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ON THE WALD-OPTIMALITY OF RANK-ORDER TESTS FOR PAIRED COMPARISONS

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1. Summary and introduction. Consider $p(\geqq 2)$ treatments in an experiment involving paired comparisons. The (i, j)th pair yields n_{ij} observations X_{ijm} with distribution functions (df's) $F_{ij}(x) = F(x-\beta_i+\beta_j)$, $(1 \leqq m \leqq n_{ij};\ 1 \leqq i < j \leqq p)$, $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_p)'$ being the vector involving treatment effects. We assume that F is symmetric about zero, i.e. $F(x)+F(-x) = 1$, for all real x. This is, for example, the situation, when one considers a replicated balanced incomplete block design with each block of size two under the usual assumption of additivity in an analysis of variance model. The null hypothesis to be tested is as follows:

(1.1) $\quad H_0'$: $F_{ij}(x) = F(x)$, for all real x, and $1 \leqq i < j \leqq p$, against all alternatives, which is equivalent to

(1.2) $\quad H_0$: $\boldsymbol{\beta} = \mathbf{0}$, against $\boldsymbol{\beta} \neq \mathbf{0}$, $\mathbf{0}$ being a p-component column vector.

A class of rank-order tests for the above problem was considered in Mehra and Puri (1967) and Puri and Sen (1969b), and asymptotic distributions of test statistics were obtained both under the null hypothesis, and under a sequence of alternatives converging to the null hypothesis at a suitable rate. However, any asymptotic optimality properties of the test procedures were not considered in either of the two papers.

For subsequent notational convenience, we first pool the $n = \sum\sum_{1 \leqq i<j \leqq p} n_{ij}$ observations, and label the mth observation for the (i, j)th pair by $\alpha = \sum_{\lambda=k+1}^{p} \sum_{k=0}^{i-1} n_{k\lambda} + \sum_{\lambda=i}^{j-1} n_{i\lambda} + m(1 \leqq m \leqq n_{ij})$, where we define $n_{0\lambda} = 0$, $n_{ii} = 0$ $(1 \leqq \lambda \leqq p$, $1 \leqq i < j \leqq p$, $1 \leqq \alpha \leqq n)$. Also, set $c_{in\alpha} = 1$, -1 or 0 according as the αth observation is from a block where the treatment i is paired with some treatment, the index of which is $>i$, $<i$, or the treatment i is not involved in the block at all $(1 \leqq \alpha \leqq n;\ 1 \leqq i \leqq p)$. Then one can write

$$(1.3) \qquad P(X_{n\alpha} \leqq x) = F_{n\alpha}(x) \text{ (say)} = F(x-\boldsymbol{\beta}'\mathbf{c}_{n\alpha}),$$
$$\mathbf{c}_{n\alpha} = (c_{1n\alpha}, \cdots, c_{pn\alpha})', \quad 1 \leqq \alpha \leqq n.$$

We may remark at this point that the paired comparison problem can be regarded as a particular case of the more general regression problem considered in Puri and

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Sen (1969a). The object of the present note is to show that, while tests considered in Puri and Sen (1969a), are, under certain conditions asymptotically optimal for the regression problem (and, hence, for the paired comparison case if the same conditions are satisfied), tests considered in Mehra and Puri (1967) or Puri and Sen (1969b) are, in general, not so. A counter-example is given in Section 3 to illustrate this fact.

2. The main results. Let $R_{n\alpha} = \frac{1}{2} + \sum_{\beta=1}^{n} u(|X_{n\alpha}| - |X_{n\beta}|)$, where $u(x) = 1, \frac{1}{2}$ or 0 according as $x >, =$ or < 0. Consider the score functions $J_n(u)$ $(0 < u < 1)$ as defined in (1.6) of Puri and Sen (1969a) (replacing both v and N_v by n). Assume $J(u) = \lim_{n\to\infty} J_n(u)$ exists for all $0 < u < 1$, and satisfies (1.7) of Puri and Sen (1969a). Consider the statistics

$$S_{ni} = n^{-\frac{1}{2}} \sum_{\alpha=1}^{n} c_{in\alpha} J_n(R_{n\alpha}/(n+1)) \operatorname{sgn} X_{n\alpha}, \qquad 1 \leqq i \leqq p, \tag{2.1}$$

where $\operatorname{sgn} x = 2u(x) - 1$, for all real x. Let

$$C_{ij,n} = n^{-1} \sum_{\alpha=1}^{n} c_{in\alpha} c_{jn\alpha}, \qquad 1 \leqq i, j \leqq p. \quad \text{Assume,}$$

$$\max_{1 \leqq \alpha \leqq n} |c_{jn\alpha}| / C_{jj,n}^{\frac{1}{2}} = o(n^{\frac{1}{2}}), \qquad 1 \leqq j \leqq p; \tag{2.2}$$

$$\lim_{n\to\infty} C_{ij,n} = \Lambda_{ij} \quad \text{exists for all} \quad 1 \leqq i, j \leqq p; \tag{2.3}$$

$$\mathbf{C}_n = ((C_{ij,n})), \ (n \geqq 1), \qquad \Lambda = ((\Lambda_{ij})), \tag{2.4}$$

$\mathbf{C}_n (n \geqq 1)$, Λ being assumed to be positive semi-definite (psd) of rank $p' (1 \leqq p' \leqq p)$. In the particular case of paired comparisons, we have,

$$C_{ii,n} = n^{-1} \sum_{j=1}^{p} n_{ij} (1 \leqq i \leqq p); \qquad C_{ij,n} = -(n_{ij}/n),$$

$1 \leqq i \neq j \leqq p$ (defining $n_{ji} = n_{ij}$, $1 \leqq i < j \leqq p$). Assuming now, $\lim_{n\to\infty}(n_{ij}/n) = \rho_{ij}$ exists for all $1 \leqq i \neq j \leqq p \ni (0 < \rho_{ij} < 1$ for all $1 \leqq i \neq j \leqq p, \sum\sum_{1 \leqq i < j \leqq p} \rho_{ij} = 1)$, we find that (2.2) and (2.3) are satisfied, and $p' = p - 1$. Define now, $A_n^2 = n^{-1} \sum_{\alpha=1}^{n} J_n^2(\alpha/(n+1)) = \int_0^1 J_n^2(u) du$, and consider the statistics

$$T_n = A_n^{-2} \mathbf{S}_n' \mathbf{C}_n^* \mathbf{S}_n \qquad (n \geqq 1), \tag{2.5}$$

where $\mathbf{S}_n = (S_{n1}, \cdots, S_{np})'$ and where $\mathbf{C}_n^*$ is a generalized inverse of $\mathbf{C}_n (n \geqq 1)$. For the testing problem (1.2), consider test procedures of the form:

(2.6) Reject H_0 if $T_n > d_n$, d_n some constant $(n \geqq 1)$.

Consider, now, the sequence of alternatives

$$H_n\colon \boldsymbol{\beta} = \boldsymbol{\beta}_n = n^{-\frac{1}{2}} \boldsymbol{\tau}, \ \boldsymbol{\tau} = (\tau_1, \cdots, \tau_p)'. \tag{2.7}$$

Denoting by $\{P_n\}$ and $\{Q_n\}$ the sequences of probability measures corresponding to the null hypothesis (depending on n only through the number of observations), and the sequence $\{H_n\}$ of alternatives, one can prove in the same way as in Hájek (1962) that the sequence $\{Q_n\}$ of probability measures is contiguous to the sequence $\{P_n\}$ of probability measures.

Suppose F admits a density f wrt Lebesgue measure. Let $f_n\ (X_{n\alpha};\ \boldsymbol{\beta})$ denote a density of $X_{n\alpha}$ when $\boldsymbol{\beta}$ is the true value of the parameter vector, i.e.,

$$f_n(X_{n\alpha};\ \boldsymbol{\beta}) = f(X_{n\alpha} - \boldsymbol{\beta}'\mathbf{c}_{n\alpha}),\ 1 \leqq \alpha \leqq n.$$

Write,

$$L_n(\mathbf{X}_n';\ \boldsymbol{\beta}) = \prod_{\alpha=1}^{n} f_n(X_{n\alpha};\ \boldsymbol{\beta}),\ (n \geqq 1).$$

We assume that $f'(x), f''(x)$ exist for almost all x, and,

$$I(f) = \int_{-\infty}^{\infty} [f'(x)/f(x)]^2 f(x)dx < \infty. \tag{2.8}$$

Let λ_n be the likelihood ratio test criterion for testing H_0 against $H_n (n \geqq 1)$. Then,

$$\begin{aligned} \lambda_n &= L_n(\mathbf{X}_n';\ \mathbf{0})/\sup_{\boldsymbol{\beta}_n \in \mathbb{R}^p} L_n(\mathbf{X}_n';\ \boldsymbol{\beta}_n) \\ &= L_n(\mathbf{X}_n';\ \mathbf{0})/\sup_{\boldsymbol{\tau} \in \mathbb{R}^p} L_n(\mathbf{X}_n';\ n^{-\frac{1}{2}}\boldsymbol{\tau}), \end{aligned} \tag{2.9}$$

$\mathbb{R}_p$ being the p-dimensional Euclidean space. Define,

$$\psi^{-1}(u) = -f'(F^{-1}((1+u)/2))/f(F^{-1}((1+u)/2)), \qquad 0 < u < 1. \tag{2.10}$$

Hence, $\int_0^1 \psi^2(u)du = I(f) < \infty$. Further let,

$$A^2(J, \psi) = [\int_0^1 J(u)\psi(u)du]^2 / \int_0^1 J^2(u)du. \tag{2.11}$$

Consider now the score functions $\psi_n(u)$ similar to $J_n(u)$, $(n \geqq 1,\ 0 < u < 1)$. Assume $\lim_{n\to\infty} \psi_n(u) = \psi(u)\ (0 < u < 1)$. Define $B_n^2 = \int_0^1 \psi_n^2(u)du$, and consider the statistics

$$U_{ni} = n^{-\frac{1}{2}} \sum_{\alpha=1}^{n} c_{in\alpha}\psi_n(R_{n\alpha}/(n+1)) \operatorname{sgn} X_{n\alpha} \qquad (1 \leqq i \leqq p;\ n \geqq 1); \tag{2.12}$$

$$\mathbf{U}_n = (U_{n1}, \cdots, U_{np})', \quad W_n = B_n^{-2}\, \mathbf{U}_n'\, \mathbf{C}_n^*\, \mathbf{U}_n \qquad (n \geqq 1). \tag{2.13}$$

Let $\hat{\boldsymbol{\beta}}_n = (\hat{\beta}_{n1}, \cdots, \hat{\beta}_{np})'$ denote the maximum likelihood estimator of $\boldsymbol{\beta}$ based on a sample of size n. Let $\boldsymbol{\tau}_n = (\tau_{n1}, \cdots, \tau_{np})'$ denote the corresponding one for τ under H_n. We assume that L_n and $\boldsymbol{\beta}_n$ satisfy the regularity assumptions of Wald (1943); for details, one may refer to pages 65–67 of [1]. We are now in a position to state the following theorem.

THEOREM 2.1. *Under the assumptions* (I)–(V), *on pages* 65–67 *of Ghosh* (1969), *and under* Ω_1, Ω_2, *and* Ω_5 *of Puri and Sen* (1969a), *it follows that*

$$W_n + 2\log\lambda_n \to 0 \qquad \textit{in } P_n\textit{-probability}. \tag{2.14}$$

PROOF. Use the notation $M_n \sim_{P_n} Z_n$ to mean that $M_n - Z_n \to 0$ in P_n-probability. Write

$$\begin{aligned} \log L(\mathbf{X}_n';\ \mathbf{0}) &= \log L(\mathbf{X}_n';\ \boldsymbol{\beta}_n) - \sum_{i=1}^{p} \hat{\beta}_{ni}[(\partial/\partial\beta_i)) \log L(\mathbf{X}_n';\ \boldsymbol{\beta})]_{\beta=\hat{\beta}_n} \\ &\quad + \tfrac{1}{2}\sum\sum_{1\leqq i,j\leqq p} \hat{\beta}_{ni}\hat{\beta}_{nj}\left[\frac{\partial^2}{\partial\beta_i\partial\beta_j}\log L(\mathbf{X}_n';\ \boldsymbol{\beta})\right]_{\beta=\beta_n^*} \end{aligned}$$

where $\boldsymbol{\beta}_n^*$ lies in the p-dimensional rectangle $[0, \hat{\boldsymbol{\beta}}_n]$. Then

$$\text{(2.15)} \qquad -2 \log \lambda_n = -2[\log L(\mathbf{X}_n'; \mathbf{0}) - \log L(\mathbf{X}_n'; \hat{\boldsymbol{\beta}}_n)]$$
$$= \hat{\boldsymbol{\beta}}_n'\left(\left(-\frac{\partial^2}{\partial\beta_i\partial\beta_j} \log L(\mathbf{X}_n'; \boldsymbol{\beta})\right)\right)_{\boldsymbol{\beta}=\boldsymbol{\beta}_n^*}\hat{\boldsymbol{\beta}}_n.$$

Now, under H_0, and under our regularity assumptions,

(i) $$n^{-1}\left[\frac{\partial^2}{\partial\beta_i\partial\beta_j} \log L(\mathbf{X}_n'; \boldsymbol{\beta})\right]_{\boldsymbol{\beta}=\boldsymbol{\beta}_n^*}$$
$$\sim_{P_n} n^{-1}\left[\frac{\partial^2}{\partial\beta_i\partial\beta_j} \log L(\mathbf{X}_n'; \boldsymbol{\beta})\right]_{\boldsymbol{\beta}=0}$$
$$\sim_{P_n} n^{-1}E_{\boldsymbol{\beta}=0}\left[-\frac{\partial^2}{\partial\beta_i\partial\beta_j} \log L(\mathbf{X}_n'; \boldsymbol{\beta})\right]$$
$$= n^{-1}E_{\boldsymbol{\beta}=0}\left[\frac{\partial}{\partial\beta_i}\log L(\mathbf{X}_n'; \boldsymbol{\beta})\frac{\partial}{\partial\beta_j}\log L(\mathbf{X}_n'; \boldsymbol{\beta})\right] = C_{ij,n}I(f),$$

(ii) $n^{\frac{1}{2}}\hat{\boldsymbol{\beta}}_n$ is bounded in P_n-probability. Hence, from (2.15), one gets,

$$\text{(2.16)} \qquad (n\hat{\boldsymbol{\beta}}_n' \, \mathbf{C}_n \, \hat{\boldsymbol{\beta}}_n)I(f) + 2 \log \lambda_n \to 0 \qquad \text{in } P_n\text{-probability.}$$

Again, $$0 = \partial \log \frac{L(\mathbf{X}_n'; \boldsymbol{\beta})}{\partial\beta_i}\bigg|_{\boldsymbol{\beta}=\hat{\boldsymbol{\beta}}_n} = \frac{\partial \log L(\mathbf{X}_n'; \boldsymbol{\beta})}{\partial\beta_i}\bigg|_{\boldsymbol{\beta}=0} + \sum_{j=1}^{p} \hat{\beta}_{nj}\frac{\partial^2 \log L(\mathbf{X}_n'; \boldsymbol{\beta})}{\partial\beta_i\partial\beta_j}\bigg|_{\boldsymbol{\beta}=\boldsymbol{\beta}_n^{**}}$$

where $\boldsymbol{\beta}_n^{**}$ lies in the p-dimensional rectangle $[0, \hat{\boldsymbol{\beta}}_n]$. Defining now,

$$\text{(2.17)} \qquad U_{ni}^* = n^{-\frac{1}{2}} \textstyle\sum_{\alpha=1}^{n} c_{in\alpha}\,\psi(H(|X_{n\alpha}|)) \operatorname{sgn} X_{n\alpha} \qquad (1 \leqq i \leqq p, n \geqq 1),$$

where $H(x) = F(x) - F(-x) = 2F(x) - 1$, for all real x, one gets

$$0 = n^{\frac{1}{2}}U_{ni}^* + \textstyle\sum_{j=1}^{p} \hat{\beta}_{nj} \displaystyle\frac{\partial^2 \log L(\mathbf{X}_n'; \boldsymbol{\beta})}{\partial\beta_i\partial\beta_j}\bigg|_{\boldsymbol{\beta}=\boldsymbol{\beta}_n^{**}}, \qquad i = 1, 2, \cdots, p.$$

But it follows in the same way as the results of Hájek (1962), (5.17) that $U_{ni}^* \sim_{P_n} U_{ni}(1 \leqq i \leqq p)$. Hence,

$$\text{(2.18)} \qquad U_n \sim_{P_n} n^{-\frac{1}{2}}\left(\left(\frac{\partial^2 \log L(\mathbf{X}_n'; \boldsymbol{\beta})}{\partial\beta_i\partial\beta_j}\right)\right)_{\boldsymbol{\beta}=\boldsymbol{\beta}_n^{**}}\hat{\boldsymbol{\beta}}_n$$
$$= \left(\left(-n^{-1}\frac{\partial^2 \log L(\mathbf{X}_n'; \boldsymbol{\beta})}{\partial\beta_i\partial\beta_j}\right)\right)(n^{\frac{1}{2}}\hat{\boldsymbol{\beta}}_n).$$

Now, $B_n^2 \to I(f)$ as $n \to \infty$. Using (2.18), Slutsky's theorem, and arguments similar as (i), (ii), one gets,

$$\text{(2.19)} \qquad W_n \sim_{P_n} I^{-1}(f)(n^{\frac{1}{2}}\hat{\boldsymbol{\beta}}_n)' \, \mathbf{C}_n I(f) \, \mathbf{C}_n^* \, \mathbf{C}_n I(f)(n^{\frac{1}{2}} \hat{\boldsymbol{\beta}}_n) = n(\hat{\boldsymbol{\beta}}_n' \, \mathbf{C}_n \, \hat{\boldsymbol{\beta}}_n)I(f);$$

(2.16) and (2.19) lead to (2.14).

Because of contiguity, an immediate consequence of the above theorem is that $W_n+2\log\lambda_n \to 0$ in Q_n-probability. Also, one can see that when $J \equiv \psi$, i.e. the assumed and true scores are equal, $T_n+2\log\lambda_n \to 0$ in P_n (and hence in Q_n)-probability. From the results of Wald (1943) page 470, we are now in a position to state the asymptotic optimality properties of the test procedure (1.2). With this end in view, the following notations are introduced.

Define the surface $S_c(\tau)$ by

$$\tau'(AI(f))\tau = c. \tag{2.20}$$

Also, for any $\boldsymbol{\tau}$ and $d > 0$, let $\omega(\boldsymbol{\tau}, d) = \{\boldsymbol{\tau}: |\tilde{\tau}-\tau| \leqq d\}$, where $\tilde{\tau}$ lies on the same surface $S_c(\boldsymbol{\tau})$ as $\boldsymbol{\tau}$; let $\omega'(\boldsymbol{\tau}, d)$ be the image of $\omega(\boldsymbol{\tau}, d)$ by transformation $\boldsymbol{\tau}^* = \boldsymbol{\beta}_\tau\boldsymbol{\tau}$, where $\boldsymbol{\beta}_\tau'\boldsymbol{\beta}_\tau = AI(f)$. Define the weight function

$$\xi(\boldsymbol{\tau}) = \lim_{d\to 0} A[\omega'(\boldsymbol{\tau}, d)]/A[\omega(\boldsymbol{\tau}, d)], \tag{2.21}$$

$A(\omega)$ denoting the area of ω. The following theorem is now an immediate consequence of Theorem VII of Wald (1943), our Theorem 2.1 and the remarks made in the previous paragraph.

THEOREM 2.2. *When* $J(u) = \psi(u)$ *for all* $0 < u < 1$, *under the assumptions* (I)–(V) *on pages* 65–67 *of Ghosh* (1969), *and, under* Ω_1, Ω_2 *and* Ω_5 *of Puri and Sen* (1969*a*) *the test proposed in* (1.2)

(a) *has asymptotically the best average power* wrt *surfaces* $S_c(\boldsymbol{\tau})$ *and weight function* $\xi(\boldsymbol{\tau})$;

(b) *has asymptotically the best constant power on the surfaces* $S_c(\boldsymbol{\tau})$;

(c) *is an asymptotically most stringent test.*

Any such properties, cannot, in general, be claimed by tests proposed in Mehra and Puri (1967), even when the assumed and true scores are equal. A counter-example is provided in the following section to illustrate this fact.

Now, if rank $\mathbf{C}_n = p'(1 \leqq p' \leqq p, n \geqq 1)$, under the sequence $\{H_n\}$ of alternatives $n(\hat{\boldsymbol{\beta}}_n{}'\, \mathbf{C}_n\, \hat{\boldsymbol{\beta}}_n)I(f)$ (and hence, $-2\log\lambda_n$) is distributed asymptotically as a noncentral chi-square with p' degrees of freedom and noncentrality parameter $\eta_1 = (\boldsymbol{\tau}'\boldsymbol{\Lambda}\boldsymbol{\tau})I(f)$. Also, proceeding as on page 6 of Puri and Sen (1969a), one can show that under $\{H_n\}$, T_n is distributed asymptotically as a noncentral chi-square with p' degrees of freedom, and, noncentrality parameter $\eta_2 = (\boldsymbol{\tau}'\boldsymbol{\Lambda}\boldsymbol{\tau})A^2(J, \psi)$. (Note that $A^2(J, \psi) = I(f)$ when $J \equiv \psi$). These results will be used in the following section.

3. A counter-example. We assume that $\rho_{ij} = \lim_{n\to\infty}(n_{ij}/n)$ exist and >0 for all $1 \leqq i < j \leqq p$; $\sum\sum_{1\leqq i<j\leqq p}\rho_{ij} = 1$. Let $\rho_{ji} = \rho_{ij}(1 \leqq i < j \leqq p)$. Then for testing (1.1) against alternatives of the form $F_{ij}(x) = F(x+n^{-\frac{1}{2}}(\tau_j-\tau_i))$, $1 \leqq i < j \leqq p$, test statistics proposed in Mehra and Puri (1967) can be formulated in either of the following two ways.

(i) "*Combined Ranking*" approach: Let $R_{n\alpha}^{(i,j)} \sum \sum_{1 \leqq r < s \leqq p} \sum_{m=1}^{n_{rs}} u(|X_{ij\alpha}| - |X_{rsm}|)$, $1 \leqq \alpha \leqq n_{ij}$, $1 \leqq i < j \leqq p$; $S_{n,ij}^* = \sum_{\alpha=1}^{n_{ij}} J_n(R_{n,\alpha}^{(i,j)}/(n+1)) \operatorname{sgn} X_{ij\alpha}$ $(1 \leqq i < j \leqq p;\ n = \sum \sum_{1 \leqq i < j \leqq p} n_{ij})$. Then the statistic proposed is $L_n^* = \sum_{i=1}^{p} \sum_{j=1 \neq i}^{p} (S_{n,ij}^*/n_{ij}^{\frac{1}{2}})^2/(pA_n^2)$, $(n \geqq 1)$.

(ii) "*Separate Ranking*" approach: Let $R_{n_{ij},\alpha}^* = \sum_{m=1}^{n_{ij}} u(|X_{ij\alpha}| - |X_{ijm}|)$, $1 \leqq \alpha \leqq n_{ij}$, $1 \leqq i < j \leqq p$; $S_{n,ij}^{**} = \sum_{\alpha=1}^{n_{ij}} J_{n_{ij}}(R_{n_{ij},\alpha}/(n_{ij}+1)) \operatorname{sgn} X_{ij\alpha}$, $1 \leqq i < j \leqq p$. The proposed statistic is

$$L_n^{**} = \sum_{i=1}^{p} \{\sum_{j=1}^{p} S_{n,ij}^{**}/(pn_{ij}A_{n_{ij}}^2)^{\frac{1}{2}}\}^2, \qquad (n \geqq 1).$$

It is shown in Mehra and Puri (1967) that L_n^* (or L_n^{**}) is distributed asymptotically under H_n as a noncentral chi-square with $(p-1)$ degrees of freedom and noncentrality parameter

$$\delta = p^{-1} \sum_{i=1}^{p} \{\sum_{j=1, \neq i}^{p} \rho_{ij}^{\frac{1}{2}} \tau_{ij}\}^2 A^2(J, \psi), \quad \tau_{ij} = \tau_j - \tau_i \qquad (1 \leqq i < j \leqq p),$$

and defining $\rho_{ji} = \rho_{ij}$ for all $1 \leqq i < j \leqq p$.
In the particular case $n_{ij} = \binom{p}{2}^{-1} n$, $(1 \leqq i < j \leqq p)$, $\rho_{ij} = \binom{p}{2}^{-1}$, $1 \leqq i < j \leqq p$. Then, it follows after some simple algebra that

$$\delta = (2/(p-1))[\sum_{i=1}^{p} (\tau_i - \bar{\tau})^2] A^2(J, \psi), \qquad \bar{\tau} = p^{-1} \sum_{i=1}^{p} \tau_i.$$

In this case, $A = (2/p)[(1-\rho)I_p + \rho J_p]$, $\rho = -(p-1)^{-1}$, I_p is a unit matrix of order p and J_p is a $p \times p$ matrix with all elements 1. Then, $\eta_2 = [2/(p-1)] \sum_{i=1}^{p} (\tau_i - \bar{\tau})^2 A^2(J, \psi) = \delta$. If, however, all pairs of treatments are not compared the same number of times, the two test procedures are not asymptotically power-equivalent, since, then δ and η_2 may differ. To see this consider the following example:

Let $p = 3$, $n_{12} = n_{13} = n/6$, $n_{23} = 2n/3$. Then, omitting some elementary algebra, we get,

$$\delta = A^2(J, \psi)[(\tfrac{1}{3})\tau_1^2 + (7/9)(\tau_2^2 + \tau_3^2) - (\tfrac{1}{3})(\tau_1\tau_2 + \tau_1\tau_3) - (11/9)\tau_2\tau_3].$$

Also, here,
$$A = \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{5}{6} & -\frac{2}{3} \\ -\frac{1}{6} & -\frac{2}{3} & \frac{5}{6} \end{bmatrix}.$$

Then $\eta_2 = A^2(J, \psi)[(\tfrac{1}{3})\tau_1^2 + (\tfrac{5}{6})(\tau_2^2 + \tau_3^2) - (\tfrac{1}{3})(\tau_1\tau_2 + \tau_1\tau_3) - (\tfrac{4}{3})\tau_2\tau_3]$.

Hence, $\eta_2 - \delta = (1/18)(\tau_2 - \tau_3)^2 A^2(J, \psi) > 0$, when $\tau_2 \neq \tau_3$ and $A^2(J, \psi) \neq 0$.

This example shows that in this particular case, the tests proposed in Mehra and Puri (1967) and Puri and Sen (1969a) are not asymptotically power-equivalent and the test proposed in Puri and Sen (1969a) is asymptotically at least as powerful as the one in Mehra and Puri (1967) (or Puri and Sen (1969b)). Thus, unlike the test proposed in Puri and Sen (1969a), the test proposed in Mehra and Puri (1967) does not possess any asymptotic optimality properties in the sense of Wald (1943).

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