The Annals of Mathematical Statistics
1971, Vol. 42, No. 2, 517–525

SOME DISTRIBUTION PROBLEMS IN THE MULTIVARIATE COMPLEX GAUSSIAN CASE

BY K. C. S. PILLAI[1] AND G. M. JOURIS
Purdue University

1. Introduction and summary. Let $\mathbf{X}_1 : p \times n$ and $\mathbf{X}_2 : p \times n$ be real random variables having the joint density function

$$(2\pi)^{-pn}|\mathbf{\Sigma}_0|^{-\frac{1}{2}n}\exp\{-\tfrac{1}{2}\operatorname{tr}\mathbf{\Sigma}_0^{-1}(\mathbf{X}-\mathbf{v})(\mathbf{X}-\mathbf{v})'\}, \qquad -\infty \leqq \mathbf{X} \leqq \infty \tag{1.1}$$

where

$$\mathbf{X} = \begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix}, \qquad \mathbf{\Sigma}_0 = \begin{pmatrix}\mathbf{\Sigma}_1 & -\mathbf{\Sigma}_2\\ \mathbf{\Sigma}_2 & \mathbf{\Sigma}_1\end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix}\boldsymbol{\mu}_1 & -\boldsymbol{\mu}_2\\ \boldsymbol{\mu}_2 & \boldsymbol{\mu}_1\end{pmatrix}\begin{pmatrix}\mathbf{M}_1\\ \mathbf{M}_2\end{pmatrix},$$

$\mathbf{\Sigma}_1 : p \times p$ is a real symmetric positive definite (pd) matrix, $\mathbf{\Sigma}_2 : p \times p$ is a real skew-symmetric matrix, $\boldsymbol{\mu}_j : p \times q$ and $\mathbf{M}_j : q \times n$ ($j = 1, 2$), are given matrices or their joint density does not contain $\mathbf{\Sigma}_1$, $\mathbf{\Sigma}_2$, $\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_2$ as parameters. Then it has been shown by Goodman [10] that the distribution of the complex matrix $\mathbf{Z} = \mathbf{X}_1 + i\mathbf{X}_2$, ($i = (-1)^{\frac{1}{2}}$), is complex Gaussian and its density function is given by

$$N_c(\boldsymbol{\mu}\mathbf{M}, \mathbf{\Sigma}) = \pi^{-pn}|\mathbf{\Sigma}|^{-n}\exp\{-\operatorname{tr}\mathbf{\Sigma}^{-1}(\mathbf{Z}-\boldsymbol{\mu}\mathbf{M})(\overline{\mathbf{Z}}-\overline{\boldsymbol{\mu}\mathbf{M}})'\} \tag{1.2}$$

where $\mathbf{\Sigma} = \mathbf{\Sigma}_1 + i\mathbf{\Sigma}_2$ is Hermitian pd, i.e. $\overline{\mathbf{\Sigma}}' = \mathbf{\Sigma}$, $\boldsymbol{\mu} = \boldsymbol{\mu}_1 + i\boldsymbol{\mu}_2$ and $\mathbf{M} = \mathbf{M}_1 + i\mathbf{M}_2$. Goodman [5], Wooding [17], James [6], Al-Ani [1], and Khatri [8], [9], [10], [11] have studied distributions derived from a sample of a complex p-variate normal distribution.

Some important concepts and necessary notation are given below.

$$\tilde{\Gamma}_m(a) = \pi^{\frac{1}{2}m(m-1)}\textstyle\prod_{i=1}^m \Gamma(a-i+1),$$

$$[a]_\kappa = \textstyle\prod_{i=1}^m (a-i+1)_{k_i} = \tilde{\Gamma}_m(a,\kappa)/\tilde{\Gamma}_m(a)$$

where $\kappa = (k_1, k_2, \cdots, k_p)$ is a partition of the integer k and

$$\tilde{\Gamma}_m(a,\kappa) = \pi^{\frac{1}{2}m(m-1)}\textstyle\prod_{i=1}^m \Gamma(a+k_i-i+1).$$

The hypergeometric functions are defined as

$${}_p\tilde{F}_q(a_1,\cdots,a_p; b_1,\cdots,b_q; \mathbf{A},\mathbf{B}) = \sum_{k=0}^{\infty}\sum_{\kappa}\frac{\prod_{i=1}^p [a_i]_\kappa \tilde{C}_\kappa(A)\tilde{C}_\kappa(\mathbf{B})}{\prod_{i=1}^q [b_i]_\kappa \tilde{C}_\kappa(\mathbf{I}_m)k!}$$

or when $\mathbf{B} = \mathbf{I}_m$ we denote it by

$${}_p\tilde{F}_q(a_1,\cdots,a_p; b_1,\cdots,b_q; \mathbf{A})$$

and $\tilde{C}_\kappa(\mathbf{A})$ is a zonal polynomial of a Hermitian matrix $\mathbf{A}$ and is given as a symmetric function of the characteristic roots of $\mathbf{A}$.

Received February 3, 1969; revised September 8, 1970.

[1] The work of this author was supported by National Science Foundation Grant No. GP-7663.

The non-central distributions of the characteristic roots concerning the classical problems of the equality of two covariance matrices, MANOVA model, and canonical correlation coefficients have been found by James [6] and Khatri [8], [10]. Here for the three cases mentioned, we give the general moment and the density which is expressed in terms of Meijer's G-function [13], [14], for $W^{(p)} = \prod_{i=1}^{p}(1-w_i)$, where the w_i, $i = 1, 2, \cdots, p$ are the characteristic roots in the above cases. The moments and densities are analogous to those given in the real case [7], [15]. Further the density functions of U and Pillai's V criteria in the complex central case are obtained for $p = 2$, and from the non-central complex multivariate F distribution various independence relationships are shown and independent beta variables are obtained.

2 Density functions of $W^{(p)}$ in the non-central case.

CASE 1. *Testing the equality of two covariance matrices.*

Let $\mathbf{X}: (p \times n_1) \sim N_c(\mathbf{0}, \mathbf{\Sigma}_1)$ and $\mathbf{Y}: (p \times n_2) \sim N_c(\mathbf{0}, \mathbf{\Sigma}_2)$ be independent and $n_1 \geqq p$. Then Khatri [10] has shown the density function of the characteristic roots, $0 < f_1 < \cdots < f_p < \infty$ of $(\mathbf{X}\bar{\mathbf{X}}')(\mathbf{Y}\bar{\mathbf{Y}}')^{-1}$ can be written as

$$(2.1) \qquad C(p)|\mathbf{\Lambda}|^{-n_1}{}_1\tilde{F}_0(n; \mathbf{I}_p - \mathbf{\Lambda}^{-1}, \mathbf{F}(\mathbf{I}_p + \mathbf{F})^{-1})\frac{|\mathbf{F}|^{n_1-p}}{|\mathbf{I}_p + \mathbf{F}|^{n}}\prod_{i>j}(f_i - f_j)^2$$

where

$$(2.2) \qquad C(p) = \frac{\pi^{p(p-1)}\tilde{\Gamma}_p(n)}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)\tilde{\Gamma}_p(p)}, \qquad n = n_1 + n_2, \qquad \mathbf{F} = \mathrm{diag}(f_1, \cdots, f_p)$$

and $\mathbf{\Lambda}$ is a diagonal matrix whose diagonal elements are the characteristic roots of $(\mathbf{\Sigma}_1\mathbf{\Sigma}_2{}^{-1})$. Transforming

$$(2.3) \qquad w_i = f_i/(1+f_i)$$

we find the density of $0 < w_1 < \cdots < w_p$ is

$$(2.4) \qquad C(p)|\mathbf{\Lambda}|^{-n_1}{}_1\tilde{F}_0(n; \mathbf{I}_p - \mathbf{\Lambda}^{-1}, \mathbf{W})|\mathbf{W}|^{n_1-p}|\mathbf{I}_p - \mathbf{W}|^{n_2-p}\prod_{i>j}(w_i - w_j)^2$$

where

$$\mathbf{W} = \mathrm{diag}(w_1, w_2, \cdots, w_p).$$

To find $E[W^{(p)}]^h$ where $W^{(p)} = \prod_{i=1}^{p}(1-w_i)$ we multiply (2.4) by $|\mathbf{I}_p - \mathbf{W}|^h$ and transform $\mathbf{T} \to \mathbf{U}\mathbf{W}\bar{\mathbf{U}}'$ where $\mathbf{U}$ is unitary, i.e. $\mathbf{U}\bar{\mathbf{U}}' = \mathbf{I}$, and $\mathbf{T}$ is Hermitian pd. Using the Jacobian of transformation given by Khatri [8]

$$(2.5) \qquad J(\mathbf{T}; \mathbf{U}, \mathbf{W}) = \prod_{i>j}(w_i - w_j)^2 h_2(\mathbf{U})$$

and integrating out $\mathbf{U}$ and $\mathbf{W}$ using

$$(2.6) \qquad \int_{U\bar{U}'=I} h_2(\mathbf{U}) = \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)} \qquad \text{and}$$

$$(2.7)\qquad \int_{I>\bar{S}=S>0} |\mathbf{S}|^{q-p}|\mathbf{I}_p-\mathbf{S}|^{n+h-q-p}\tilde{C}_\kappa(\mathbf{S})\,d\mathbf{S} = \frac{\tilde{\Gamma}_p(q,\kappa)\tilde{\Gamma}_p(n+h-q)\tilde{C}_\kappa(\mathbf{I}_p)}{\tilde{\Gamma}_p(n+h,\kappa)},$$

we get after simplifying

$$(2.8)\qquad E[W^{(p)}]^h = |\Lambda|^{-n_1}\frac{\tilde{\Gamma}_p(n)\tilde{\Gamma}_p(n_2+h)}{\tilde{\Gamma}_p(n_2)\tilde{\Gamma}_p(n+h)}\,{}_2\tilde{F}_1(n,n_1;n+h;\mathbf{I}_p-\Lambda^{-1}).$$

Before finding the density of $W^{(p)}$, below are stated some needed results on Mellin's transforms [2], [3], [4], and Meijer's G-function [13], [14].

If s is any complex variate and $f(x)$ is a function of a real variable x, such that

$$(2.9)\qquad F(s) = \int_0^\infty x^{s-1}f(x)\,dx$$

exists, then under certain regularity conditions

$$(2.10)\qquad f(x) = (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} x^{-s}F(s)\,ds.$$

$F(s)$ is called the Mellin transform of $f(x)$ and $f(x)$ is the inverse Mellin transform of $F(s)$. Meijer defined the G-function by

$$(2.11)\qquad G_{p,q}^{m,n}\left(x\,\middle|\,{a_1,\cdots,a_p \atop b_1,\cdots,b_q}\right) = (2\pi i)^{-1}\int_C \frac{\prod_{j=1}^m\Gamma(b_j-s)\prod_{j=1}^n(1-a_j+s)}{\prod_{j=m+1}^q\Gamma(1-b_j+s)\prod_{j=n+1}^p\Gamma(a_j-s)}x^s\,ds$$

where C is a curve separating the singularities of $\prod_{j=1}^m\Gamma(b_j-s)$ from those of $\prod_{j=1}^n\Gamma(1-a_j+s)$, $q \geqq 1$, $0 \leqq n \leqq p \leqq q$, $0 \leqq m \leqq q$; $x \neq 0$ and $|x| < 1$ if $q = p$; $x \neq 0$ if $q > p$. Using (2.9) and (2.10) we see from (2.8) that the density of $(W^{(p)})$ has the form

$$(2.12)\qquad \begin{aligned} f(W^{(p)}) = C_p\sum_{k=0}^\infty\sum_\kappa \frac{[n]_\kappa[n_1]_\kappa}{k!}\tilde{C}_\kappa(\mathbf{I}_p-\Lambda^{-1})\{W^{(p)}\}^{n_2-p} \\ \cdot(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}\{W^{(p)}\}^{-r}\frac{\prod_{i=1}^p\Gamma(r+b_i)}{\prod_{i=1}^p\Gamma(r+a_i)}dr \end{aligned}$$

where

$$(2.13)\qquad C_p = \frac{\tilde{\Gamma}_p(n)}{\tilde{\Gamma}_p(n_2)}|\Lambda|^{-n_1}, \qquad b_i = i-1, \qquad a_i = n_1+k_{p-i+1}+b_i.$$

Noting that the integral in (2.12) is in the form of Meijer's G-function we can write the density of $W^{(p)}$ as

$$(2.14)\qquad \begin{aligned} f(W^{(p)}) = C_p\sum_{k=0}^\infty\sum_\kappa \frac{[n]_\kappa[n_1]_\kappa}{k!}\tilde{C}_\kappa(\mathbf{I}_p-\Lambda^{-1})\{W^{(p)}\}^{n_2-p} \\ \cdot G_{p,p}^{p,0}\left(W^{(p)}\,\middle|\,{a_1,\cdots,a_p \atop b_1,\cdots,b_p}\right). \end{aligned}$$

Using the fact that

$$(2.15)\qquad \begin{aligned} G_{2,2}^{2,0}\left(x\,\middle|\,{a_1,a_2 \atop b_1,b_2}\right) = \frac{x^{b_1}(1-x)^{a_1+a_2-b_1-b_2-1}}{\Gamma(a_1+a_2-b_1-b_2)} \\ \cdot{}_2F_1(a_2-b_2,a_1-b_2;a_1+a_2-b_1-b_2;1-x), \quad 0<x<1, \end{aligned}$$

we find the density of $W^{(2)}$ to be

$$f(W^{(2)}) = C_2 \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[n]_\kappa [n_1]_\kappa}{k!} \tilde{C}_\kappa(\mathbf{I}_2 - \mathbf{\Lambda}^{-1}) \{W^{(2)}\}^{n_2-2} \tag{2.16}$$
$$\cdot \frac{\{1-W^{(2)}\}^{2n_1+k-1}}{\Gamma(2n_1+k)} \, {}_2F_1(n_1+k_1, n_1+k_2-1; 2n_1+k; 1-W^{(2)})$$

where $\kappa = (k_1, k_2)$. Using the results of Consul [4] for $p = 3$ and Al-Ani [1] for $p = 4$ we could also write out the densities of $W^{(3)}$ and $W^{(4)}$.

CASE 2. *MANOVA model.* Suppose $\mathbf{X}: (p \times m) \sim N_c(\boldsymbol{\mu}, \mathbf{\Sigma})$ and $\mathbf{Y}: (p \times n) \sim N_c(\mathbf{0}, \mathbf{\Sigma})$ are independent with $m \geqq p$. Then the joint density of the characteristic roots $0 < f_1 < \cdots < f_p$ of $(\mathbf{X}\bar{\mathbf{X}}')(\mathbf{Y}\bar{\mathbf{Y}}')^{-1}$ is given by Khatri [10] as

$$C'(p) \exp(-\operatorname{tr} \mathbf{\Omega}) {}_1F_1(m+n; m; \mathbf{\Omega}, (\mathbf{I}_p + \mathbf{F}^{-1})^{-1}) \frac{|\mathbf{F}|^{m-p}}{|\mathbf{I}_p + \mathbf{F}|^{m+n}} \prod_{i>j} (f_i - f_j)^2 \tag{2.17}$$

where

$$C'(p) = \frac{\tilde{\Gamma}_p(m+n)\pi^{p(p-1)}}{\tilde{\Gamma}_p(m)\tilde{\Gamma}_p(n)\tilde{\Gamma}_p(p)}, \qquad \mathbf{F} = \operatorname{diag}(f_1, \cdots, f_p)$$

and $\mathbf{\Omega} = \operatorname{diag}(\omega_1, \cdots, \omega_p)$ where ω_i are the characteristic roots of $\boldsymbol{\mu}\bar{\boldsymbol{\mu}}'\mathbf{\Sigma}^{-1}$. Now proceeding as in the previous case we obtain $E[W^{(p)}]^h$, $W^{(p)} = \prod_{i=1}^p (1-w_i)$ where $w_i = f_i/(1+f_i)$

$$E[W^{(p)}]^h = \exp(-\operatorname{tr} \mathbf{\Omega}) \frac{\tilde{\Gamma}_p(m+n)\tilde{\Gamma}_p(n+h)}{\tilde{\Gamma}_p(n)\tilde{\Gamma}_p(m+n+h)} {}_1\tilde{F}_1(m+n; m+n+h; \mathbf{\Omega}). \tag{2.18}$$

Using Mellin's transform and Meijer's G-function as in the previous case we get the density of $W^{(p)}$ as

$$f(W^{(p)}) = \exp(-\operatorname{tr} \mathbf{\Omega}) \frac{\tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(n)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[m+n]_\kappa \tilde{C}_\kappa(\mathbf{\Omega})}{k!} \{W^{(p)}\}^{n-p} \tag{2.19}$$
$$\cdot G_{p,p}^{p,0}\left(W^{(p)} \,\middle|\, \begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_p \end{matrix}\right)$$

where

$$a_i = m + k_{p-i+1} + b_i, \qquad b_i = i - 1.$$

As in the covariance model case, we could also obtain the density explicitly for $p = 2, 3, 4$.

CASE 3. *Canonical correlation.* Let

$$\begin{bmatrix} \mathbf{X}: p \times n \\ \mathbf{Y}: q \times n \end{bmatrix} \sim N_c\left[\mathbf{0}, \begin{pmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \bar{\mathbf{\Sigma}}_{12}' & \mathbf{\Sigma}_{22} \end{pmatrix}\right] \tag{2.20}$$

$n \geqq p+q$ and $q \geqq p$. Then the joint density of the characteristic roots $0 < r_1^2 < \cdots < r_p^2$ of $(\mathbf{X}\bar{\mathbf{Y}}')(\mathbf{Y}\bar{\mathbf{Y}}')^{-1}(\mathbf{Y}\bar{\mathbf{X}}')(\mathbf{X}\bar{\mathbf{X}}')^{-1}$ is given by Khatri [10] as

$$C''(p) |\mathbf{I}_p - \mathbf{P}^2|^n {}_2\mathbf{F}_1(n, n; q; \mathbf{P}^2, \mathbf{R}^2) |\mathbf{R}^2|^{q-p} |\mathbf{I}_p - \mathbf{R}^2|^{n-q-p} \prod_{i>j} (r_i^2 - r_j^2)^2 \tag{2.21}$$

where

$$(2.22) \qquad C''(p) = \frac{\tilde{\Gamma}_p(n)\pi^{p(p-1)}}{\tilde{\Gamma}_p(n-q)\tilde{\Gamma}_p(q)\tilde{\Gamma}_p(p)}, \qquad \mathbf{R}^2 = \operatorname{diag}(r_1^2, \cdots, r_p^2)$$

and $\mathbf{P}^2 = \operatorname{diag}(\rho_1, \cdots, \rho_p)$ where ρ_i are the characteristic roots of $\mathbf{\Sigma}_{11}^{-1}\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{12}'$. Proceeding as in the previous cases we find $E[W^{(p)}]^h$, $W^{(p)} = \prod_{i=1}^{p}(1-r_i^2)$, by substituting in (2.8) as follows

$$(2.23) \qquad (n_1, n_2, \mathbf{\Lambda}) \to (n, n-q, (\mathbf{I}_p - \mathbf{P}^2)^{-1}).$$

Further, the density of $W^{(p)}$ is obtained from (2.14) by making the above substitution and letting

$$a_i = q + k_{p-i+1} + b_i, \qquad b_i = i-1.$$

As in the other cases the densities could be written out explicitly for $p = 2, 3, 4$.

3. The density function of Pillai's V-statistic in the central case for two roots. If $\mathbf{P}^2 = \mathbf{0}$ in (2.19) we have the density function of the characteristic roots $r_1^2, r_2^2, \cdots, r_p^2$ in the central case. Letting $p = 2$ we have

$$(3.1) \qquad f_1(r_1^2, r_2^2) = C''(2)|\mathbf{R}^2|^{q-2}|\mathbf{I}_2 - \mathbf{R}^2|^{n-q-2}(r_1^2 - r_2^2)^2.$$

Let $V = r_1^2 + r_2^2$ and $G = r_1^2 r_2^2$, $0 < V < 1$. To find the density function of V we make the above transformation and find

$$(3.2) \qquad f_2(V, G) = C''(2)G^{q-2}(1-V+G)^{n-q-2}(V^2-4G)^{\frac{1}{2}}.$$

Integrating G between the limits 0 to $V^2/4$, [16] and writing $(1-V+G)^{n-q-2}$ as a finite series we have

$$(3.3) \qquad f(V) = C''(2)\textstyle\sum_{r=0}^{n-q-2}\binom{n-q-2}{r}(1-V)^{n-q-r-2}\left\{\int_0^{V^2/4} G^{q+r-2}(V^2-4G)^{\frac{1}{2}}\,dG\right\}.$$

Integrating the expression in the brackets by parts we find the density function of V to be

$$(3.4) \qquad \begin{aligned} f_3(V) = C''(2)\sum_{r=0}^{n-q-2}&\binom{n-q-2}{r}(1-V)^{n-q-r-2} \\ &\cdot\frac{(q+r-2)!\,V^{2(q+r)-1}}{2^{q+r-1}3\cdot 5\cdots(2(q+r)-1)}, \qquad 0 < V < 1.\end{aligned}$$

To obtain the density function of V in the range $1 \leqq V \leqq 2$ we change $r_i^2 \to 1-r_i^2$ in (3.1) and transform as before to get

$$(3.5) \qquad f_4(V, G) = C''(2)(1-V+G)^{q-2}G^{n-q-2}(V^2-4G)^{\frac{1}{2}}.$$

Writing $(1-V+G)^{q-2}$ as a series and integrating G between the limits 0 to $V^2/4$ we have

$$(3.6) \qquad f_5(V) = C''(2)\textstyle\sum_{r=0}^{q-2}\binom{q-2}{r}(1-V)^{q-r-2}\int_0^{V^2/4} G^{n+r-q-2}(V^2-4G)^{\frac{1}{2}}\,dG.$$

Evaluating the integral by parts yields

$$(3.7)\quad f_5(V) = C''(2)\sum_{r=0}^{q-2}\binom{q-2}{r}(1-V)^{q-r-2}\frac{(n+r-q-2)!\,V^{2(n+r-q)-1}}{2^{n+r-q-1}3\cdot 5\cdots(2(n+r-q)-1)}.$$

Transforming $V' = 2-V$, $1 \leqq V \leqq 2$ we find

$$(3.8)\quad f_6(V) = C''(2)\sum_{r=0}^{q-2}\binom{q-2}{r}(V-1)^{q-r-2}\frac{(n+r-q-2)!(2-V)^{2(n+r-q)-1}}{2^{n+r-q-1}3\cdot 5\cdots(2(n+r-q)-1)},\qquad 1 \leqq V \leqq 2.$$

By making the following changes in the parameters in (3.1)

$$(q, n-q, r_i^2) \to (m, n, w_i) \quad \text{or} \quad (q, n-q, r_i^2) \to (n_1, n_2, w_i)$$

we obtain the central density of the characteristic roots in the MANOVA or equality of two covariance matrices cases, respectively. Thus the results of this section and the next are not restricted to the canonical correlation case, but extend to the two cases mentioned above as well.

4. The density function of the U-statistic in the central case for two roots. To obtain the density function of U we make the transformation in (3.1)

$$r_i^2 = \lambda_i(1+\lambda_i)^{-1}$$

and find

$$(4.1)\quad g_1(\lambda_1, \lambda_2) = C''(2)|\mathbf{Q}|^{q-2}|\mathbf{I}_2+\mathbf{Q}|^{-n}(\lambda_1-\lambda_2)^2$$

where $\mathbf{Q} = \operatorname{diag}(\lambda_1, \lambda_2)$. Letting $U = \lambda_1+\lambda_2$ and $G = \lambda_1\lambda_2$ we see the joint density of U and G can be put in the form

$$(4.2)\quad g_2(U, G) = C''(2)G^{q-2}(1+U/2)^{-2n}(U^2-4G)^{\frac{1}{2}}\left[1-\frac{U^2-4G}{4(1+U/2)^2}\right]^{-n}.$$

Writing the part in brackets as a series and integrating G between the limits 0 to $U^2/4$ yields

$$(4.3)\quad g_3(U) = C''(2)(1+U/2)^{-2n}\sum_{r=0}^{\infty}\frac{(-1)^r\binom{-n}{r}}{4^r(1+U/2)^{2r}}\cdot\left\{\int_0^{U^2/4}(U^2-4G)^{r+\frac{1}{2}}G^{q-2}\,dG\right\}.$$

Integrating the expression in the brackets by parts, we find the density of U for $p = 2$ is

$$(4.4)\quad g_3(U) = C''(2)\sum_{r=0}^{\infty}(-1)^r\frac{\binom{-n}{r}(q-2)!\,U^{2r+2(q-3)+5}}{4^{r+q-1}(1+U/2)^{2r+2n}}\cdot\frac{1}{(r+3/2)(r+5/2)\cdots[r+(2(q-3)+5)/2]}.$$

5. Complex multivariate beta distribution and independent beta variables. If $\mathbf{X}:(p\times m)$ and $\mathbf{Y}:(p\times n)$ are independent complex matrix variates $p \geqq m$, whose columns are independent complex p-variate with covariance matrix $\mathbf{\Sigma}$, and if $E(\mathbf{X}) = \boldsymbol{\mu}$ and $E(\mathbf{Y}) = \mathbf{0}$, then the distribution of

$$\mathbf{F} = \bar{\mathbf{X}}'(\mathbf{Y}\bar{\mathbf{Y}}')^{-1}\mathbf{X} \tag{5.1}$$

depends on parameters

$$\mathbf{\Omega} = \bar{\boldsymbol{\mu}}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu} \tag{5.2}$$

and is [6]

$$f(\mathbf{F}) = k_1 \exp(-\operatorname{tr}\mathbf{\Omega})\,{}_1\tilde{F}_1(m+n;\,p;\,\mathbf{\Omega}(\mathbf{I}_m+\mathbf{F}^{-1})^{-1}) \cdot |\mathbf{F}|^{p-m}|\mathbf{I}_m+\mathbf{F}|^{-(m+n)}(d\mathbf{F}), \tag{5.3}$$

where

$$k_1 = \frac{\tilde{\Gamma}_m(m+n)}{\tilde{\Gamma}_m(p)\tilde{\Gamma}_m(m+n-p)}. \tag{5.4}$$

Making the transformation

$$\mathbf{L} = (\mathbf{I}_m + \mathbf{F}^{-1})^{-1} \tag{5.5}$$

in (5.3) and noting [8]

$$J(\mathbf{F};\,\mathbf{L}) = |\mathbf{I}_m - \mathbf{L}|^{-2m} \tag{5.6}$$

we have

$$f(\mathbf{L}) = k_1 \exp(-\operatorname{tr}\mathbf{\Omega})\,{}_1\mathbf{F}_1\,(m+n;\,p;\,\mathbf{\Omega}\mathbf{L})|\mathbf{L}|^{p-m}|\mathbf{I}_m-\mathbf{L}|^{n-p}(d\mathbf{L}). \tag{5.7}$$

Now consider the linear case i.e. when $\mathbf{\Omega}$ has only one non-zero characteristic root, say, λ^2. Proceeding in a manner similar to that of Khatri and Pillai [12] let

$$\mathbf{L} = \begin{array}{c} \begin{pmatrix} l_{11} & \bar{\mathbf{l}}' \\ \mathbf{l} & \mathbf{L}_{11} \end{pmatrix}\begin{array}{l} 1 \\ m-1 \end{array} \\ \begin{array}{cc} 1 & m-1 \end{array} \end{array}, \qquad \mathbf{L}_{22} = \mathbf{L}_{11} - \mathbf{l}\bar{\mathbf{l}}'/l_{11} \tag{5.8}$$

and note that $|\mathbf{L}| = l_{11}|\mathbf{L}_{22}|$ and

$$|\mathbf{I}_m - \mathbf{L}| = (1-l_{11})|\mathbf{I}_{m-1} - \mathbf{L}_{22} - \mathbf{l}\bar{\mathbf{l}}'/[l_{11}(1-l_{11})]|. \tag{5.9}$$

Now it can be shown that l_{11} and $\{\mathbf{L}_{22},\, \mathbf{v} = \mathbf{l}/[l_{11}(1-l_{11})]^{\frac{1}{2}}\}$ are independently distributed and their respective distributions are

$$f_1(l_{11}) = [\beta(p,\, m+n-p)]^{-1}\, e^{-\lambda^2} \cdot {}_1\tilde{F}_1(m+n;\,p;\,\lambda^2 l_{11}) l_{11}^{p-1}(1-l_{11})^{m+n-p-1} \tag{5.10}$$

and

$$f_2(\mathbf{L}_{22},\mathbf{v}) = k_2|\mathbf{L}_{22}|^{p-m}|\mathbf{I}_{m-1} - \mathbf{L}_{22} - \mathbf{v}\bar{\mathbf{v}}'|^{n-p}, \tag{5.11}$$

where

$$k_2 = k_1 \beta(p, m+n-p). \tag{5.12}$$

For further independence, we can use the transformation

$$\mathbf{u} = (\mathbf{I}_{m-1} - \mathbf{L}_{22})^{-\frac{1}{2}}\mathbf{v}.$$

With Jacobian of transformation $|\mathbf{I}_{m-1} - \mathbf{L}_{22}|^{-1}$ it can be shown that $\mathbf{u}$ and $\mathbf{L}_{22}$ are independently distributed and that their respective distributions are

$$f_3(\mathbf{u}) = \pi^{-(m-1)}[\Gamma(m+n-p)/\Gamma(n-p+1)](1-\bar{\mathbf{u}}'\mathbf{u})^{n-p} \quad \text{and} \tag{5.13}$$

$$f_4(\mathbf{L}_{22}) = k_3|\mathbf{L}_{22}|^{p-1-(m-1)}|\mathbf{I}_{m-1} - \mathbf{L}_{22}|^{n-(p-1)}, \quad \text{where} \tag{5.14}$$

$$k_3 = \pi^{(p-1)}[\Gamma(n-p+1)/\Gamma(m+n-p)]k_2.$$

Notice that $\mathbf{L}_{22}: (m-1) \times (m-1)$ is the central complex multivariate beta distribution with $(p-1)$ instead of p. Making the transformation

$$x_i = u_i/(1 - \bar{u}_1 u_1 - \cdots - \bar{u}_{i-1}u_{i-1})^{\frac{1}{2}}, \qquad i = 1, 2, \cdots m-1, u_0 = 0 \tag{5.15}$$

in (5.13) with Jacobian of transformation $\prod_{i=1}^{m-1} (1 - \bar{x}_i x_i)^{m-i-1}$, we obtain the density of $\mathbf{X} = (x_1, x_2, \cdots, x_{m-1})'$ as

$$f(\mathbf{X}) = \pi^{-(m-1)} \prod_{i=1}^{m-1} \frac{\Gamma(m+n-p-i+1)}{\Gamma(m+n-p-i)} (1 - \bar{x}_i x_i)^{m+n-p-i-1}. \tag{5.16}$$

After making the transformation of $x_j = a_j + ib_j$ to polar coordinates (r_j, θ_j), we find with $\mathbf{r} = (r_1, \cdots, r_{m-1})'$

$$f(\mathbf{r}) = \prod_{i=1}^{m-1} \frac{\Gamma(m+n-p-i+1)}{\Gamma(m+n-p-i)} (1 - r_i^2)^{m+n-p-i-1} 2r_i \, dr_i. \tag{5.17}$$

Finally the transformation $w_i' = r_i^2$ yields independent real beta variates and their respective densities are given by

$$f_i(w_i') = [\beta(1, m+n-p-i)]^{-1}(1 - w_i')^{m+n-p-i-1}. \tag{5.18}$$

REFERENCES

[1] Al-Ani, S. (1968). Some distribution problems concerning characteristic roots and vectors in multivariate analysis. Mimeograph Series No. 162, Department of Statistics, Purdue Univ.

[2] Consul, P. C. (1966). On some inverse Mellin integral transforms. *Acad. Roy. Belg. Bull. Cl. Sci.* **52** 547–561.

[3] Consul, P. C. (1967). On the exact distributions of likelihood ratio criteria for testing independence of sets of variates under the null hypothesis. *Ann. Math. Statist.* **38** 1160–1169.

[4] Consul, P. C. (1967). On the exact distribution of the W criterion for testing the sphericity in a p-variate normal distribution. *Ann. Math. Statist.* **38** 1170–1174.

[5] Goodman, N. R. (1963). Statistical analysis based on a certain multivariate complex Gaussian distribution (an introduction). *Ann. Math. Statist.* **34** 152–176.

[6] James, A. T. (1964). Distributions of matrix variates and latent roots derived from normal samples. *Ann. Math. Statist.* **35** 475–501.

[7] Jouris, G. M. (1968). On the non-central distributions of Wilks'-Λ for tests of three hypotheses. Mimeograph Series No. 164, Department of Statistics, Purdue Univ.

[8] Khatri, C. G. (1965). Classical statistical analysis based on a certain multivariate complex Gaussian distribution. *Ann. Math. Statist.* **36** 98–114.

[9] Khatri, C. G. (1965). A test for reality of a covariance matrix in certain complex Gaussian distributions. *Ann. Math. Statist.* **36** 115–119.

[10] Khatri, C. G. (1969). Noncentral distributions of the *i*th largest characteristic roots of three matrices concerning complex multivariate normal populations. *Ann. Inst. Statist. Math.* **21** 23–32.

[11] Khatri, C. G. (1968). On the moments of two matrices in three situations for complex multivariate normal populations. Mimeograph Series No. 161, Department of Statistics, Purdue Univ.

[12] Khatri, C. G. and Pillai, K. C. S. (1965). Some results on the non-central multivariate beta distribution and moments of traces of two matrices. *Ann. Math. Statist.* **36** 1511–1520.

[13] Meijer, C. S. (1946). On the *G*-function. II, III, IV. *Nederl. Akad. Wetensch. Proc.* **49** 344–356, 475–469, 632–641.

[14] Meijer, C. S. (1946). On the *G*-function I. *Indag. Math.* **8** 124–134.

[15] Pillai, K. C. S., Al-Ani, S. and Jouris, G. M. (1969). On the distributions of the ratios of the roots of a covariance matrix and Wilks' criterion for tests of three hypotheses. *Ann. Math. Statistist.* **40** 2033–2040.

[16] Pillai, K. C. S. and Jayachandran, K. (1967). Power comparisons of tests of two multivariate hypotheses based on four criteria. *Biometrika* **54** 195–210.

[17] Wooding, R. A. (1956). The multivariate distribution of complex normal variables. *Biometrika* **43** 212–215.