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ON THE UNIMODALITY OF L FUNCTIONS[1]

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It is shown that an L function is unimodal if its Lévy spectral function has support on $(-\infty, 0]$ or on $[0, \infty)$, and that this implies that every L function is the convolution of at most two unimodal L functions. Other results concerning the unimodality of L functions and other infinitely divisible distribution functions are also obtained.

1. Introduction and summary. A distribution function $F(x)$ is said to be an L function if there exists a sequence of independent random variables $X_1, \cdots, X_n, \cdots$ such that for suitable constants A_n and $B_n > 0$ the random variables

$$Y_n = (X_1 + \cdots + X_n)/B_n - A_n$$

have the property that $F_{Y_n} \to_c F$ and in addition the random variables

$$X_{n,j} = X_j/B_n \ (1 \leqq j \leqq n)$$

are an infinitesimal system. A discussion of some properties of L functions can be found in ([5] Chapter 6). In the Russian edition of this book, published in 1949, there appeared a theorem due to Gnedenko, stating that every L function is unimodal. (A distribution function F is said to be unimodal if there exists an x_0 such that F is convex at all $x < x_0$ and concave at all $x > x_0$.) However, Gnedenko's proof, although correct, made use of an incorrect theorem due to A. L. Lapin which states that the convolution of two unimodal distribution functions is also unimodal. A counter-example of Lapin's theorem, constructed by K. L. Chung, appeared in [2] and [5].

Since the proof of Gnedenko's theorem depended upon Lapin's theorem, the validity of Gnedenko's theorem was now in doubt. A. Wintner had shown by 1938 ([12] Theorem 11.4, page 30) that the convolution of two symmetric unimodal distribution functions is unimodal. In 1956 he used this theorem to show that every symmetric L function is unimodal (see [13] Appendix II, pages 840–842]). In 1957 I. A. Ibragimov published a paper [7] in which he gave examples of L functions that were not unimodal. However, in 1967 T. C. Sun [9] showed that the L functions that Ibragimov had constructed were indeed unimodal. Thus the question as to whether or not every L function is unimodal was again open.

In this paper it is shown that an L function $F(x)$ is unimodal if its Lévy spectral function $M(u)$ has support on $(-\infty, 0]$ or on $[0, \infty)$, and that this implies that every

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L function is the convolution of at most two unimodal L functions. Other results concerning the unimodality of L functions and other infinitely divisible distribution functions are also obtained.

2. A lemma. Let $0 = p_0 < p_1 < \cdots < p_k < \infty$. Let $\lambda_1, \cdots, \lambda_k$ be positive constants. For $1 \leqq i \leqq k$ let $\lambda_0(x) = \sum_{j=i}^{k} \lambda_j$ if $p_{i-1} < x \leqq p_i$ and let $\lambda_0(x) = 0$ if $x > p_k$. Let $M_0(u) = -\int_u^\infty \lambda_0(x)/x\, dx$ if $u > 0$ and let $M_0(u) = 0$ if $u < 0$. Let γ be a constant and let

$$\hat{f}_0(t) = \exp\{i\gamma t + \int_{+0}^{+\infty} (e^{iut} - 1 - iut(1+u^2)^{-1})\, dM_0(u)\}.$$

It can easily be shown that $M_0(u)$ is a Lévy spectral function and that $\hat{f}_0(t)$ is the characteristic function of an L function $F_0(x)$. It will now be shown that $F_0(x)$ is unimodal.

LEMMA 1. *The L function $F_0(x)$ is a unimodal distribution function.*

PROOF. Without loss of generality, γ can be chosen so that

$$\hat{f}_0(t) = \exp\{\int_{+0}^{+\infty} (e^{iut} - 1)\, dM_0(u)\}.$$

Let $\lambda = \sum_{i=1}^{k} \lambda_i$ and let $\hat{f}_{\lambda_i}(t) = \exp\{\lambda_i \int_0^t (e^{iu} - 1)/u\, du\}$ for $1 \leqq i \leqq k$. Since

$$|\hat{f}_{\lambda_i}(t)| = \exp\{\lambda_i \int_0^1 (\cos u - 1)/u\, du + \lambda_i \int_1^t (\cos u)/u\, du - \lambda_i \int_1^t (1/u)\, du\}$$

and $\hat{f}_0(t) = \prod_{i=1}^{k} \hat{f}_{\lambda_i}(p_i t)$ it follows that

$$|\hat{f}_0(t)| = 0(|t|^{-\lambda}) \qquad \text{as} \quad |t| \to \infty. \tag{1}$$

Every nondegenerate L function is absolutely continuous (see [4] page 338). Thus $F_0(x)$ is absolutely continuous. Let $f(x)$ denote the density function of $F_0(x)$. It follows from (1) that

$$F(x) - F(0) = (2\pi)^{-1} \int_{-\infty}^{\infty} ((e^{-itx} - 1)/-it) \hat{f}_0(t)\, dt. \tag{2}$$

For $x > 0$ let $s = tx$. Then (2) is equivalent to

$$F(x) - F(0) = (2\pi)^{-1} \int_{-\infty}^{\infty} ((e^{-is} - 1)/-is) \exp\{\sum_{i=1}^{k} \lambda_i \int_0^{p_i s/x} (e^{iu} - 1)/u\, du\}\, ds. \tag{3}$$

Both sides of (3) can be differentiated yielding

$$\begin{aligned} f(x) &= (2\pi)^{-1} \int_{-\infty}^{\infty} ((e^{-is} - 1)/-is) \hat{f}_0(s/x) [\sum_{i=1}^{k} (\lambda_i/x)(1 - e^{ip_i s/x})]\, ds \\ &= \sum_{i=1}^{k} (\lambda_i/x) \{[F_0(x) - F_0(x - p_i)] - [F_0(0) - F_0(-p_i)]\}. \end{aligned} \tag{4}$$

The same formula can be obtained for $x < 0$. Since $f(x) \geqq 0$ for all x, it follows that $F_0(0) - F_0(-p_i) = 0$ for $1 \leqq i \leqq k$ and $f(x) = 0$ for $x < 0$. Thus if $x > 0$,

$$xf'(x) = (\lambda_1 + \cdots + \lambda_k - 1) f(x) - \lambda_1 f(x - p_1) - \cdots - \lambda_k f(x - p_k). \tag{5}$$

From this it follows that $f(x)$ is continuous at all $x \neq 0$, and $f'(x)$ is continuous at all x except $x = 0, p_1, \cdots, p_k$.

If $\lambda \leqq 1$ then $f'(x) \leqq 0$ for $x > 0$. Thus $F_0''(x) = 0$ for $x < 0$ and $F_0''(x) \leqq 0$ for $x > 0$. It follows that $F_0(x)$ is unimodal with a mode at 0.

If $\lambda > 1$ then $xf'(x) = (\lambda-1)f(x)$ for $0 < x < p_1$. It follows that if $0 < x < p_1$ then

(6) $$f(x) = cx^{\lambda-1} \qquad \text{where } c > 0.$$

From (5) and (6) it follows that $f(x)$ is continuous at 0 and $f'(x)$ is continuous for $x > 0$. Also $f'(x) > 0$ for $0 < x < p_1$. From (4) it follows that $f(x) \to 0$ as $x \to \infty$. Thus $f(x)$ has at least one relative maximum in the interval (p_1, ∞). Let $A = \{x: f(x)$ has a relative maximum and $x > 0\}$ and let $x_0 = \inf A$. Two cases must be considered.

Case 1. Suppose x_0 is an isolated point of A or $x_0 = \min A$.

In this case let $x_1 = x_0$. It will be shown by contradiction that $f(x)$ is non-increasing on the interval (x_1, ∞). Assume to the contrary that $f(x)$ is not non-increasing on the interval (x_1, ∞). Then $f(x)$ has at least one relative minimum in the interval (x_1, ∞). Let $x_2 = \inf \{x: f(x)$ has a relative minimum and $x > x_1\}$. Then $x_2 > x_1 > p_1$ and $f'(x_2) = 0$ by the continuity of $f'(x)$. Also

(7) $$f(x) \quad \text{is strictly increasing on} \quad (0, x_1),$$

(8) $$(\lambda_1+\cdots+\lambda_k-1)f(x_1)-\lambda_1 f(x_1-p_1)-\cdots-\lambda_k f(x_1-p_k) = 0,$$

(9) $$f(x) \quad \text{is strictly decreasing on} \quad (x_1, x_2),$$

(10) $$(\lambda_1+\cdots+\lambda_k-1)f(x_2)-\lambda_1 f(x_2-p_1)-\cdots-\lambda_k f(x_2-p_k) = 0.$$

If $x_2-p_k \geqq x_1$ then $x_2-p_i \geqq x_1$ for $1 \leqq i \leqq k$ and this fact along with (9) and (10) will give a contradiction. If $x_2-p_i \geqq x_1$ for $1 \leqq i \leqq m$ and $x_2-p_i < x_1$ for $m+1 \leqq i \leqq k$, where $m < k$, then (9) and (10) imply that

(11) $$\lambda_{m+1}f(x_2-p_{m+1})+\cdots+\lambda_k f(x_2-p_k) < (\lambda_{m+1}+\cdots+\lambda_k-1)f(x_2).$$

Since $x_1-p_i < x_2-p_i < x_1$ for $m+1 \leqq i \leqq k$, (7) implies that

(12) $$f(x_1-p_i) < f(x_2-p_i) \qquad \text{for } m+1 \leqq i \leqq k.$$

It also follows from (7) that

(13) $$f(x_1-p_i) < f(x_1) \qquad \text{for } 1 \leqq i \leqq m.$$

Combining (9), (11), (12), and (13) yields

(14) $$\lambda_1 f(x_1-p_1)+\cdots+\lambda_k f(x_1-p_k) < (\lambda_1+\cdots+\lambda_k-1)f(x_1)$$

and this statement contradicts (8). Thus $f(x)$ is non-increasing on the interval (x_1, ∞).

Case 2. Suppose x_0 is a limit point of A.

In this case it is possible to choose x_1 and x_2 such that

(15) $f(x)$ has a relative maximum at x_1 and $f(x)$ has a relative minimum at x_2,

(16) $$f(x_1) \geqq f(x_2),$$

(17) $$x_2 > x_1 > x_0 \quad \text{and} \quad x_2-p_1 < x_0.$$

Since $f(x)$ is strictly increasing on $(0, x_0)$,

$$f(x_1-p_i) < f(x_2-p_i) \qquad \text{for } 1 \leqq i \leqq k. \tag{18}$$

It follows from (15) that (8) and (10) hold in this case, and the statements (8), (10), (16) and (18) yield a contradiction. Thus $f(x)$ has a unique relative maximum at x_0 and $F_0(x)$ is unimodal with a mode at x_0. □

3. Main results. The main theorem of this paper can now be proved.

THEOREM 1. *If $F(x)$ is an L function with a Lévy spectral function $M(u)$ such that $M(u) = 0$ for $u < 0$ or $M(u) = 0$ for $u > 0$, then $F(x)$ is unimodal.*

PROOF. It will be assumed that $M(u) = 0$ for $u < 0$. The proof is similar if $M(u) = 0$ for $u > 0$. By ([3] Lemma 3, page 188), $M(u)$ is absolutely continuous on $(0, \infty)$. Thus it follows from ([5] Theorem 1, page 149) that the characteristic function $\hat{f}(t)$ of $F(x)$ can be written in the form

$$\hat{f}(t) = \exp\{i\gamma t-\sigma^2t^2/2+\textstyle\int_{+0}^{\infty}(e^{iut}-1-iut(1+u^2)^{-1})(\lambda(u)/u)\,du\}$$

where $\lambda(u) = uM'(u)$ is non-increasing on $(0, \infty)$. Two cases must be considered.

Case 1. $\sigma^2 = 0$. It can be assumed without loss of generality that $\gamma = 0$. It is possible to construct a sequence of non-increasing step functions $\{\lambda_n(u)\}$ such that $0 \leqq \lambda_1(u) \leqq \cdots \leqq \lambda_n(u) \leqq \cdots$ and $\lambda_n(u) \to \lambda(u)$ as $n \to \infty$ for $u > 0$. For each value of n let $F_n(x)$ denote the L function with characteristic function

$$\hat{f}_n(t) = \exp\{\textstyle\int_{+0}^{\infty}(e^{iut}-1-iut(1+u^2)^{-1})(\lambda_n(u)/u)\,du\}.$$

Let $G_n(u)$ and $G(u)$ denote the Lévy–Khintchine functions of $F_n(x)$ and $F(x)$ respectively. It is easy to see that $G_n(u) \to G(u)$ as $n \to \infty$ for all values of u, From ([11] Theorem 1, pages 101–102) it follows that $F_n \to_c F$. By Lemma 1, $F_n(x)$ is unimodal for each value of n. By a theorem of A. L. Lapin ([5] Theorem 4, page 160), if a sequence of unimodal distribution functions converges completely to a distribution function, then the limit function is unimodal. Thus $F(x)$ is unimodal and the theorem is proved in the case when $\sigma^2 = 0$.

Case 2. $\sigma^2 > 0$. In this case $F(x)$ is the convolution of a normal distribution and an L function without a normal component. By a theorem of I. A. Ibragimov ([6] page 255), the convolution of a normal distribution function and any unimodal distribution function is unimodal. Thus, the fact that $F(x)$ is unimodal follows from Ibragimov's theorem and the above proof. □

COROLLARY 1. *Every L function is the convolution of two unimodal L functions.*

4. The general problem. It has been shown that every L function is the convolution of two unimodal L functions. The question remains as to whether or not every L function is unimodal. Let $F_0(x)$ be an L function without a normal component and with a Lévy spectral function $M_0(u)$. Assume that $\lambda_0(u) = uM_0'(u)$ is a step function with a finite number of jumps that occur at both positive and negative values of u. If it could be shown that $F_0(x)$ is unimodal, the proof of Theorem 1 could be generalized and it could be shown that every L function is unimodal.

Let $f(x)$ denote the density function of $F_0(x)$. It can be shown that $f(x)$ satisfies the difference-differential equation (5) for all values of $x \neq 0$. However, in this more general case, $p_i > 0$ for some values of i and $p_i < 0$ for other values of i. The proof of Lemma 1 depends strongly on the fact that when $M_0(u) = 0$ for $u < 0$, then (5) has only positive lags and $f(x) = 0$ for $x < 0$. Thus it is not possible to generalize the proof of Lemma 1. However, it is possible to prove a weaker result.

Without loss of generality, choose the centering constant of $F_0(x)$ so that $F_0(x)$ has characteristic function

$$\hat{f}_0(t) = \exp\{\textstyle\int_{-\infty}^{-0} + \int_{+0}^{+\infty} (e^{iut}-1)\, dM_0(u)\}.$$

If $\lambda_0(+0) \leqq 1$ and $|\lambda_0(-0)| \leqq 1$, then $F_0(x) = F_1 * F_2(x)$ where $F_1(x)$ and $F_2(x)$ are unimodal L functions with modes at 0 such that $F_1(x)$ has support on the positive axis and $F_2(x)$ has support on the negative axis. It follows easily that $F_0(x)$ is unimodal with a mode at 0. Thus it is possible to prove the following theorem:

THEOREM 2. *Let $F(x)$ be an L function with a Lévy spectral function $M(u)$. Let $\lambda(u) = uM'(u)$. If $\lambda(+0) \leqq 1$ and $|\lambda(-0)| \leqq 1$ then $F(x)$ is unimodal.*

5. Other unimodal infinitely divisible distribution functions.

P. Medgyessy has proved ([8] Theorem 2, page 444) that if $F(x)$ is a symmetric infinitely divisible distribution function with a Lévy spectral function $M(u)$ such that $M(u)$ is concave on $(0, \infty)$, then $F(x)$ is unimodal. It follows that there exist symmetric unimodal infinitely divisible distribution functions that are not L functions. It will now be shown that there exist unimodal infinitely divisible distribution functions that have Lévy spectral functions with support on the positive axis and are not L functions.

THEOREM 3. *Let $\{Y, X_1, X_2, \cdots\}$ be independent random variables. Let $F(x)$ be an exponential distribution function with density function $f(x) = 0$ if $x < 0$ and $f(x) = e^{-x}$ if $x > 0$. Assume that the X_i's have distribution function $F(x)$ and that Y has a Poisson distribution function with expectation λ. Let $Z_\lambda = X_1 + \cdots + X_Y$ and let $H_\lambda(x)$ be the distribution function of Z_λ. The distribution function $H_\lambda(x)$ is infinitely divisible. If $0 < \lambda \leqq 2$ then $H_\lambda(x)$ is unimodal with a unique mode at 0. If $\lambda > 2$ then $H_\lambda(x)$ is not unimodal.*

PROOF. Let $E(x)$ denote the distribution function degenerate at 0, and let $f^{*n}(x)$ denote the convolution of $f(x)$ with itself n times. By a lemma of H. G. Tucker

([10] Lemma 3, page 1126), $H_\lambda(x)$ is infinitely divisible with a jump at 0 and is absolutely continuous elsewhere. By the same lemma

$$\begin{aligned} H_\lambda(x) &= 0 \quad \text{if} \quad x < 0, \\ &= e^{-\lambda}E(x) + \textstyle\int_0^x \left[\sum_{n=1}^\infty ((e^{-\lambda}\lambda^n/n!) f^{*n}(t))\right] dt \\ &= e^{-\lambda}E(x) + e^{-\lambda} \textstyle\int_0^x \left[e^{-t}\sum_{n=1}^\infty (\lambda^n t^{n-1}/n!(n-1)!)\right] dt \quad \text{if} \quad x > 0. \end{aligned}$$

If $0 < \lambda \leqq 2$, then $H_\lambda''(x) < 0$ for $0 < x < \infty$. Thus $H_\lambda(x)$ is unimodal with a unique mode at 0. If $\lambda > 2$ then $H_\lambda''(+0) = \lambda(\lambda/2 - 1) > 0$. It follows that $H_\lambda(x)$ is not unimodal if $\lambda > 2$. □

It should be pointed out that $H_{\frac{3}{2}}(x)$ is unimodal and $H_3(x) = H_{\frac{3}{2}} * H_{\frac{3}{2}}(x)$ but $H_3(x)$ is not unimodal. Thus it has been shown that the convolution of two unimodal infinitely divisible distribution functions is not necessarily unimodal. Since $H_\lambda(x)$ has support on the positive axis for all values of λ, it follows from a theorem of G. Baxter and J. M. Shapiro ([1] Theorem 2, page 254) that the Lévy spectral function of $H_\lambda(x)$ has support on the positive axis for all values of λ. The following corollary follows immediately from this previous statement and Theorems 1 and 3.

COROLLARY 2. *The class of L functions with Lévy spectral functions that have support on the positive axis is properly contained in the class of unimodal infinitely divisible distribution functions with Lévy spectral functions that have support on the positive axis. The latter class of distribution functions is not closed under the operation of convolution.*

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