epsilon].$$

Now again consider the symmetrized random variables X_k^s and S_n^s:

$$\begin{aligned} P[S_n^s > \epsilon(n \lg n)^{\frac{1}{2}}] &\geqq P[S_n^s > \epsilon(2^{i+1} \lg 2^{i+1})^{\frac{1}{2}}] \\ &\geqq P[(S_{2^i}^s > \epsilon(2^{i+1} \lg 2^{i+1})^{\frac{1}{2}}) \quad \text{and} \quad \textstyle\sum_{k=2^i+1}^n X_k^s \geqq 0] \\ &\geqq \tfrac{1}{2}P[S_{2^i}^s > \epsilon(2^{i+1} \lg 2^{i+1})^{\frac{1}{2}}] \\ &\geqq \tfrac{1}{2}P[S_{2^i}^s > 2\epsilon(2^i \lg 2^i)^{\frac{1}{2}}]. \end{aligned}$$

The above inequalities are due to the symmetry and independence. By the above and the symmetrization inequalities, one obtains

$$\textstyle\sum_{n=2}^\infty \lg n(n^{-1})P[|S_n| > \epsilon(n \lg n)^{\frac{1}{2}}] < \infty \quad \text{for all} \quad \epsilon > 0 \quad \text{implies}$$

$$\begin{aligned} \text{(B)} \quad \infty > \textstyle\sum_{n=2}^\infty \lg n(n^{-1})P[S_n^s > \epsilon(n \lg n)^{\frac{1}{2}}] \\ = \textstyle\sum_{i=1}^\infty \sum_{n=2^i}^{2^{i+1}-1} \lg n(n^{-1})P[S_n^s > \epsilon(n \lg n)^{\frac{1}{2}}] \\ \geqq \tfrac{1}{4}\textstyle\sum_{i=2}^\infty \lg 2^i P[S_{2^i}^s > 2\epsilon(2^i \lg 2^i)^{\frac{1}{2}}]. \end{aligned}$$

With the above inequality the convergence of 3) for symmetrized random variables may be demonstrated.

$$\begin{aligned} &\textstyle\sum_{n=2}^\infty n^{-1}P[\sup_{k\geqq n} S_k^s(k \lg k)^{-\frac{1}{2}} > \epsilon] \\ &\quad\leqq \textstyle\sum_{i=1}^\infty P[\sup_{k\geqq 2^i} S_k^s(k \lg k)^{-\frac{1}{2}} > \epsilon] \\ &\quad\leqq \textstyle\sum_{i=1}^\infty \sum_{j=i}^\infty P[\max_{2^j \leqq k < 2^{j+1}} S_k^s(k \lg k)^{-1} > \epsilon] \\ &\quad\leqq \textstyle\sum_{i=1}^\infty \sum_{j=i}^\infty P[\max_{2^j \leqq k < 2^{j+1}} S_k^s > \epsilon(2^j \lg 2^j)^{\frac{1}{2}}] \\ &\quad\leqq 2\textstyle\sum_{i=1}^\infty \sum_{j=i}^\infty P[S_{2^{j+1}}^s > \epsilon(2^j \lg 2^j)^{\frac{1}{2}}]. \end{aligned}$$

Here the last inequality comes about from Levy's inequalities [13], p. 247, and the earlier ones from the subadditivity of the measure. Now this last sum is exactly

$$\begin{aligned} &= 2\textstyle\sum_{j=1}^\infty jP[S_{2^{j+1}}^s > \epsilon(2^j \lg 2^j)^{\frac{1}{2}}] \\ &\leqq 2(\lg 2)^{-1}\textstyle\sum_{j=2}^\infty \lg 2^{j+1}P[S_{2^{j+1}}^s > \tfrac{1}{2}\epsilon(2^{j+1} \lg 2^{j+1})^{\frac{1}{2}}] \\ &\leqq c\textstyle\sum_{n=2}^\infty \lg n(n^{-1})P[S_n^s > \tfrac{1}{4}\epsilon(n \lg n)^{\frac{1}{2}}] < \infty. \end{aligned}$$

Here the last inequality is an application of (B). Hence 3) is satisfied for symmetrized variables.

Again by the symmetrization inequalities [13], p. 247,

$$\begin{aligned} &\textstyle\sum_{n=2}^\infty n^{-1}P[\sup_{k\geqq n} S_k^s(k \lg k)^{-\frac{1}{2}} > \epsilon] < \infty \\ &\qquad\Rightarrow \textstyle\sum_{n=2} n^{-1}P[\sup_{k\geqq n} |(S_k - \mu(S_k))(k \lg k)^{-\frac{1}{2}}| > \epsilon] < \infty. \end{aligned}$$

Now by an argument as in (A) it can be shown that the median term does not affect the convergence, and one obtains that

$$\sum_{n=2}^{\infty} n^{-1} P[\sup_{k \geqq n} |S_k(k \lg k)^{-\frac{1}{2}}| > 2\epsilon] < \infty$$

for all ϵ. Thus 1) and 2) $\Rightarrow$ 3).

In order to show 3) $\Rightarrow$ 1), apply Lemma 2. First it must be shown, however, that

$$\sum_{k=1}^{\infty} P[|X_1| > \epsilon(k \lg k)^{\frac{1}{2}}] < \infty.$$

Let $A_k = [|X_k| > \epsilon(k \lg k)^{\frac{1}{2}}]$. Note that $P(A_k) = P[|X_1| > \epsilon(k \lg k)^{\frac{1}{2}}]$ by the identical distribution of X_k's, and also that the A_k's are independent events.

$$\bigcup_{k=m+1}^{\infty} A_k \subset \bigcup_{k=m}^{\infty} [|S_k| > \tfrac{1}{2}\epsilon(k \lg k)^{\frac{1}{2}}],$$

thus

$$P[\bigcup_{k=m+1}^{\infty} A_k] \leqq P[\bigcup_{k=m}^{\infty}[|S_k| > \tfrac{1}{2}\epsilon(k \lg k)^{\frac{1}{2}}]]$$
$$= P[\sup_{k \geqq m} |S_k(k \lg k)^{-\frac{1}{2}}| > \tfrac{1}{2}\epsilon] \to_{m\to\infty} 0.$$

From hypothesis 3) and the fact that the last term is non-increasing in m, it must tend to zero; see [1], Lemma p. 113. That is, $P(\limsup A_k) = 0$.

Now apply the Borel zero-one criterion [13], p. 228, and

$$\sum_{k=1}^{\infty} P[|X_1| > \epsilon(k \lg k)^{\frac{1}{2}}] = \sum_{k=1}^{\infty} P[|X_k| > \epsilon(k \lg k)^{\frac{1}{2}}] = \sum_{k=1}^{\infty} P[A_k] < \infty.$$

To complete the proof of 3) $\Rightarrow$ 1), let N be such that $\sum_{n=N}^{\infty} P[|X_n| > \epsilon(n \lg n)^{\frac{1}{2}}] \leqq \delta < 1$. Then

$$\begin{aligned}
\infty &> \sum_{n=N}^{\infty} n^{-1} P[\sup_{k \geqq n} |S_k(k \lg k)^{-\frac{1}{2}}| > \tfrac{1}{2}\epsilon] \\
&\geqq \sum_{n=N}^{\infty} n^{-1} P[\bigcup_{k=0}^{\infty} [|X_{n+k}| > \epsilon((n+k) \lg (n+k))^{\frac{1}{2}}]] \\
&\geqq c \sum_{n=N}^{\infty} n^{-1} \sum_{k=0}^{\infty} P[|X_{n+k}| > \epsilon((n+k) \lg (n+k))^{\frac{1}{2}}] \\
&\geqq c' \sum_{k=N}^{\infty} P[|X_1| > \epsilon(k \lg k)^{\frac{1}{2}}] \sum_{n=2}^{k} n^{-1} \\
&\geqq c'' \sum_{k=N}^{\infty} P[|X_1| > \epsilon(k \lg k)^{\frac{1}{2}}] \lg k
\end{aligned}$$

where the third inequality is of course obtained by Lemma 2.

Since this last series converges, Lemma 1 is applied once more taking $\alpha = 0$, and the desired result $EX_1^2 < \infty$ is obtained. From the hypothesis 3) one easily obtains $S_n/n \to_{\text{a.s.}} 0$, but since $EX^2 < \infty$, the strong law of large numbers yields $S_n/n \to_{\text{a.s.}} EX_1$ and hence $EX_1 = 0$. Thus 3) $\Rightarrow$ 1), and the proof of Theorem 1 is complete.

In Theorem 2 random variables with several finite moments are considered. The result demonstrates the conditions that a prescribed convergence rate imposes on the individual random variables.

THEOREM 2. *If the series*

$$\sum_{n=1}^{\infty} n^{(c_0^2/2)-1} (\lg n)^{(c_0^2/2)+1} P[|S_n| > c(n \lg n)^{\frac{1}{2}}]$$

converges for all $c > c_0 > 0$ *then* $EX_1 = 0$ *and* $E|X_1|^{c_0^2+2} < \infty$. *Also for* $c_0 \leqq 1$ *then* $EX_1^2 \leqq 1$ *and for* $c_0 > 1$, $EX_1^2 \leqq c_0^2$.

PROOF. Take $\varphi_n = (\lg n)^{\frac{1}{2}+c_0^2/4}$ and apply Theorem 5 to obtain $EX_1 = 0$ and $EX_1^2 < \infty$. Thus (3) of Theorem 2 can be used to give $\sum P[|X_n| > 2(n \lg n)^{\frac{1}{2}}] < \infty$ by the zero-one argument given previously. Since the sequence of above terms is monotone decreasing one obtains $nP[|X_n| > \epsilon(n \lg n)^{\frac{1}{2}}] \to 0$ for all $\epsilon > 0$. Let $M_n = \sum [|S_n| > \epsilon(n \lg n)^{\frac{1}{2}}]$ and notice $P[M_n] \to 0$ as in the proof to Theorem 5 with $\varphi_n = (\lg n)^{\frac{1}{2}}$. There is an N such that for all $n \geqq N$ and all k, $1 \leqq k \leqq n$, $P[|S_n - X_k| < \epsilon(n \lg n)^{\frac{1}{2}}] > \lambda > 0$. Now fix $n \geqq N$, let $A_k = [|X_k| > 2\epsilon(n \lg n)^{\frac{1}{2}}]$, $B_k = [|S_n - X_k| < \epsilon(n \lg n)^{\frac{1}{2}}]$ for $1 \leqq k \leqq n$ and proceed as in [3] using the fact that $nP[A_n] \to 0$. That is, since $A_k \cap B_k \subset M_n$, $P[M_n] \geqq P[\bigcup_{k=1}^{n} (A_k \cap B_k)] \geqq \gamma n P[A_n]$, $\gamma > 0$ and independent of n. Now since the random variables are identically distributed one obtains

$$\sum n^{c_0^2/2}(\lg n)^{(c_0^2/2)+1}P[|X_1| > 2c(n \lg n)^{\frac{1}{2}}] < \infty.$$

An application of Lemma 1 gives the desired result $E|X_1|^{c_0^2+2} < \infty$.

In showing $EX_1^2 \leqq 1$ for $c_0 \leqq 1$ Katz' extension of the Berry-Esseen theorem may be used since the existence of $c_0^2 + 2$ moments has been established. Let $\sigma^2 = EX_1^2$.

$$\begin{aligned} P[|S_n| > c(n \lg n)^{\frac{1}{2}}] &= P[|S_n\sigma^{-1}n^{-\frac{1}{2}}| > c\sigma^{-1}(\lg n)^{\frac{1}{2}}] \\ &\geqq 2\Phi(-c\sigma^{-1}(\lg n)^{\frac{1}{2}}) - |P[|S_n\sigma^{-1}n^{-\frac{1}{2}}| > c\sigma^{-1}(\lg n)^{\frac{1}{2}}] \\ &\qquad - 2\Phi(-c\sigma^{-1}(\lg n)^{\frac{1}{2}}))|. \end{aligned}$$

Now by [10], one obtains

$$|P[|S_n\sigma^{-1}n^{-\frac{1}{2}}| > c\sigma^{-1}(\lg n)^{\frac{1}{2}}] - 2\Phi(-c\sigma^{-1}(\lg n)^{\frac{1}{2}})| \leqq k/n^{c_0^2/2}$$

with $k > 0$. Also using the approximation to the tail of the normal distribution [5], p. 166:

$$2\Phi(-c\sigma^{-1}(\lg n)^{\frac{1}{2}}) \geqq d(\lg n)^{-\frac{1}{2}}n^{-c^2/2\sigma^2}, \qquad d > 0.$$

Now from the above inequalities

$$P[|S_n| > c(n \lg n)^{\frac{1}{2}}] \geqq d(\lg n)^{-\frac{1}{2}}n^{-c^2/2\sigma^2}(1 - k(\lg n)^{\frac{1}{2}}d^{-1}n^{-(c_0^2/2)+(c^2/2\sigma^2)}).$$

Under the assumption $\sigma^2 > 1$, choose a $c > c_0$ suh that $c/\sigma < c_0$. Then for all sufficiently large n the last expression above majorizes $dn^{-c^2/2\sigma^2}/2(\lg n)^{\frac{1}{2}}$ and the contradiction can now be obtained:

$$\begin{aligned} \sum_{n=N}^{\infty} n^{(c_0^2/2)-1}(\lg n)^{(c_0^2/2)+1}P[|S_n| > c(n \lg n)^{\frac{1}{2}}] \\ \geqq \tfrac{1}{2}d\sum_{n=N}^{\infty} n^{(c_0^2/2)-(c^2/2\sigma^2)-1}(\lg n)^{(c_0^2+1)/2} = \infty. \end{aligned}$$

Thus $EX_1^2 \leqq 1$ for $c_0 \leqq 1$.

If $c_0 \geqq 1$ the convergence rate certainly implies

$$\sum n^{-\frac{1}{2}}P[|S_n| > c(n \lg n)^{\frac{1}{2}}] < \infty.$$

The Berry-Esseen theorem then gives

$$\sum n^{-(\frac{1}{2}+\epsilon)} \sup |P[S_n\sigma^{-1}n^{-\frac{1}{2}} < x] - \Phi(x)| < \infty$$

for all $\epsilon > 0$. Thus

$$\sum n^{-(\frac{1}{2}+\epsilon)}|P[S_n\sigma^{-1}n^{-\frac{1}{2}} > c\sigma^{-1}(\lg n)^{\frac{1}{2}}] - (1 - \Phi(c\sigma^{-1}(\lg n)^{\frac{1}{2}}))| < \infty.$$

Then $\sum n^{-(\frac{1}{2}+\epsilon)}P[|S_n| > c(n \lg n)^{\frac{1}{2}}] < \infty$ if and only if $\sum n^{-(\frac{1}{2}+\epsilon)}(1 - \Phi(c\sigma^{-1}(\lg n)^{\frac{1}{2}}) < \infty$ which in turn converges and diverges with

$$\sum n^{-(\frac{1}{2}+\epsilon)}(\lg n)^{-\frac{1}{2}} \exp(-\tfrac{1}{2}c^2\sigma^{-2} \lg n) = \sum (\lg n)^{-\frac{1}{2}}n^{-(\frac{1}{2}+\epsilon+\frac{1}{2}c^2/\sigma^2)}.$$

Thus $2\epsilon + c^2(\sigma^{-2}) > 1$ for all $\epsilon > 0$ and $c^2 \geqq \sigma^2$ for all $c > c_0$. This implies $c_0^2 \geqq \sigma^2$ which completes the theorem.

Theorem 3 demonstrates how the moment conditions on the random variables imply a convergence rate for $P[\sup_{k\geqq n} |S_k(k \lg k)^{-\frac{1}{2}}| > c]$. The following proposition is needed.

Proposition 1. *Let $a > -1$, $b \geqq 0$, and $c_0 \geqq 0$. If $EX_1 = 0$ and $EX_1^2 < \infty$,*

$$\sum_{n=1}^{\infty} n^a(\lg n)^b P[\sup_{k\geqq n} |S_k(k \lg k)^{-\frac{1}{2}}| > c] < \infty$$

for all $c > c_0$ if and only if

$$\sum_{n=1}^{\infty} n^a(\lg n)^b P[|S_n| > c(n \lg n)^{\frac{1}{2}}] < \infty$$

for all $c > c_0$.

Proof. Fix $c > c_0$ and choose $\alpha > 1$ such that $c/\alpha^4 > c_0$.

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^a(\lg n)^b P[\sup_{k\geqq n} |S_k(k \lg k)^{-\frac{1}{2}}| > c] \\
(1)\qquad &\leqq c\sum_{i=1}^{\infty} \alpha^{i(a+1)}(\lg \alpha^i)^b P[\sup_{k\geqq \alpha^i} |S_k(k \lg k)^{-\frac{1}{2}}| > c] \\
&\leqq c_1\sum_{i=1}^{\infty} \alpha^{i(a+1)}i^b\sum_{j=i}^{\infty} P[\max_{\alpha^j\leqq k<\alpha^{j+1}} |S_k(k \lg k)^{-\frac{1}{2}}| > c] \\
&\leqq c_1\sum_{i=1}^{\infty} \alpha^{i(a+1)}i^b\sum_{j=i}^{\infty} P[\max_{\alpha^j\leqq k<\alpha^{j+1}} \\
&\qquad \cdot|[S_k - \mu(S_k - S_{[\alpha^{j+1}]})](\alpha^j \lg \alpha^j)^{-\frac{1}{2}}| \\
&\qquad > c - \max_{\alpha^j\leqq k<\alpha^{j+1}} |\mu(S_k - S_{[\alpha^{j+1}]})|(\alpha^j \lg \alpha^j)^{-\frac{1}{2}}] \\
(2)\qquad &\leqq c_2\sum_{i=1}^{\infty} \alpha^{i(a+1)}i^b\sum_{j=i}^{\infty} P[\max_{\alpha^j\leqq k<\alpha^{j+1}} |(S_k - \mu(S_k - S_{[\alpha^{j+1}]})) \\
&\qquad \cdot(\alpha^j \lg \alpha^j)^{-\frac{1}{2}}| > c\alpha^{-1}] \\
(3)\qquad &\leqq c_3\sum_{i=1}^{\infty} \alpha^{i(a+1)}i^b\sum_{j=i}^{\infty} P[|S_{[\alpha^{j+1}]}| > c\alpha^{-2}(\alpha^{j+1} \lg \alpha^{j+1})^{\frac{1}{2}}] \\
&= c_3\sum_{j=1}^{\infty} P[|S_{[\alpha^{j+1}]}| > c\alpha^{-2}(\alpha^{j+1} \lg \alpha^{j+1})^{\frac{1}{2}}]\sum_{i=1}^{j} \alpha^{i(a+1)}i^b \\
(4)\qquad &\leqq c_4\sum_{j=1}^{\infty} \alpha^{j(a+1)}j^b P[|S_{[\alpha^{j+1}]}| > c\alpha^{-2}(\alpha^{j+1} \lg \alpha^{j+1})^{\frac{1}{2}}] \\
(5)\qquad &\leqq c_5\sum_{j=1}^{\infty} \alpha^{ja}j^b\sum_{n=[\alpha^j]}^{[\alpha^{j+1}]-1} P[|S_n| > c\alpha^{-4}(n \lg n)^{\frac{1}{2}}] \\
(6)\qquad &\leqq c_6\sum_{n=1}^{\infty} n^a(\lg n)^b P[|S_n| > c\alpha^{-4}(n \lg n)^{\frac{1}{2}}] < \infty.
\end{aligned}$$

Step (1) obtains as follows. If $\alpha^i \leqq n$ then

$$P[\sup_{k\geqq n} |S_k(k \lg k)^{-\frac{1}{2}}| > c] \leqq P[\sup_{k\geqq \alpha^i} |S_k(k \lg k)^{-\frac{1}{2}}| > c].$$

Thus

$$\sum_{n=[\alpha^i]}^{[\alpha^{i+1}]} n^a(\lg n)^b P[\sup_{k\geqq n} |S_k(k \lg k)^{-\frac{1}{2}}| > c]$$
$$\leqq P[\sup_{k\geqq \alpha^i} |S_k(k \lg k)^{-\frac{1}{2}}| > c]\sum_{n=[\alpha^i]}^{[\alpha^{i+1}]} n^a(\lg n)^b$$
$$\leqq c\alpha^{i(a+1)}(\lg \alpha^i)^b P[\sup_{k\geqq \alpha^i} |S_k(k \lg k)^{-\frac{1}{2}}| > c].$$

(2) is obtained as in step (A) of Theorem 1.

Step (3) follows by Levy's inequalities, Loève [13], p. 247, and

$$c\alpha^{-2}(\alpha^{j+1} \lg \alpha^{j+1})^{\frac{1}{2}} \leqq c\alpha^{-1}(\alpha^j \lg \alpha^j)^{\frac{1}{2}}, \quad \text{for all} \quad j \geqq (\alpha - 1)^{-1}.$$

(4) follows from

$$\sum_{i=1}^{j} \alpha^{i(a+1)} i^b = O(\alpha^{j(a+1)} j^b).$$

Another application of Levy's inequalities yields (5). For $\alpha^j \leqq n < \alpha^{j+1}$,

$$P[|S_{[\alpha^j]}| > c\alpha^{-2}(\alpha^j \lg \alpha^j)^{\frac{1}{2}}]$$
$$\leqq P[\max_{\alpha^j \leqq k \leqq n} |S_k - \mu(S_k - S_n)| > c\alpha^{-2}(\alpha^j \lg \alpha^j)^{\frac{1}{2}} - \max_{\alpha^j \leqq k \leqq n} |\mu(S_k - S_n)|]$$
$$\leqq P[\max_{\alpha^j \leqq k \leqq n} |S_k - \mu(S_k - S_n)| > c\alpha^{-3}(\alpha^j \lg \alpha^j)^{\frac{1}{2}}]$$
$$\leqq 2P[|S_n| > c\alpha^{-4}(n \lg n)^{\frac{1}{2}}].$$

Thus,

$$P[|S_{[\alpha^j]}| > c\alpha^{-2}(\alpha^j \lg \alpha^j)^{\frac{1}{2}}]$$
$$\leqq 2\alpha^{-j}(\alpha - 1)^{-1}\sum_{n=[\alpha^j]}^{[\alpha^{j+1}]} P[|S_n| > c\alpha^{-4}(n \lg n)^{\frac{1}{2}}].$$

Finally, for $\alpha^j \leqq n < \alpha^{j+1}$, (6) follows from

$$\alpha^{ja} j^b \leqq cn^a(\lg n)^b.$$

The converse of the proposition is obvious.

Using the above result, the proof of Theorem 3 is reduced to showing that

$$\sum_{n=1}^{\infty} n^{(c_0^2/2)-1}(\lg n)^{(c_0-1)/2} P[|S_n| > c(n \lg n)^{\frac{1}{2}}] < \infty.$$

THEOREM 3. *If* $EX_1 = 0$, $EX_1^2 \leqq 1$, *and* $E|X_1|^{c_0^2+2} < \infty$ *for some integer* $c_0^2 \geqq 1$, *then*

$$\sum_{n=1}^{\infty} n^{(c_0^2/2)-1}(\lg n)^{(c_0^2-1)/2} P[\sup_{k\geqq n} |S_k(k \lg k)^{-\frac{1}{2}}| > c]$$

converges for all $c > c_0$.

PROOF. Setting $t_n = c\sigma^{-1}(\lg n)^{\frac{1}{2}}$ with $\sigma^2 = EX_1^2$, one may proceed as in Theorem 2:

$$P[|S_n| > c(n \lg n)^{\frac{1}{2}}] = P[|S_n(\sigma^{-1}n^{-\frac{1}{2}})| > t_n]$$
$$\leqq |P[|S_n\sigma^{-1}n^{-\frac{1}{2}}| > t_n] - 2\Phi(-t_n)| + 2\Phi(-t_n).$$

Now because $t_n = c\sigma^{-1}(\lg n)^{\frac{1}{2}} \geqq ((1+\delta)c_0^2 \lg n)^{\frac{1}{2}}$ for some $\delta > 0$, an application of an inequality due to Esseen [4], p. 73, gives

$$|P[|S_n\sigma^{-1}n^{-\frac{1}{2}}| > t_n] - 2\Phi(-t_n)| \leqq c_1/(1 + t_n^{c_0^2+2})n^{c_0^2/2}.$$

Again by the approximation to the normal distribution in the tail,

$$2\Phi(-t_n) \leqq c_2 t_n^{-1}e^{-t_n^2/2}.$$

Thus, replacing t_n by $c\sigma^{-1}(\lg n)^{\frac{1}{2}}$, each term of the original series may be bounded:

$$n^{(c_0^2/2)-1}(\lg n)^{(c_0^2-1)/2}P[|S_n| > c(n \lg n)^{\frac{1}{2}}]$$
$$\leqq c_3(n^{-1}(\lg n)^{-3/2} + n^{-[(c^2/2\sigma^2)-(c_0^2/2)+1]}(\lg n)^{-[(c_0^2/2)+1]})$$

for all sufficiently large n. This sequence is summable for all $c > c_C$; thus

$$\sum n^{(c_0^2/2)-1}(\lg n)^{(c_0^2-1)/2}P[|S_n| > c(n \lg n)^{\frac{1}{2}}]$$

converges for all $c > c_0$. The hypotheses of Proposition 1 are satisfied and the theorem is proven.

3. Generalizations. In this section sequences $\{\varphi_n\}$ with certain properties of $(\lg n)^{\frac{1}{2}}$ are considered. By restricting the rate of increase of $\{\varphi_n\}$ it is possible to prove theorems analogous to those of the previous part. Lemmas 3 and 4 are extensions of the first lemma of the previous section.

LEMMA 3. *Let X be a random variable and $\{\varphi_n\}$ a non-decreasing positive sequence. Then*

$$EX^2 < \infty \Rightarrow \sum \varphi_n^2 P[|X| > n^{\frac{1}{2}}\varphi_n] < \infty.$$

PROOF. Take $\psi(n\varphi_n^2) = n$ and proceed as in Lemma 1.

With the above a convergence rate is established for sequences $\{\varphi_n\}$ that may increase somewhat less rapidly than $(\lg n)^{\frac{1}{2}}$.

THEOREM 4. *Let $\{\varphi_n\}$ be a non-decreasing positive sequence such that $\sum (1/n\varphi_n^4) < \infty$.*

If $EX_1 = 0$ and $EX_1^2 < \infty$, then $\sum \varphi_n^2 n^{-1}P[|S_n| > n^{\frac{1}{2}}\varphi_n] < \infty$. If $\limsup (n/\varphi_n^2) = 0$ *the centering at expectations is unnecessary.*

PROOF. As in Theorem 1, using Lemma 3 one may show that the random variables may be truncated at $n^{\frac{1}{2}}\varphi_n$. Next, by much the same argument as the one used in Theorem 1, it is shown that the convergence of the series of symmetrized random variables is equivalent to the conclusion of the theorem. Again it must be shown that

$$|\mu(S_n)/\varphi_n n^{\frac{1}{2}}| \to 0.$$

With this information Markov's inequality is applied to the truncated symmetrized random variables; i.e.,

$$P[|S_{nn}^s| > \varphi_n n^{\frac{1}{2}}] \leqq E(S_{nn}^s)^6 n^{-3}\varphi_n^{-6}.$$

Then this expectation is expanded and, as in Theorem 1, the hypothesis

$\sum n^{-1}\varphi_n^{-4} < \infty$ is just right to insure that $\sum \varphi_n^2 n^{-1} P[|S_{nn}^s| > \varphi_n n^{\frac{1}{2}}] < \infty$,

which has been shown to be equivalent to the original conclusion. To demonstrate the last remark of the theorem, assume that lim sup $(n/\varphi_n^2) = 0$ and $EX_1^2 < \infty$. Then

$$\sum_{n=1}^{\infty} \varphi^2(n) n^{-1} P[|S_n - E(S_n)| > \tfrac{1}{2} n^{\frac{1}{2}} \varphi_n] < \infty,$$

but, as in (A) of Theorem 1, it is shown that this implies

$$\sum \varphi^2(n) n^{-1} P[|S_n| > n^{\frac{1}{2}} \varphi_n] < \infty$$

with $EX_1 \neq 0$.

Before proving a converse to Theorem 4, the following lemmas are needed.

Lemma 4. *Let X be a random variable and $\{\varphi_n\}$ a non-decreasing positive sequence such that*

$$n\varphi_n^2 = O(\textstyle\sum_{k=1}^{n} \varphi_k^2).$$

Then

$$EX^2 < \infty \Leftrightarrow \textstyle\sum_{n=1}^{\infty} \varphi_n^2 P[|X| > \varphi_n n^{\frac{1}{2}}] < \infty.$$

Proof. Let $n\varphi_n^2 \leqq K \sum_{k=1}^{n} \varphi_k^2$ for all n. Then

$$\begin{aligned} EX^2 &\leqq \textstyle\sum_{n=1}^{\infty} n\varphi^2(n) P[(n-1)\varphi_{n-1}^2 < X^2 \leqq n\varphi_n^2] \\ &\leqq K \textstyle\sum_{n=1}^{\infty} (\sum_{k=1}^{n} \varphi_k^2) P[(n-1)\varphi_{n-1}^2 < X^2 \leqq n\varphi_n^2] \\ &= K \textstyle\sum_{n=1}^{\infty} \varphi_n^2 P[X^2 > (n-1)\varphi_{n-1}^2] \end{aligned}$$

which converges by hypothesis since $\varphi_n^2 \leqq k[(n-1)/(n-k)]\varphi_{n-1}^2$ for $n > k$. The converse is, of course, Lemma 3.

Lemma 5. *Let $\{\varphi_n\}$ be a positive non-decreasing sequence. Then*

$$n\varphi_n = O(\textstyle\sum_{k=1}^{n} \varphi_k) \Leftrightarrow \varphi_{2n} = O(\varphi_n).$$

Proof. Let

$$n\varphi_n = O(\textstyle\sum_{k=1}^{n} \varphi_k).$$

Then there is a constant L, $0 < L < 1$, such that

$$0 < Ln\varphi_n \leqq \textstyle\sum_{k=1}^{n} \varphi_k \leqq n\varphi_n$$

because φ_n is non-decreasing. Now choose α such that $1 < \alpha < 1/(1 - L)$ and again let $[x]$ denote the greatest integer function. Then

$$L[\alpha n]\varphi_{[\alpha n]} \leqq \textstyle\sum_{k=1}^{[\alpha n]} \varphi_k \qquad \text{and} \qquad -n\varphi_n \leqq -\sum_{k=1}^{n} \varphi_k .$$

By adding the above inequalities and simplifying, one obtains

$$\varphi_{[\alpha n]} \leqq \varphi_n/(\alpha(L-1)+1) \leqq k\varphi_n .$$

To show that $\varphi_{2n} = O(\varphi_n)$, one obtains for sufficiently large n

$$\varphi_n \geqq \varphi_{[\alpha n]}k^{-1} \geqq \varphi[\alpha[\alpha n]]k^{-2} \geqq \cdots \geqq \varphi[\alpha[\cdots[\alpha n]]k^{-r} \geqq \varphi_{2n}k^{-r},$$

independently of n.

For the converse, assume the existence of some $k > 0$ such that $\varphi_{2n} \leqq k\varphi_n$ for all n; then

$$n\varphi_{2n} \leqq nk\varphi_n \leqq k\sum_{k=n+1}^{2n} \varphi_k \leqq k\sum_{k=1}^{2n} \varphi_k .$$

THEOREM 5. *Let* $\{\varphi_n\}$ *be a positive non-decreasing sequence such that* $n\varphi_n^2 = O(\sum_{k=1}^{n} \varphi_k^2)$. *If*

$$\sum_{n=1}^{\infty} \varphi_n^2(n^{-1})P[|S_n| > \epsilon n^{\frac{1}{2}}\varphi_n]$$

converges for all $\epsilon > 0$, *then* EX_1^2 *is finite. Furthermore, if* lim sup $(n/\varphi_n^2) > 0$ *then* $EX_1 = 0$.

PROOF. By the symmetrization inequalities,

$$\sum_{n=1}^{\infty} \varphi_n^2 n^{-1}P[|S_n| > \epsilon n^{\frac{1}{2}}\varphi_n] < \infty \Rightarrow \sum_{n=1}^{\infty} \varphi_n^2 n^{-1}P[S_n^s > \epsilon n^{\frac{1}{2}}\varphi_n] < \infty$$

where S_n^s is, of course, the sum of the symmetrized random variables. It will now be shown that the convergence of the latter series gives $E(X_1^s)^2 < \infty$, and thus $EX_1^2 < \infty$.

First one must show that $S_n^s\varphi_n^{-1}n^{-\frac{1}{2}} \to 0$ in probability. This is, of course, clear from the hypothesis if $\varphi_n^2 > \delta n$ infinitely often for some $\delta > 0$. If $S_n^s\varphi_n^{-1}n^{-\frac{1}{2}}$ does not converge to zero, there is an $\epsilon_0 > 0$ and a subsequence $\{n_k\}$ such that $n_{k+1} > 2n_k$ and

$$P[S_{n_k}^s > \epsilon_0\varphi_{n_k}n_k^{\frac{1}{2}}] > \epsilon_0 .$$

Let j be such that $n_k \leqq j < 2n_k$, and apply Lemma 5 to obtain the existence of K such that $\varphi_{2n}^2 \leqq K\varphi_n^2$ for all n. Then

$$(2K)^{\frac{1}{2}}\varphi(n_k)n_k^{\frac{1}{2}} \geqq \varphi(2n_k)(2n_k)^{\frac{1}{2}} > \varphi(j)j^{\frac{1}{2}}.$$

Hence

$$\begin{aligned} P[S_j^s > \epsilon_0(2k)^{-\frac{1}{2}}\varphi(j)j^{\frac{1}{2}}] &\geqq P[S_j^s > \epsilon_0(2K)^{-\frac{1}{2}}\varphi(2n_k)(2n_k)^{\frac{1}{2}}] \\ &\geqq P[S_j^s > \epsilon_0\varphi(n_k)n_k^{\frac{1}{2}}] \\ &\geqq \tfrac{1}{2}P[S_{n_k}^s > \epsilon_0\varphi(n_k)n_k^{\frac{1}{2}}] \\ &> \epsilon_0/2. \end{aligned}$$

The above obtains by symmetry and independence. Then we have

$$\begin{aligned} \sum_{n=1}^{\infty} \varphi^2(n)n^{-1}P[S_n^s > \epsilon_0(2K)^{-\frac{1}{2}}n^{\frac{1}{2}}\varphi(n)] & \\ &\geqq \sum_{k=1}^{\infty} \sum_{i=n_k}^{2n_k} \varphi^2(i)i^{-1}P[S_i^s > \epsilon_0(2K)^{-\frac{1}{2}}i^{\frac{1}{2}}\varphi(i)] \\ &\geqq \tfrac{1}{2}\epsilon_0\sum_{k=1}^{\infty} \sum_{i=n_k}^{2n_k} \varphi^2(i)i^{-1} \\ &\geqq \tfrac{1}{2}\epsilon_0\sum_{k=1}^{\infty} \tfrac{1}{2}\varphi^2(n_k)n_k^{-1}\cdot n_k = \infty \end{aligned}$$

because φ^2 is non-decreasing. Thus this contradiction shows that $S_n^s(\varphi(n)n^{\frac{1}{2}})^{-1} \to 0$ in probability. Now by the corollary to the degenerate convergence criterion [13], p. 317, one obtains

$$nP[X_n^s > \epsilon\varphi(n)n^{\frac{1}{2}}] \to 0,$$

and now Erdös' method is used to obtain $E(X^s)^2 < \infty$. Taking

$$A_k = [X_k^s > 2\epsilon\varphi(n)n^{\frac{1}{2}}]$$

and

$$B_k = [\sum_{i \neq k}^n X_i^s < \epsilon\varphi(n)n^{\frac{1}{2}}],$$

one proceeds just as in Theorem 2 and obtains

$$\begin{aligned}\infty &> \sum_{n=1}^\infty \varphi^2(n)n^{-1}P[S_n^s > \epsilon n^{\frac{1}{2}}\varphi(n)]\\ &\geqq c\sum_{n=1}^\infty \varphi^2(n)P[X_1^s > 2\epsilon n^{\frac{1}{2}}\varphi(n)].\end{aligned}$$

By Lemma 4, since $n\varphi^2(n) = O(\sum_{k=1}^n \varphi^2(k))$ and ϵ was arbitrary, $E(X_1^s)^2 < \infty$. Thus $EX_1^2 < \infty$.

To show the last statement in the theorem, it is known that $S_n/n \to EX_1$ and $S_n/(n^{\frac{1}{2}}\varphi(n)) \to 0$, but

$$S_n/(n^{\frac{1}{2}}\varphi(n)) - EX_1 n^{\frac{1}{2}}/(\varphi(n)) \to 0.$$

Hence if $n/\varphi^2(n) > \epsilon$ infinitely often, i.e., lim sup $(n/\varphi^2(n)) > 0$, then EX_1 must be zero.

The combination of these results gives an extension of Theorem 1 to a more general class of sequences. For example taking $\varphi_n = n^{\frac{1}{2}}$ in Theorems 4 and 5 gives the previously mentioned result of Erdös [3].

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