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A NOTE ON WEAK CONVERGENCE OF EMPIRICAL PROCESSES FOR SEQUENCES OF ϕ-MIXING RANDOM VARIABLES

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Two results of Billingsley [*Convergence of Probability Measures* (1968), Wiley, New York] on weak convergence of empirical distribution functions for sequences of ϕ-mixing random variables are shown to hold under weaker conditions.

1. Summary and introduction. In this note, Theorems 22.1 and 22.2 of Billingsley (1968) on the weak convergence of empirical distribution functions (df) to appropriate Gaussian random functions are derived under less stringent regularity conditions. The proofs are based on a basic lemma comparable to his Lemma 22.1 on page 195.

Let $\{\xi_i\}$ be a strictly stationary ϕ-mixing sequence of random variables defined on a probability space $(\Omega, \mathscr{B}, P)$. Thus, if $\mathscr{M}_{-\infty}^{k}$ and $\mathscr{M}_{k+n}^{\infty}$ be respectively the σ-fields generated by $\{\xi_i, i \leqq k\}$ and $\{\xi_j, j \geqq k+n\}$, and if $E_1 \in \mathscr{M}_{-\infty}^{k}$ and $E_2 \in \mathscr{M}_{k+n}^{\infty}$, then for all $k(-\infty < k < \infty)$ and $n(\geqq 0)$,

$$|P(E_1 \cap E_2) - P(E_1)P(E_2)| \leqq \phi_n P(E_1), \quad \phi_n \geqq 0, \tag{1.1}$$

where $1 \geqq \phi_0 \geqq \phi_1 \geqq \cdots$, and $\lim_{n\to\infty} \phi_n = 0$. Assuming that ξ_i has a continuous df $F(x)$ and working with $\xi_i^* = F(\xi_i)$, $\forall i$, we define the empirical df by

$$F_n(t) = n^{-1} \sum_{i=1}^{n} c(t - \xi_i^*), \qquad 0 \leqq t \leqq 1, \tag{1.2}$$

where $c(u)$ is 1 or 0 according as u is $\geqq$ or < 0. An interesting result on the weak convergence of

$$Y_n(t) = n^{\frac{1}{2}}[F_n(t) - t], \qquad 0 \leqq t \leqq 1, \tag{1.3}$$

to a Gaussian random function $Y(t)$, $0 \leqq t \leqq 1$, has been established by Billingsley (1968, page 197) under the condition that $\sum n^2 \phi_n^{\frac{1}{2}} < \infty$; the same condition has also been used by him in proving the next Theorem (on page 200) on functions of ϕ-mixing processes. It is shown here that for both the theorems, the condition $\sum n^2 \phi_n^{\frac{1}{2}} < \infty$ can be replaced by a weaker condition that $\sum n \phi_n^{\frac{1}{2}} < \infty$.

Our results are based on a basic lemma comparable to his Lemma 22.1 (on page 195).

2. A basic lemma. Let $\{Z_i\}$ be stationary and ϕ-mixing with $E(Z_i) = 0$, $E(Z_i^2) = \tau$, $P\{|Z_i| > 1\} = 0$, and

$$E|Z_i| \leqq c\tau, \qquad c < \infty. \tag{2.1}$$

Note that (2.1) holds with $c = 2$ when the Z_i are Bernoullian variables, centered at expectations. Let then $S_n = Z_1 + \cdots + Z_n$, $n \geqq 1$.

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LEMMA 2.1. *If* $A_\phi^{(1)} = \sum_{n+1}^{\infty} n\phi_n^{\frac{1}{2}} < \infty$ *and* (2.1) *holds, then for all* $n \geqq 1$,

$$E(S_n^4) \leqq K_\phi[n^2\tau^2 + n\tau], \qquad K_\phi < \infty, \tag{2.2}$$

where K_ϕ *depends on* ϕ *only.*

PROOF. Let $A_\phi^{(0)} = \sum_{n=0}^{\infty} \phi_n^{\frac{1}{2}}$ and $A_\phi^* = \sum_{n=0}^{\infty} (n+1)^2\phi_n$. Then,

$$A_\phi^{(1)} < \infty \Rightarrow A_\phi^{(0)} < \infty \quad \text{and} \quad A_\phi^* < \infty. \tag{2.3}$$

We denote by $\sum_n$ the summation over all $i, j, k \geqq 0$ for which $i+j+k \leqq n$, and let $\sum_n^{(1)}$, $\sum_n^{(2)}$ and $\sum_n^{(3)}$ be respectively the components of $\sum_n$ for which $i \geqq (j, k)$, $j \geqq (i, k)$ and $k \geqq (i, j)$. Then we have on noting that these components are overlapping that

$$\begin{aligned} E(S_n^4) &\leqq 4!n\{\textstyle\sum_n |E(Z_0 Z_i Z_{i+j} Z_{i+j+k})|\} \\ &\leqq 24n\{\textstyle\sum_n^{(1)} + \sum_n^{(2)} + \sum_n^{(3)} |E(Z_0 Z_i Z_{i+j} Z_{i+j+k})|\}. \end{aligned} \tag{2.4}$$

Since the $|Z_i|$ are all bounded, on using (20.28) of Billingsley (1968, page 171) and our (2.1), we have

$$\begin{aligned} \textstyle\sum_n^{(1)} |E(Z_0 Z_i Z_{i+j} Z_{i+j+k})| &\leqq 2\textstyle\sum_n^{(1)} \phi_i E|Z_0| \\ &\leqq 2c\tau \textstyle\sum (i+1)^2 \phi_i \leqq 2c\tau A_\phi^*. \end{aligned} \tag{2.5}$$

Again, on using Lemma 20.1 (page 170) and (20.28) (page 171) of Billingsley (1968), along with the Hölder inequality, we obtain

$$\begin{aligned} &\textstyle\sum_n^{(2)} |E(Z_0 Z_i Z_{i+j} Z_{i+j+k})| \\ &\quad \leqq \textstyle\sum_n^{(2)} |E(Z_0 Z_i) E(Z_0 Z_k)| + 2\sum_n^{(2)} \phi_j E|Z_0 Z_i| \\ &\quad \leqq 4\tau^2 \textstyle\sum_n^{(2)} \phi_i^{\frac{1}{2}} \phi_k^{\frac{1}{2}} + 2\tau \sum_n^{(2)} \phi_j \\ &\quad \leqq 8n\tau^2 (\textstyle\sum_{i=0}^{n} \phi_i^{\frac{1}{2}})^2 + 2\tau (\sum_{j=0}^{n} (j+1)^2 \phi_j) \\ &\quad \leqq 8n\tau^2 (A_\phi^{(0)})^2 + 2\tau A_\phi^*. \end{aligned} \tag{2.6}$$

Finally, by (20.28) (page 171) of Billingsley (1968) and our (2.1), we have

$$\begin{aligned} &\textstyle\sum_n^{(3)} |E(Z_0 Z_i Z_{i+j} Z_{i+j+k})| \\ &\quad \leqq \textstyle\sum_n^{(3)} \{|E(Z_p Z_i Z_{i+j}) E(Z_{i+j+k})| + 2\phi_k E|Z_0 Z_i Z_{i+j}|\} \\ &\quad = 2\textstyle\sum_n^{(3)} \phi_k E|Z_0 Z_i Z_{i+j}| \\ &\quad \leqq 2\textstyle\sum_n^{(3)} \phi_k E|Z_0| \leqq 2c\tau \sum_n^{(3)} \phi_k \\ &\quad \leqq 2c\tau\{\textstyle\sum (k+1)^2 \phi_k\} \leqq 2c\tau A_\phi^*. \end{aligned} \tag{2.7}$$

Thus, (2.2) follows directly from (2.3) through (2.7).

3. The main results. Let $g_t(\xi_i^*) = [c(t - \xi_i^*) - t]$, $0 \leqq t \leqq 1$, $i \geqq 0$, and for $0 \leqq s \leqq t \leqq 1$, let

$$\gamma_{st} = E\{g_s(\xi_0^*) g_t(\xi_0^*)\} + \textstyle\sum_{k=1}^{\infty} E\{g_s(\xi_0^*) g_t(\xi_k^*) + g_s(\xi_k^*) g_t(\xi_0^*)\}. \tag{3.1}$$

Defining then $Y_n(t)$, ξ_i and ξ_i^* as in Section 1, we have the following.

THEOREM 3.1. *If* $\{\xi_i\}$ *be stationary and* ϕ*-mixing with* $\sum n\phi_n^{\frac{1}{2}} < \infty$, *then* $[Y_n(t): 0 \leqq t \leqq 1]$ *converges in law (as* $n \to \infty$*) to* $[Y(t): 0 \leqq t \leqq 1]$ *where* $Y(t)$ *is Gaussian with* $E(Y(t)) = 0$ *and* $E[Y(s)Y(t)] = \gamma_{st}$, $0 \leqq s \leqq t \leqq 1$.

OUTLINE OF THE PROOF. Let $Z_i = g_t(\xi_i^*) - g_s(\xi_i^*)$, $i = 1, \cdots, n$, where $0 \leqq s < t \leqq 1$. Since $c(t-\xi_i^*) - c(s-\xi_i^*)$ is a Bernoullian variable, (2.1) holds with $c = 2$, and $E(Z_i) = 0$, $E(Z_i^2) = (t-s)(1-t+s) \leqq (t-s)$. Hence, by Lemma 2.1, for all $n \geqq 1$, $0 \leqq s < t \leqq 1$,

$$E\left|\textstyle\sum_{i=1}^{n} [g_t(\xi_i^*) - g_s(\xi_i^*)]\right|^4 \leqq K_\phi[n^2(t-s)^2 + n|t-s|]. \tag{3.2}$$

Therefore, for every $\varepsilon(0 < \varepsilon < 1)$, if $t-s > \varepsilon/n$, we have from (1.2), (1.3) and (3.2) that for all n,

$$E|Y_n(t) - Y_n(s)|^4 \leqq 2\varepsilon^{-1}K_\phi(t-s)^2. \tag{3.3}$$

Now, (3.3) agrees with (22.15) of Billingsley (1968, page 198) which is the basic inequality in the proof of his Theorem 22.1. The rest of the proof is identical to the proof of his theorem, and hence, is omitted.

REMARK. By virtue of our Lemma 2.1, it follows that Theorem 22.2 of Billingsley (1968, page 200) on the weak convergence of the empirical df for functions of ϕ-mixing processes also remains true if his condition $\sum n^2\phi_n^{\frac{1}{2}} < \infty$ be replaced by our less restrictive condition that $\sum n\phi_n^{\frac{1}{2}} < \infty$.

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REFERENCE

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