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MAXIMUM LIKELIHOOD ESTIMATION OF A TRANSLATION PARAMETER OF A TRUNCATED DISTRIBUTION[1]

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Let $f_\theta(x) = f(x - \theta)$, $\theta, x \in R$, where $f(x) = 0$ for $x \leqq 0$ and let $\hat{\theta}_n$ be the maximum likelihood estimate (MLE) of θ based on a sample of size n. If $\alpha = \lim f'(x)$ exists as $x \to 0$, and $0 < \alpha < \infty$, then under some regularity conditions, it is shown that $\alpha_n(\hat{\theta}_n - \theta)$ has an asymptotic standard normal distribution where $2\alpha_n^2 = \alpha n \log n$ and that if θ is regarded as a random variable with a prior density, then the posterior distribution of $\alpha_n(\theta - \hat{\theta}_n)$ converges to normality in probability.

1. Introduction. Let f be a uniformly continuous density which vanishes on $(-\infty, 0]$ and consider the one parameter family of densities f_θ, $\theta \in R$, defined by

$$f_\theta(x) = f(x - \theta)\,, \qquad x \in R\,. \tag{1.1}$$

If $X_1, \cdots, X_n$ is a random sample from f_θ for some unknown value of θ, then the likelihood function

$$L_n(t) = \prod_{i=1}^n f_t(X_i)\,, \qquad t \in R\,,$$

will attain its maximum at some point interior to the interval $(-\infty, M_n)$ where $M_n = \min(X_1, \cdots, X_n)$. Thus, MLE's exist. That is, there is a random variable $\hat{\theta}_n = \hat{\theta}_n(X_1, \cdots, X_n)$ for which $L_n(\hat{\theta}_n) = \max_t L_n(t)$ w.p. 1. Moreover, if the entropy

$$\int_0^\infty -\log f(x) \cdot f(x)\, dx$$

is finite, then it follows easily from the result of Wald [11] that $\hat{\theta}_n$ is a consistent estimate of θ. Since the support of f_θ depends on θ, the classical results on asymptotic normality of MLE's (e.g., [1] or [10]) do not apply in our case, however.

Recently, LeCam [7] has obtained a general theorem (Proposition 6) from which the following result may be deduced (Lemma 4.3 of [5] or its generalization Proposition 7 of [7] are useful in checking the conditions imposed by Proposition 6).

PROPOSITION 1.1. *Let f be absolutely continuous with derivative f' and let the Fisher Information*

$$\sigma^2 = \int_0^\infty \left(\frac{f'(x)^2}{f(x)}\right) dx \tag{1.2}$$

be finite. If $\hat{\theta}_n$ is any consistent sequence of MLE'*s then $n^{\frac{1}{2}}\sigma(\hat{\theta}_n - \theta)$ has an asymptotic standard normal distribution.*

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Even this result does not cover all interesting subcases of (1.1), however, for if $\liminf f'(x) > 0$ as $x \to 0$, then it is easily verified that σ^2 must be infinite.

Here we consider the asymptotic behavior of $\hat{\theta}_n - \theta$ in the case that $\alpha = \lim f'(x)$ exists as $x \to 0$ with $0 < \alpha < \infty$. Subject to some regularity conditions to be detailed in Sections 2 and 4, our results may be summarized as follows. Let $2\alpha_n^2 = \alpha n \log n$, $n \geqq 1$. Then, (1) $\alpha_n(\hat{\theta}_n - \theta)$ has an asymptotic standard normal distribution; and (2) if θ is regarded as a random variable with a prior density, then the posterior distribution of $\alpha_n(\theta - \hat{\theta}_n)$ converges in probability to normality.

It is interesting that in both cases, $\sigma^2 < \infty$ and $0 < \alpha < \infty$, the MLE $\hat{\theta}_n$ converges to θ strictly faster than does the minimum M_n, thus justifying the additional work required to compute an MLE (See Lemma 2.1). In particular, if f is a log normal density, then M_n converges more slowly than any power of $1/n$, while $\hat{\theta}_n$ converges at the rate $1/n^{\frac{1}{2}}$ by Proposition 1.1.

Fisher [4] conjectured the correct asymptotic distribution for $n^{\frac{1}{2}}(\hat{\theta}_n - \theta)$ in the special case of a Gamma distribution for which $\sigma^2 < \infty$ and the fact that $n^{\frac{1}{2}}(\hat{\theta}_n - \theta) = o_p(1)$ in the case of a Gamma distribution for which $0 < \alpha < \infty$. Polfeldt has considered variance bounds [8] and systematic statistics [9] in a non-regular case similar to ours, and Dawid [2] has considered the asymptotic shape of posterior distributions in cases where the support of the density may depend on the unknown parameter.

Our results concerning maximum likelihood estimates are stated precisely in Section 2 and proved in Section 3. Posterior distributions are discussed in Section 4, and Section 5 consists of several concluding remarks.

2. Conditions and theorems. Throughout this section and the next, f, θ, f_θ, and $X_1, \cdots, X_n$ will be as described in the introduction. That is, θ is fixed, and $X_1, \cdots, X_n$ are independent with common density f_θ, which is defined by (1.1). In the statement of our results, it will be convenient to refer to the following conditions.

P (positivity): the support of f is the interval $(0, b)$ where $0 < b \leqq \infty$ and by support we mean the set of $x \in R$ for which $f(x) > 0$.

D (differentiability): f is continuously differentiable on $(0, b)$ with derivative f', $\alpha = \lim f'(x)$ exists as $x \to 0$, and $0 < \alpha < \infty$. Moreover, f' is absolutely continuous on every compact subinterval of $(0, b)$ with derivative f'', and $\lim xf''(x) = 0$ as $x \to 0$.

Let $g(x) = \log f(x)$, $0 < x < b$. If condition D is satisfied, then g will be continuously differentiable on $(0, b)$ with derivative $g' = f'/f$, and g' will be absolutely continuous on every compact subinterval of $(0, b)$ with derivative $g'' = (ff'' - f'^2)/f^2$. We also need

I (integrability): for every $\delta > 0$, $\int_\delta^b g'(x)^2 f(x)\,dx < \infty$.

B_1 (boundedness): for every a, $0 < a < b$, there are $\eta > 0$ and $0 < \delta < \min(a, \eta)$ for which

$$\int_a^{b-\eta} \sup_{|t| \leq \delta} |g''(x - t)| f(x)\,dx < \infty ,$$

where $b - \eta = \infty$ if $b = \infty$.

B_2: $b < \infty$, $\limsup g''(x) < \infty$ as $x \to b$ (may be $-\infty$), and there are $\pi > 0$, $\beta < 1$, and an increasing function h on $(0, \eta\beta^{-1})$ for which $g''(x) \geqq h(b - x)$, $b - \eta < x < b$, and

$$\int_{b-\eta}^b h(\beta(b - x)) f(x)\,dx > -\infty .$$

If $b = \infty$, then the η in condition B_1 is irrelevant. If g'' is continuous and B_1 holds for some $a > 0$, then it also holds for every choice of $0 < a < b$. An easily checked condition which implies B_2 will be given below.

Let $N_n = \max(X_1, \cdots, X_n)$ and let conditions P and D be satisfied. We will say that an estimate $\varphi_n = \varphi_n(X_1, \cdots, X_n)$ of θ satisfies the likelihood equation if the relations $N_n - b < \varphi_n < M_n$ and $L_n'(\varphi_n) = 0$ are both satisfied w.p. 1 for every θ. Clearly, any MLE satisfies the likelihood equation.

THEOREM 2.1. *Let conditions* P, D, I, *and* B_1 *hold with* $b = \infty$ *and let* $2\alpha_n^2 = \alpha n \log n$, $n \geqq 1$. *If* $\hat\theta_n$ *denotes any consistent sequence of roots of the likelihood equation, then* $\alpha_n(\hat\theta_n - \theta)$ *has an asymptotic standard normal distribution as* $n \to \infty$.

THEOREM 2.2. *Let conditions* P, D, I, B_1, *and* B_2 *hold with* $b < \infty$ *and let* $2\alpha_n^2 = \alpha n \log n$, $n \geqq 1$. *If* $\hat\theta_n$ *denotes any consistent sequence of roots of the likelihood equation, then* $\alpha_n(\hat\theta_n - \theta)$ *has an asymptotic standard normal distribution as* $n \to \infty$. *Moreover, if* P *and* D *hold with* $b < \infty$ *and if* $f''(x) = (b - x)^\gamma L(b - x)$ *where* $\gamma > -1$ *and* $L(x)$ *varies slowly as* $x \to 0$, *then condition* B_2 *is satisfied.*

Condition I has an information theoretic interpretation. Condition B_1 and the requirement of a second derivative in D have no nice statistical interpretation, but conditions of their type are quite common in work with MLE's. The novel conditions are B_2 and the requirement that $xf''(x) \to 0$ as $x \to 0$. The latter merely rules out wild oscillations by f'' near zero, and the former is, at least, reasonably general, as indicated in Theorem 2.2. In particular, conditions D, I, and B_1 are satisfied by any density of the form $f(x) = cxe^{-x^\gamma}$, $x > 0$, or $f(x) = cx/(1 + x)^{2+\gamma}$, $x > 0$, for any $\gamma > 0$; and D, I, B_1, and B_2 are satisfied by any density of the form $f(x) = cx(b - x)^{1+\gamma}$, $0 < x < b$, for any $\gamma > 0$ and any $b > 0$.

We will prove Theorems 2.1 and 2.2 in the next section. We conclude this one by recording the asymptotic behavior of M_n and N_n and thereby justifying the comparison between $\hat\theta_n$ and M_n which we made in the introduction.

LEMMA 2.1. *Let* f *be absolutely continuous. If* $\alpha = \lim f'(x)$ *exists and is finite* *as* $x \to 0$, *then* $\lim \Pr(n^{\frac{1}{2}}(M_n - \theta) > t) = \exp(-\frac{1}{2}\alpha t^2)$ *as* $n \to \infty$ *for all* $t > 0$; *and if* $\sigma^2 < \infty$, *then* $\lim \Pr(n^{\frac{1}{2}}(M_n - \theta) > t) = 1$ *as* $n \to \infty$ *for all* $t > 0$.

PROOF. Let F denote the distribution function of f. Then,

$$\Pr(n^{\frac{1}{2}}(M_n - \theta) > t) = (1 - F(tn^{-\frac{1}{2}}))^n \tag{2.1}$$

for all $t > 0$. Thus, the first assertion of the lemma follows from the easily verified fact that if $f'(x) \to \alpha < \infty$ as $x \to 0$, then $F(x) = \frac{1}{2}\alpha x^2 + o(x^2)$ as $x \to 0$. Similarly, the second will follow if we can show that the finiteness of σ^2 implies $F(x) = o(x^2)$ as $x \to 0$. This follows from

$$\begin{aligned} F(x) &= \textstyle\int_0^x (x - y) f'(y)\, dy \\ &\leqq \left(\textstyle\int_0^x \left(\frac{f'(y)^2}{f(y)}\right) dy\right)^{\frac{1}{2}} \left(\textstyle\int_0^x (x - y)^2 f(y)\, dy\right)^{\frac{1}{2}}, \end{aligned} \tag{2.2}$$

which is bounded by $o(1)xF^{\frac{1}{2}}(x)$ as $x \to 0$ if $\sigma^2 < \infty$.

Since only the local integrability of f'^2/f was used in (2.2), we also have (by interchanging the roles of 0 and b)

LEMMA 2.2. *Let condition* P *be satisfied with* $b < \infty$, *let* f *be absolutely continuous, and let condition* I *be satisfied. Then, as* $n \to \infty$, $\lim \Pr(n^{\frac{1}{2}}(b - N_n) > t) = 1$ *for all* $t > 0$ *where* $N_n = \max(X_1, \cdots, X_n)$.

3. Proofs. In this section we will prove Theorems 2.1 and 2.2 Conditions P, D, and I will be assumed throughout, but conditiions B_1 and B_2 will only be imposed as needed. We may, of course, and will restrict our attention to the case $\theta = 0$.

We begin by remarking that since $f'(x) \to \alpha$ and $xf''(x) \to 0$ as $x \to 0$, we must also have $f(x) \sim \alpha x$, $g'(x) \sim 1/x$, and $g''(x) \sim -1/x^2$ as $x \to 0$. These relations will be used without further comment. Let $G_n(t) = \log L_n(t)$, $N_n - b < t < M_n$. Then,

LEMMA 3.1. *If* $b = \infty$ *and* B_1 *is satisfied, then for sufficiently small* $\delta > 0$, *there are events* C_n, $n \geqq 1$, *for which* $\lim P(C_n) = 1$ *as* $n \to \infty$ *and for each* $n \geqq 1$, C_n *implies* $G_n''(t) < 0$ *for* $-\delta \leqq t < M_n$. *Moreover, if* $b < \infty$, *and if* B_1 *and* B_2 *are both satisfied, then there are events* D_n, $n \geqq 1$, *for which* $\lim P(D_n) = 1$ *as* $n \to \infty$ *and for each* $n \geqq 1$, D_n *implies* $G_n''(t) < 0$ *for* $N_n - b < t < M_n$.

PROOF. Let $a > 0$ be so small that $g''(x) \leqq -1/2x^2$ for $0 < x \leqq 2a$, let $0 < \delta < a$, and for $0 < c < d \leqq \infty$ let $\sum_c^d$ denote summation over all $i = 1, \cdots, n$ for which $c < X_i < d$. If $b = \infty$ and B_1 is satisfied, then the event $M_n \leqq \delta$ implies that

$$\frac{1}{n} G_n''(t) \leqq \left(\frac{-1}{2n}\right) \textstyle\sum_0^a (X_i + \delta)^{-2} + \left(\frac{1}{n}\right) \textstyle\sum_a^\infty \sup_{|t| \leq \delta} |g''(X_i - t)|\,, \tag{3.1}$$

for $-\delta \leqq t < M_n$. The right side of (3.1) converges in probability as $n \to \infty$ to

$$-\tfrac{1}{2}\int_0^a (x + \delta)^{-2} f(x)\,dx + \int_a^\infty \sup_{|t|\leqq\delta} |g''(x - t)| f(x)\,dx\,, \tag{3.2}$$

which, in turn, diverges to $-\infty$ as $\delta \to 0$. The first assertion of the lemma follows on taking C_n to be the intersection of the event $M_n \leqq \delta$ with the event that the right side of (3.1) is less than one half of (3.2) for sufficiently small δ.

To establish the second assertion of the lemma suppose that $b < \infty$ and that B_1 and B_2 both hold. Then, there are $A > 0$ and $\eta > 0$ for which $g''(x) \leqq A$ for $b - 2\eta \leqq x < b$. Let a be as above and let $\delta < \min(a, \eta)$. Then, $M_n \leqq \delta$ and $b - N_n \leqq \delta$ imply that

$$\frac{1}{n} G_n''(t) \leqq \left(\frac{-1}{2n}\right) \sum_0^a (X_i + \delta)^{-2} + \left(\frac{1}{n}\right) \sum_a^{b-\eta} \sup_{|t|\leqq\delta} |g''(X_i - t)| + \left(\frac{1}{n}\right) \sum_{b-\eta}^{\eta} A\,, \tag{3.3}$$

for $N_n - b < t < M_n$. Since the right side of (3.3) also diverges to $-\infty$ as $n \to \infty$ and $\delta \to 0$ (in that order), the second assertion of the lemma follows easily.

Thus, if $b = \infty$ and B_1 is satisfied, then with probability approaching one, G_n' will be a decreasing function $(-\delta, M_n)$ for sufficiently small positive δ. Since, by assumption, the $\hat\theta_n$ of Theorem 2.1 is a consistent sequence of roots of the equation $G_n'(\hat\theta_n) = 0$, it follows easily that

$$\Pr(\alpha_n \hat\theta_n \leqq t) = \Pr(G_n'(t\alpha_n^{-1}) \leqq 0) + o(1)\,, \tag{3.4}$$

as $n \to \infty$. Here we include in the event $G_n'(t\alpha_n^{-1}) \leqq 0$ the requirement that $G_n'(t\alpha_n^{-1})$ be defined—that is, the requirement that $t\alpha_n^{-1} < M_n$. By Lemma 2.1, the latter requirement is satisfied with probability approaching one. Similarly, if $b < \infty$, and if both B_1 and B_2 are satisfied with probability approaching one, G_n' will be a decreasing function on $(N_n - b, M_n)$, so that (3.4) holds in this case too. (In this case the event $G_n'(t\alpha_n^{-1}) \leqq 0$ includes the requirement that $N_n - b < t\alpha_n^{-1} < M_n$. This requirement is also satisfied with probability approaching one by Lemmas 2.1 and 2.2.) Therefore, either Theorem 2.1 or 2.2 may be established by showing that its hypotheses imply that $\alpha_n^{-1}G_n'(t\alpha_n^{-1})$ is asymptotically normal with mean $-t$ and variance one. This will be deduced from the expansion

$$\alpha_n^{-1}G_n'(t\alpha_n^{-1}) = \alpha_n^{-1}G_n'(0) + \alpha_n^{-2} \int_0^t G_n''(s\alpha_n^{-1})\,ds\,, \tag{3.5}$$

(valid for $N_n - b < t\alpha_n^{-1} < M_n$) by considering $G_n'(0)$ and $G_n''(s\alpha_n^{-1})$ separately.

LEMMA 3.2. *$\alpha_n^{-1}G_n'(0)$ has an asymptotic standard normal distribution as $n \to \infty$.*

PROOF. Let $Y_i = g'(X_i)$, $i = 1, \cdots, n$. Then, $Y_1, \cdots, Y_n$ are independent and identically distributed, $E(Y_i) = 0$, and $G_n'(0) = -(Y_1 + \cdots + Y_n)$. Therefore, by Theorem 2 of [3], page 546, it will suffice to show that $\mu(y) \sim a \log y$ as $y \to \infty$, where

$$\mu(y) = \int_{B_y} Y_1^2 \, dP$$

and B_y is the event that $|Y_1| \leqq y$, $y > 0$. To see this let $0 < \varepsilon < 1$ be given, let $\delta > 0$ be so small that $|f(x) - \alpha x| \leqq \alpha\varepsilon x$ and $|xg'(x) - 1| \leqq \varepsilon$ for $0 < x \leqq \delta$, and let A be the event that $X_1 \leqq \delta$. Then, A implies $Y_1 \leqq (1 + \varepsilon)/X_1$, and AB_y implies $(1 - \varepsilon)/y \leqq X_1 \leqq \delta$. Since also $f(x) \leqq \alpha(1 + \varepsilon)x$ for $0 < x \leqq \delta$, we have, as $y \to \infty$,

$$\begin{aligned} \mu(y) &\leqq \int_{AB_y} X_1^2 \, dP + \int_{A^c} Y_1^2 \, dP \\ &\leqq \int_{(1-\varepsilon)/y}^{\delta} \alpha(1 + \varepsilon)^3 x^{-1} \, dx + \int_{\delta}^{b} g'(x)^2 f(x) \, dx \\ &= \alpha(1 + \varepsilon)^3 \log y + O(1), \end{aligned}$$

where we have used condition I in the final step. Similarly,

$$\mu(y) \geqq \alpha(1 - \varepsilon)^3 \log y + O(1) \qquad \text{as} \quad y \to \infty .$$

The lemma follows easily.

LEMMA 3.3. *Let $0 < \delta < b$ and define $Z_i = X_i^{-1}$ if $0 < X_i < \delta$ and $Z_i = 0$ if $X_i \geqq \delta$, $i = 1, \cdots, n$. Then,*

$$\alpha_n^{-2} \sum_{i=1}^{a} Z_i^2 \to 1 ,$$

in probability as $n \to \infty$.

PROOF. As in the previous lemma, it follows from Theorem 2 of [3], page 546, that

$$\left(\frac{1}{n}\right) \sum_{i=1}^{n} Z_i^2 - nb_n$$

has an asymptotic stable distribution, where $b_n = E(\sin(Z_1^2/n))$, $n \geqq 1$. Therefore, it will suffice to show that $nb_n \sim \frac{1}{2}\alpha \log n$ as $n \to \infty$. To see this let $\varepsilon > 0$ be given, let $\delta' < \delta$ be so small that $|f(x) - \alpha x| \leqq \alpha\varepsilon x$ for $0 < x \leqq \delta'$, and let $a > 0$ be so large that $|\sin x - x| \leqq \varepsilon x$ for $0 < x \leqq a^{-2}$. Then, as $n \to \infty$,

$$\begin{aligned} nb_n &= \int_{a/n^{\frac{1}{2}}}^{\delta'} n \sin(1/nx^2) f(x) \, dx + O(1) \\ &\leqq \int_{a/n^{\frac{1}{2}}}^{\delta'} \alpha(1 + \varepsilon)^2 x^{-1} \, dx + O(1) \\ &= \tfrac{1}{2}\alpha(1 + \varepsilon)^2 \log n + O(1) . \end{aligned}$$

Similarly, $nb_n \geqq \frac{1}{2}\alpha(1 - \varepsilon)^2 \log n + O(1)$ as $n \to \infty$. The lemma follows.

In our next two lemmas we will use the notation

$$r_n(\theta, k) = \sup_{|t| \leqq k} |\alpha_n^{-2} G_n''(\theta + t\alpha_n^{-1}) + 1| , \tag{3.6}$$

where the supremum is understood to be infinite if either $k\alpha_n^{-1} \geqq M_n - \theta$ or $-k\alpha_n^{-1} \leqq N_n - b - \theta$.

LEMMA 3.4. *If $b = \infty$, and condition* B_1 *is satisfied, then* $r_n(0, k) \to 0$ *in probability as* $n \to \infty$ *for any* $k > 0$.

PROOF. Given $0 < \varepsilon < 1$ and $k > 0$, let $a > 0$ be so small that $|x^2 g''(x) + 1| \leqq \varepsilon$ for $0 < x \leqq 2a$, and let $\delta_n = k\alpha_n^{-1}$. If $M_n \geqq \delta_n/\varepsilon$, which it is with probability approaching one, then for $|t| \leqq k$ and $\delta_n < a$,

$$\begin{aligned}\alpha_n^{-2}G_n''(t\alpha_n^{-1}) &\leqq -(1-\varepsilon)\alpha_n^{-2} \textstyle\sum_0^a (X_i - t\alpha_n^{-1})^{-2} \\ &\quad + \alpha_n^{-2} \textstyle\sum_a^\infty \sup_{|t| \leqq \delta_n} |g''(X_i - t)| \\ &\leqq -(1-\varepsilon)(1+\varepsilon)^{-2}\alpha_n^{-2} \textstyle\sum_0^a X_i^{-2} + o_p(1) \\ &\to -(1-\varepsilon)(1+\varepsilon)^{-2},\end{aligned}$$

in probability as $n \to \infty$. Here we have used condition B_1 in the final steps. A similar lower bound may be obtained to complete the proof.

LEMMA 3.5. *If $b < \infty$, and if both* B_1 *and* B_2 *are satisfied, then* $r_n(0, k) \to 0$ *in probability as* $n \to \infty$ *for any* $k > 0$.

PROOF. Let $0 < \varepsilon < 1$ and $k > 0$ be given, let η, β, and h be as in the statement of B_2, let $\eta_0 = \frac{1}{2}\eta$, and let a and δ_n be as in the previous lemma. Then, as in the proof of the previous lemma, $b - N_n \geqq \delta_n/(1-\beta)$ and $M_n \leqq \delta_n/\varepsilon$ hold with probability approaching one and imply that for $|t| \leqq k$ and $\delta_n \leqq \min(a, \eta_0)$,

$$\begin{aligned}\alpha_n^{-2}G_n''(t\alpha_n^{-1}) &\geqq -(1+\varepsilon)(1-\varepsilon)^{-2} + o_p(1) \\ &\quad + \alpha_n^{-2} \textstyle\sum_{b-\eta_0}^b g''(X_i - t\alpha_n^{-1}),\end{aligned} \tag{3.7}$$

where $o_p(1)$ is uniform in $|t| \leqq k$. By B_2, $g''(X_i - t\alpha_n^{-1}) \geqq h(b - X_i + t\alpha_n^{-1}) \geqq h(b - X_i - \delta_n) \geqq h(\beta(b - X_i))$ for $b - \eta_0 < X_i < b$. Therefore, with probability approaching one, the last member of (3.7) is at least

$$\alpha_n^{-2} \textstyle\sum_{b-\eta_0}^b h(\beta(b - X_i)),$$

which is $o_p(1)$ by condition B_2. Therefore, with probability approaching one, $\alpha_n^{-2}G_n''(t\alpha_n^{-1}) \geqq -(1+\varepsilon)(1-\varepsilon)^{-2} + o_p(1)$, where $o_p(1)$ is uniform in $|t| \leqq k$. A similar upper bound may be obtained to complete the proof (cf. (3.3)).

The asymptotic normality of $\alpha_n(\hat{\theta}_n - \theta)$ now follows easily. For example, it follows from Lemma 3.2, Lemma 3.4, and equation (3.5) that $\alpha_n^{-1}G_n'(t\alpha_n^{-1})$ is asymptotically normal with mean $-t$ and variance one, $t \in R$, under the hypotheses of Theorem 2.1, so that Theorem 2.1 follows from equation (3.4), as explained above. The asymptotic normality assertion in Theorem 2.2 may be established similarly, so it remains only to prove the final assertion of Theorem 2.2.

LEMMA 3.6. *If $b < \infty$, and if $f''(x) = (b - x)^\gamma L(b - x)$ where $\gamma > -1$ and $L(x)$ varies slowly as $x \to 0$, then condition* B_2 *is satisfied.*

PROOF. It follows from Section 8.9 of [3] that $f'(x) \sim -(b-x)^{1+\gamma} \times L(b-x)/(1+\gamma)$ and $f(x) \sim (b-x)^{2+\gamma}L(b-x)/(1+\gamma)(2+\gamma)$ as $x \to b$. Therefore, $g''(x) \sim -(2+\gamma)/(b-x)^2$ as $x \to b$, and in particular, $\limsup g''(x) \leqq 0$ as $x \to b$. Moreover, letting $h(x) = -c/x^2$, it follows that $g''(x) \geqq h(b-x)$, $b - \eta < x < b$, for c sufficiently large and η sufficiently small, and it is easily verified that

$$\int_{b-\eta}^{b} h(\beta(b-x))f(x)\,dx = -c\beta^{-2}\int_{b-\eta}^{b}(b-x)^{-2}f(x)\,dx\,,$$

which is finite for all $\beta < 1$ (cf. Section 8.9 of [3]).

4. Posterior distributions. In this section we take a Bayesian approach by placing a prior density q over the possible states of nature. That is, we invent a random variable $\boldsymbol{\theta}$ with density q, and we suppose that conditionally given $\boldsymbol{\theta} = \theta$, $X = (X_1, \cdots, X_n)$ is a random sample from f_θ. Conditional probabilities given $\boldsymbol{\theta} = \theta$ will now be denoted by P_θ and unconditional probabilities by Pr. Thus,

$$\Pr(\boldsymbol{\theta} \in A, X \in B) = \int_A P_\theta(X \in B)q(\theta)\,dt\,,$$

for Borel sets A and B. The conclusions of Theorems 2.1 and 2.2 may now be written

$$\lim P_\theta(\alpha_n(\hat{\theta}_n - \theta) \in J) = \Phi(J) \quad \text{as} \quad n \to \infty\,, \tag{4.1}$$

for every $\theta \in R$ and every interval $J \subset R$, where Φ denotes the standard normal distribution—that is,

$$\Phi(J) = \int_J e^{-\frac{1}{2}x^2}/(2\pi)^{\frac{1}{2}}\,dx\,.$$

THEOREM 4.1. *Let conditions* P, D, I, *and* B_1 *be satisfied, and if $b < \infty$, let condition* B_2 *be satisfied also. Suppose that the support of q is an interval and that q is continuous on its support. Let $\hat{\theta}_n$ be any consistent sequence of roots of the likelihood equation and let Q_n denote the conditional distribution of $\alpha_n(\boldsymbol{\theta} - \hat{\theta}_n)$ given $X_1, \cdots, X_n$, where $2\alpha_n^2 = \alpha n \log n$, $n \geqq 1$. Then, as $n \to \infty$, $\lim Q_n(J) = \Phi(J)$ in probability for every finite interval J.*

In the proof of Theorem 4.1, it will be convenient to have the following lemma (see (3.6)).

LEMMA 4.1. *If the hypotheses of Theorem* 4.1 *are satisfied, then $r_n(\hat{\theta}_n, k) \to 0$ in probability as $n \to \infty$ for all $k > 0$.*

PROOF. It follows immediately from Lemmas 3.4 and 3.5 that $r_n(\theta, k) \to 0$ in P_θ probability as $n \to \infty$ for every $\theta \in R$ and $k > 0$. Therefore, since

$$P_\theta(r_n(\hat{\theta}_n, k) \geqq \varepsilon) \leqq P_\theta(r_n(\theta, k + j) \geqq \varepsilon) + P_\theta(\alpha_n|\hat{\theta}_n - \theta| \geqq j)\,,$$

for every $\theta \in R$ and $k, j, \varepsilon > 0$, it follows from Theorems 2.1 and 2.2 that $r_n(\hat{\theta}_n, k) \to 0$ in P_θ probability for every $\theta \in R$ and $k > 0$. Finally, since

$$\Pr(r_n(\hat{\theta}_n, k) \geqq \varepsilon) = \int_{-\infty}^{\infty} P_\theta(r_n(\hat{\theta}_n, k) \geqq \varepsilon)q(\theta)\,d\theta\,,$$

for k, $\varepsilon > 0$, the lemma now follows from the bounded convergence theorem.

To prove Theorem 4.1, let J be any finite interval, let $k > 0$ be so large that $J \subset [-k, k]$, and define $\varepsilon = \varepsilon(k)$ by $\Phi([-k, k]) = 1 - \varepsilon$. Further, let $\delta = r_n(\hat{\theta}_n, k)$ and observe that $|W_n(t) + \frac{1}{2}t^2| \leqq \frac{1}{2}\delta t^2$ for $|t| \leqq k$, where $W_n(t) = G_n(\hat{\theta}_n + t\alpha_n^{-1}) - G_n(\hat{\theta}_n)$, $t \in R$, and $\log 0 = -\infty$. Finally, let

$$q_n^+ = \sup_{|t| \leqq k} q(\hat{\theta}_n + t\alpha_n^{-1}) , \qquad q_n^- = \inf_{|t| \leqq k} q(\hat{\theta}_n + t\alpha_n^{-1}) .$$

Then, it follows easily from the conditions imposed on q and the consistency of $\hat{\theta}_n$ that $q_n^-/q_n^+ \to 1$ in probability $n \to \infty$.

Let q_n denote the conditional density of $\alpha_n(\boldsymbol{\theta} - \hat{\theta})$ given $X_1, \cdots, X_n$. Then, for $|t| \leqq k$

$$(4.2) \qquad \begin{aligned} q_n(t) &= c_n^{-1} q(\hat{\theta}_n + t\alpha_n^{-1}) e^{W_n(t)} \\ &\leqq c_n^{-1} q_n^+ \exp(-\tfrac{1}{2}(1 - \delta)t^2) , \end{aligned}$$

where

$$\begin{aligned} c_n &= \textstyle\int_{-\infty}^{\infty} q(\hat{\theta}_n + s\alpha_n^{-1}) e^{W_n(s)}\, ds \\ &\geqq \textstyle\int_{-k}^{k} q_n^- \exp(-\tfrac{1}{2}(1 + \delta)s^2)\, ds = q_n^- d_n , \qquad \text{say} . \end{aligned}$$

Now, $d_n \to (1 - \varepsilon)(2\pi)^{\frac{1}{2}}$ in probability as $n \to \infty$ by Lemma 4.1. Therefore, as $n \to \infty$,

$$\begin{aligned} Q_n(J) &= \textstyle\int_J q_n(t)\, dt \\ &\leqq (q_n^+/q_n^- d_n) \textstyle\int_J \exp(-\tfrac{1}{2}(1 - \delta)t^2)\, dt \\ &\to \Phi(J)/(1 - \varepsilon) \end{aligned}$$

in probability. Since k may be made arbitrarily large, it now follows that $\lim \Pr(Q_n(J) \geqq \Phi(J) + \eta) = 0$ for every $\eta > 0$. Now suppose that

$$\limsup \Pr(Q_n(J) \leqq \Phi(J) - \eta_0) > 0$$

for some $\eta_0 > 0$. Then, by what has just been shown, we would have $\liminf E(Q_n(J)) < \Phi(J)$. The latter conclusion is impossible, however, in view of equation (4.1) and the bounded convergence theorem. Theorem 4.1 follows.

Asymptotic normality of posterior distributions may also be deduced if the Fisher Information σ^2 is finite. The result follows easily from the work of LeCam.

THEOREM 4.2. *Let f be absolutely continuous with derivative f' and let the Fisher Information σ^2 be finite. Suppose also that the support of q is an interval and that q is continuous on its support. Let $\hat{\theta}_n$ be any consistent sequence of* MLE*'s and let Q_n denote the conditional distribution of $n^{\frac{1}{2}}\sigma(\boldsymbol{\theta} - \hat{\theta}_n)$ given $X_1, \cdots, X_n$. Then, as $n \to \infty$, $\lim Q_n(J) = \Phi(J)$ in probability for every finite interval J.*

PROOF. Let $V_n(t) = G_n(\hat{\theta}_n + t/\sigma n^{\frac{1}{2}}) - G_n(\hat{\theta}_n)$, $t \in R$. Then, it follows easily from the proof of Proposition 6 of [7] that $\sup_{|t| \leqq k} |V_n(t) + \frac{1}{2}t^2| \to 0$ in probability for every $k > 0$. Using this fact, Theorem 4.2 may be proved by an argument which is virtually identical to that given in the proof of Theorem 4.1.

5. Concluding remarks. If $b < \infty$, then the roles of 0 and b may be reversed in Theorems 2.2 and 4.1. Simply let $Y_i = b - X_i$, $i = 1, \cdots, n$.

Under the hypotheses of either Theorems 2.1 or 2.2, $\alpha_n(\hat{\theta}_n - \theta) - \alpha_n^{-1}G_n'(\theta) = o_p(1)$ as $n \to \infty$ (cf. [10], page 285). To see this recall that $r_n(\hat{\theta}_n, k) \to 0$ in (P_θ) probability for every $k > 0$, observe that

$$G_n'(\theta) = -\int_\theta^{\hat{\theta}_n} G_n''(t)\, dt ,$$

and use the fact that $\hat{\theta}_n - \theta = O_p(\alpha_n^{-1})$.

The conditions imposed on q in Theorems 4.1 and 4.2 may be weakened. It is sufficient that there be an open set $U \subset R$ for which q is continuous and positive on U and $\Pr(\boldsymbol{\theta} \in U) = 1$.

Also, a stronger conclusion is possible in Theorems 4.1 and 4.2 than was given. For example, in Theorem 4.1, we may conclude that

$$\sup_{|t| \leq k} |q_n(t) - e^{-\frac{1}{2}t^2}/(2\pi)^{\frac{1}{2}}| \to 0$$

in probability for any $k < 0$. To see this from (4.2) and Lemma 4.1, it will suffice to show that $c_n \to (2\pi)^{\frac{1}{2}}$ in probability. If this were not the case, then, in view of Lemma 4.1, we could not have $Q_n(J) \to \Phi(J)$ in probability for any non-degenerate interval J.

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