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ON THE RATE OF CONVERGENCE FOR THE LAW OF LARGE NUMBERS[1]

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Introduction. Let $X_1 X_2, \ldots$ be a sequence of independent random variables with a common expectation μ. If $t > 1$ and $E|X_i|^t \leqq C$ for all i then it is known $\sum_1^n X_i/n \to \mu$ with probability 1. The main result of this paper is that

$$q_n(\epsilon) = P[\sup_{k \geqq n} |\sum_1^k X_i - k\mu|/k \geqq \epsilon] = O(1/n^{t-1}) \tag{1}$$

for each $\epsilon > 0$. Assuming that $X_1, X_2, \cdots$ are also identically distributed, Baum and Katz [1] showed, among other things, that $EX_1 = \mu$ and $E|X_1|^t < \infty$ if, and only if,

$$\sum_1^\infty n^{t-2} q_n(\epsilon) < \infty \quad \text{for each} \quad \epsilon > 0. \tag{2}$$

We note that (2) implies that $q_n(\epsilon) = o(1/n^{t-1})$ (e.g., see the lemma on page 113 of [1]).

THEOREM. *Let $X_1, X_2, \cdots$ be a sequence of independent random variables with a common expectation μ. If $t > 1$ and $E|X_i|^t \leqq C$ for all i then, for each $\epsilon > 0$,*

$$P[\sup_{k \geqq n} |\sum_1^k X_i - k\mu|/k \geqq \epsilon] = O(1/n^{t-1}).$$

The first step in the proof is a generalization of the Hájék-Rényi inequality [2].

LEMMA. *Suppose $r \geqq 1$. Let $Y_1, Y_2, \cdots$ be a sequence of independent random variables with $EY_i = 0$ and $E|Y_i|^r < \infty$ for all i. Put $S_k = Y_1 + \cdots + Y_k$ for all k. Then for each nonincreasing sequence $\{c_k\}$ os positive constants, for each choice of integers $0 < n < m \leqq \infty$ and for each $\epsilon > 0$*

$$\epsilon^r P[\sup_{n \leqq k \leqq m} c_k |S_k| \geqq \epsilon] \leqq c_n^r E|S_n|^r + \sum_{n+1}^m c_k^r [E|S_k|^r - E|S_{k-1}|^r]. \tag{3}$$

The usual Hájek-Rényi inequality is obtained from (3) when $r = 2$. The proof of (3) follows the proof for $r = 2$ given in [2] after noting that if $j \leqq k$ then $E|S_k|^r I_A \geqq E|S_j|^r I_A$ where A is any event defined by $Y_1, \cdots, Y_j$. (See Loève [3], problem 2, page 263).

PROOF OF THE THEOREM. We may take $\mu = 0$, $\epsilon < 1$, and $C > 1$ without loss of generality.

$$\begin{aligned} &P\{\textstyle\bigcup_{k=n}^\infty [|\sum_1^k X_i| \geqq k\epsilon]\} \\ (4) \quad &= P\{\textstyle\bigcup_{k=n}^\infty [|\sum_1^k X_i| \geqq k\epsilon]; (\bigcup_1^n [|X_k| > n-1]) \cup (\bigcup_{n+1}^\infty [|X_k| > k-1])\} \\ &\quad + P\{\textstyle\bigcup_{k=n}^\infty [|\sum_1^k X_i| \geqq k\epsilon]; \bigcap_1^n [|X_k| \leqq n-1]; \bigcap_{n+1}^\infty [|X_k| \leqq k-1]\}. \end{aligned}$$

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The first term of the right-hand side of (4) is bounded by

$$P\{(\textstyle\bigcup_1^n [|X_k| > n - 1]) \cup (\bigcup_{n+1}^\infty [|X_k| > k - 1])\}$$
$$\leqq nC/(n-1)^t + \textstyle\sum_{n+1}^\infty C/(k-1)^t = O(1/n^{t-1}).$$

For the second term of the right-hand side of (4) let

$$\begin{aligned} Y_j &= X_j I_{[|X_j| \leqq n-1]} - EX_j I_{[|X_j| \leqq n-1]}, && 1 \leqq j \leqq n \\ &= X_j I_{[|X_j| \leqq j-1]} - EX_j I_{[|X_j| \leqq j-1]}, && j \geqq n+1 \end{aligned}$$

so that $Y_1, Y_2, \cdots$ is a sequence of independent random variables where, for all i, $EY_i = 0$, $E|Y_i|^t \leqq 2^t C$ and $E|Y_i|^r < \infty$ for all $r \geqq 1$. Choose N such that

$$|EX_j I_{[|X_j| \leqq k-1]}| = |EX_j I_{[|X_j| > k-1]}| \leqq C/(k-1)^{t-1} < \tfrac{1}{2}\epsilon$$

for all j and for all $k \geqq N$. Thus for $n \geqq N$

$$\{\textstyle\bigcup_{k=n}^\infty [|\sum_1^k X_i| \geqq k\epsilon];\ \bigcap_1^n [|X_k| \leqq n-1];\ \bigcap_{n+1}^\infty [|X_k| \leqq k-1]\}$$
$$\subseteq \textstyle\bigcup_n^\infty [|S_k| \geqq k\frac{1}{2}\epsilon]$$

where $S_k = Y_1 + \cdots + Y_k$ for all k. We now use the lemma to show that $P\{\bigcup_n^\infty [|S_k| \geqq k\epsilon]\} = O(1/n^{t-1})$ for each $\epsilon > 0$.

The case $1 < t \leqq 2$ follows easily from the lemma by taking $r = 2$, $\{c_k\} = \{1/k\}$ and using the argument given below for $t > 2$. Thus assume $t > 2$ and put $\{c_k\} = \{1/k\}$ and $r = 2b$ where b is the smallest integer $\geqq t$. We see from (3) that

$$(\epsilon)^{2b} P\{\textstyle\bigcup_n^\infty [|S_k| \geqq k\epsilon]\} \leqq ES_n^{2b}/n^{2b} + \sum_{n+1}^\infty [ES_k^{2b} - ES_{k-1}^{2b}]/k^{2b}.$$

The proof will be completed if we can show that there exist constants A and B such that (a) $ES_n^{2b} \leqq An^{2b-t+1}$ and (b) $ES_k^{2b} - ES_{k-1}^{2b} \leqq Bk^{2b-t}$ for all $k \geqq n+1$.

Suppose $k \geqq n \geqq N$. Then

$$ES_k^{2b} = \textstyle\sum_{\tau=1}^b \sum_{\delta_1,\cdots,\delta_\tau \geqq 2;\delta_1+\cdots+\delta_\tau=2b} C_{\delta_1,\cdots,\delta_\tau}^{2b} \sum_{1 \leqq i_1 < \cdots < i_\tau \leqq k} EY_{i_1}^{\delta_1} \cdots EY_{i_\tau}^{\delta_\tau}$$

For $i \leqq k$, $E|Y_i|^\delta \leqq 2^t C$ (or $2^t Ck^{\delta - t}$) if $2 \leqq \delta \leqq t$ (or $\delta > t$) so that

$$|\textstyle\sum_{1 \leqq i_1 < \cdots < i_\tau \leqq k} EY_{i_1}^{\delta_1} \cdots EY_{i_\tau}^{\delta_\tau}| \leqq k^\tau (2^t C)^\tau k^\alpha = (2^t C)^\tau k^{\alpha + \tau},$$

where $\alpha = \sum_{i:\delta_i > t} (\delta_i - t)$. It is easy to see that $c + \tau \leqq 2b - t + 1$ so that

$$ES_k^{2b} \leqq \{\textstyle\sum_{\tau=1}^b \sum_{\delta_1,\cdots,\delta_\tau \geqq 2;\delta_1+\cdots+\delta_\tau=2b} C_{\delta_1,\cdots,\delta_\tau}^{2b} (2^t C)^\tau\} k^{2b-t+1}. \tag{5}$$

The proof of (a) now follows from (5) by taking $k = n$.

To prove (b) note that $ES_k^{2b} - ES_{k-1}^{2b} = \sum_{j=2}^{2b} C_j^{2b} EY_k^j ES_{k-1}^{2b-j}$. Because $E|Y_k^j| \leqq 2^t C$ for $2 \leqq j < b$ (or $\leqq 2^t Ck^{j-t}$ if $j \geqq b$) and $E|S_{k-1}^{2b-j}| \leqq 2^t C(k-1)^{2b-j}$ for $b < j \leqq 2b$ we conclude that $|EY_k^j ES_{k-1}^{2b-j}|$ is

$$\leqq (2^t C)^2 E|S_{k-1}^{2b-j}| \leqq (2^t C)^2 ES_{k-1}^{2b-2}, \qquad 2 \leqq j < b; \tag{6a}$$

$$\leqq (2^t C)k^{b-t} E|S_{k-1}^b|, \qquad j = b; \tag{6b}$$

$$\leqq (2^t C)^2 (k-1)^{2b-j} \leqq (2^t C)^2 k^{2b-t}, \qquad b < j \leqq 2b. \tag{6c}$$

Using (5) there is a constant D such that $ES_{k-1}^{2b-2} \leqq ES_k^{2b-2} \leqq (\mathrm{ES}_k^{2b})^{(1-1/b)} \leqq Dk^{2b-t}$. Similarly, $E|S_{k-1}^b| \leqq E|S_k|^b \leqq (ES_k^{2b})^{\frac{1}{2}} \leqq D'k^b$ for some constant D'. Finally then, we see from (6) that there is a constant H such that

$$EY_k^j ES_{k-1}^{2b-j}| \leqq Hk^{2b-t}, \qquad 2 \leqq j \leqq 2b,$$

and (b) is proven.

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