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OPTIMAL PORTFOLIO IN PARTIALLY OBSERVED STOCHASTIC VOLATILITY MODELS

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We address the maximization problem of expected utility from terminal wealth. The special feature of this paper is that we consider a financial market where price process of risky assets follows a stochastic volatility model and we require that investors observe just the vector of stock prices. Using stochastic filtering techniques and adapting martingale duality methods in this partially observed incomplete model, we characterize the value function and the optimal portfolio policies. We study in detail the Bayesian case, when risk premia of the stochastic volatility model are unobservable random variables with known prior distribution. We also consider the case of unobservable risk premia modelled by linear Gaussian processes.

1. Introduction. We consider an incomplete financial model with one bond and n risky assets. The price process S of the risky securities follows a stochastic volatility model, where the volatility is influenced by some latent process Y. In such a context, we solve the portfolio optimization problem when investors want to maximize the expected utility from terminal wealth, assuming that they can observe only the stock prices. This situation is referred as partial information in contrast to the case of full information.

The utility maximization problem with full information has been studied extensively in the literature. Originally introduced by Merton (1971) in the context of constant coefficients and treated by Markovian methods via the Bellman equation of dynamic programming, it was developed for general processes by the martingale duality approach. For the case of complete markets, we refer to Karatzas, Lehoczky and Shreve (1987), Cox and Huang (1989). For the case of incomplete and/or constrained markets, we refer to Karatzas, Lehoczky, Shreve and Xu (1991), He and Pearson (1991) and Cvitanić and Karatzas (1992). Models with partial observation are essentially studied in the literature in a complete market framework. Detemple (1986), Dothan and Feldman (1986), Gennotte (1986) used dynamic programming methods in a linear Gaussian filtering. Lakner (1995, 1998) solved the optimization problem via a martingale approach and worked out the special case of the linear Gaussian model. Also in the setting of complete model and using duality methods, Karatzas and Xue (1991) and Karatzas and Zhao (1998) focus on the

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Bayesian case. We mention that Frey and Runggaldier (1999) and Lasry and Lions (1999) studied hedging problems in finance under restricted information.

In this paper, we combine stochastic filtering techniques and a martingale duality approach to characterize the value function and the optimal portfolio of the utility maximization problem. The paper is organized as follows. In Section 2, we describe the model and formulate the optimization problem. In Section 3, we use results from filtering theory to reduce the optimization problem with partial information to the case of a model where all coefficients are adapted to the observation process. We use in Section 4 the martingale duality approach for the utility maximization problem. A key point is to obtain a representation formula for the minimal hedging cost of a claim in terms of suitable equivalent martingale measures adapted to stock price filtration. We work out the specific examples of logarithmic and power utility functions. In Section 5, we obtain explicit formulas for the optimal portfolio, optimal wealth process and value function of the utility maximization problem in a Bayesian setting, that is, when the risk premia of the stochastic volatility model are unobservable independent random variables with known probability distribution. In Section 6, explicit formulas are also obtained when the risk premia are Gaussian processes modelled by a system of linear stochastic differential equations.

2. The model. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space equipped with a filtration $\mathbb{F} = \{\mathscr{F}_t, 0 \le t \le T\}$, where $T > 0$ is a fixed time horizon. We assume that $\mathscr{F}_T = \mathscr{F}$. We consider a financial market which consists of one risk-free asset, whose price process is assumed for simplicity to be equal to 1 at each date, and n risky assets with n-dimensional price process $S = (S^1, \dots, S^n)'$ (sign $'$ denotes the transposition) whose dynamics is governed by

$$dS_t = \mu_t dt + \sigma(t, S_t, Y_t) dW_t, \tag{2.1}$$

$$dY_t = \eta_t dt + \rho(t, S_t, Y_t) dW_t + \gamma(t, S_t, Y_t) dB_t. \tag{2.2}$$

Here W and B are independent Brownian motion under P with respect to $\mathbb{F}$, valued, respectively, in $\mathbb{R}^n$ and $\mathbb{R}^d$. The initial prices S_0^i and the initial values Y_0^i are deterministic constants. The drift $\mu = \{\mu_t, 0 \le t \le T\}$ (resp. $\eta = \{\eta_t, 0 \le t \le T\}$) is an $\mathbb{R}^n$-valued (resp. $\mathbb{R}^d$-valued) adapted process. The known functions $\sigma(t, s, y)$, $\rho(t, s, y)$ and $\gamma(t, s, y)$ are measurable mappings from $[0, T] \times \mathbb{R}^n \times \mathbb{R}^d$ into $R^{n\times n}$, $\mathbb{R}^{d\times n}$ and $\mathbb{R}^{d\times d}$. We shall make the following standing assumption.

ASSUMPTION 2.1. (i) The functions $\sigma(t, ., .), \rho(t, ., .)$ and $\gamma(t, ., .)$ are Lipschitz in $(s, y) \in \mathbb{R}^n \times \mathbb{R}^{d-n}$, uniformly in $t \in [0, T]$.

(ii) For all (t, s, y), the $n \times n$ and $d \times d$ matrices $\sigma(t, s, y), \gamma(t, s, y)$ are nonsingular.

(iii) The function $\sigma\sigma'$ is continuous in $(t, s, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^d$ and for all (t, s), the function $\sigma\sigma'(t, s, \cdot)$ is one-to-one from $\mathbb{R}^d$ into a subset Σ of the set of $n \times n$ positive definite matrices, and its inverse function, denoted $\varsigma(t, s, \cdot)$, is continuous with respect to $(t, s, z) \in [0, T] \times \mathbb{R}^n \times \Sigma$.

We now consider agents in this market who can observe neither the Brownian motions W and B nor the drift μ and η, but only the stock price process S. We shall denote by $\mathbb{G} = \{\mathscr{G}_t,\ 0 \le t \le T\}$ the P-augmentation of the filtration generated by the price process S.

A portfolio is an $\mathbb{R}^n$-valued $\mathbb{G}$-adapted process $\theta = \{\theta_t = (\theta_t^1, \dots, \theta_t^n)',\ 0 \le t \le T\}$ such that

$$\int_0^T \left|\sigma(t, S_t, Y_t)'\theta_t\right|^2 dt < \infty \quad \text{a.s.}$$

Given an initial wealth $x \ge 0$, the wealth process corresponding to a portfolio θ is defined by $X_0^{x,\theta} = x$ and

$$\begin{aligned} dX_t^{x,\theta} &= \theta_t' dS_t \\ &= \theta_t'\mu_t dt + \theta_t'\sigma(t, S_t, Y_t)\, dW_t. \end{aligned} \tag{2.3}$$

We regard θ_t^i as the number of shares invested in the ith stock at time t. Given $x \ge 0$, we denote by $\mathscr{A}(x)$ the set of portfolios θ such that

$$X_t^{x,\theta} \ge 0 \quad \text{a.s., } 0 \le t \le T. \tag{2.4}$$

A function U: $(0, \infty) \to \mathbb{R}$ will be called utility function if it is strictly increasing, strictly concave, of class C^1, and satisfies

$$U'(0^+) := \lim_{x \downarrow 0} U'(x) = \infty, \qquad U'(\infty) := \lim_{x \to \infty} U'(x) = 0.$$

The optimization problem is to maximize the expected utility from terminal wealth over the class $\mathscr{A}(x)$ of admissible portfolios, provided that the expectation is well defined. More precisely, the value function of this problem is defined by

$$V(x) = \sup_{\theta \in \mathscr{A}_0(x)} E\left[U(X_T^{x,\theta})\right], \qquad x > 0, \tag{2.5}$$

where $\mathscr{A}_0(x)$ is the class of processes $\theta \in \mathscr{A}(x)$ that satisfy

$$E\left[U^-(X_T^{x,\theta})\right] < \infty.$$

3. Filtering. Let us define the processes

$$\begin{aligned} \lambda_t &:= \sigma(t, S_t, Y_t)^{-1}\mu_t, \\ \alpha_t &:= \gamma(t, S_t, Y_t)^{-1}(\eta_t - \rho(t, S_t, Y_t)\lambda_t), \end{aligned}$$

assumed to satisfy the integrability condition

$$\int_0^T |\lambda_t|^2 + |\alpha_t|^2 dt < \infty \quad \text{a.s.} \tag{3.1}$$

Consider the positive local martingale defined by $L_0 = 1$ and $dL_t = -L_t(\lambda_t' dW_t + \alpha_t' dB_t)$. It is explicitly given by

$$L_t = \exp\left(-\int_0^t \lambda_u'\, dW_u - \int_0^t \alpha_u'\, dB_u - \tfrac{1}{2}\int_0^t |\lambda_u|^2 + |\alpha_u|^2\, du \right). \tag{3.2}$$

We shall make the usual standing assumption on filtering theory.

ASSUMPTION 3.1. The process L is a martingale; that is, $E[L_T] = 1$.

Under this last assumption, one can define a probability measure equivalent to P on $(\Omega, \mathscr{F})$ characterized by

$$\frac{d\widetilde{P}}{dP}\bigg|_{\mathscr{F}_t} = L_t, \qquad 0 \le t \le T. \tag{3.3}$$

By Girsanov's theorem, the n-dimensional process

$$\widetilde{W}_t = W_t + \int_0^t \lambda_u \, du \tag{3.4}$$

and the d-dimensional process

$$\widetilde{B}_t = B_t + \int_0^t \alpha_u \, du \tag{3.5}$$

are independent $(\widetilde{P}, \mathbb{F})$-Brownian motion. The dynamics of (S, Y) under $\widetilde{P}$ is given by

$$dS_t = \sigma(t, S_t, Y_t) d\widetilde{W}_t, \tag{3.6}$$

$$dY_t = \rho(t, S_t, Y_t) d\widetilde{W}_t + \gamma(t, S_t, Y_t) d\widetilde{B}_t. \tag{3.7}$$

The following result is essential in solving our optimization problem.

LEMMA 3.1. *Under Assumptions* 2.1 *and* 3.1, *the filtration* $\mathbb{G}$ *is the augmented filtration of* $(\widetilde{W}, \widetilde{B})$.

PROOF. Let $\mathbb{F}^{S,Y}$ be the augmented filtration of (S, Y). Obviously, $\mathbb{G} \subset \mathbb{F}^{S,Y}$. By (3.6), the quadratic variation process of S (which is equal to the sharp bracket process of S since S is continuous) is given by

$$\langle S, S' \rangle_t = \int_0^t \sigma\sigma'(u, S_u, Y_u) du, \qquad 0 \le t \le T. \tag{3.8}$$

From the continuity of the function $\sigma\sigma'$ and of the processes S and Y, it follows that the process $\{\sigma\sigma'(t, S_t, Y_t),\ 0 \le t \le T\}$ is $\mathbb{G}$-adapted. Moreover, by Assumption 2.1(iii), the process Y satisfies

$$Y_t = \varsigma(t, S_t, \sigma\sigma'(t, S_t, Y_t)), \qquad 0 \le t \le T,$$

which implies that it is also $\mathbb{G}$-adapted. Therefore, we get $\mathbb{G} = \mathbb{F}^{S,Y}$.

Let $\widetilde{\mathbb{F}}$ be the augmented filtration of $(\widetilde{W}, \widetilde{B})$. From (3.6) and (3.7), we have

$$\widetilde{W}_t = \int_0^t \sigma^{-1}(u, S_u, Y_u) \, dS_u, \tag{3.9}$$

$$\widetilde{B}_t = \int_0^t \gamma^{-1}(t, S_t, Y_t)\big(dY_u - \rho(u, S_u, Y_u)\sigma^{-1}(u, S_u, Y_u) \, dS_u\big), \tag{3.10}$$

for all $t \in [0, T]$, which implies that $\widetilde{\mathbb{F}} \subset \mathbb{F}^{S,Y} = \mathbb{G}$. Conversely, under Assumption 2.1 (i), by Protter [(1990), Theorem V.3.7], the unique solution (S, Y) of the system of s.d.e. (3.6) and (3.7) is $\widetilde{\mathbb{F}}$-adapted, hence $\mathbb{G} = \mathbb{F}^{S,Y} \subset \widetilde{\mathbb{F}}$ and finally $\mathbb{G} = \widetilde{\mathbb{F}}$. $\square$

REMARK 3.1. Arguments in the proof of this last lemma show that $\mathbb{G}$ is actually equal to the augmented filtration of (S, Y) and correspondence relations between $(\widetilde{W}, \widetilde{B})$ and (S, Y) are obtained through (3.6) and (3.7) and (3.9) and (3.10).

We now make the standing assumption on the risk premia processes (λ, α) of the stochastic volatility model.

ASSUMPTION 3.2. For all $t \in [0, T]$, $E|\lambda_t| + E|\alpha_t| < \infty$.

Let us then introduce measurable versions of the conditional processes (λ, α),

$$\tilde{\lambda}_t := E[\lambda_t|\mathscr{G}_t],$$
$$\tilde{\alpha}_t := E[\alpha_t|\mathscr{G}_t].$$

Consider the processes (N, M) defined by

$$N_t := \widetilde{W}_t - \int_0^t \tilde{\lambda}_u \, du,$$
$$M_t := \widetilde{B}_t - \int_0^t \tilde{\alpha}_u \, du.$$

These are the so-called innovation processes of filtering theory. By classical results in filtering theory [see, e.g., Pardoux (1989), Proposition 2.27], we have the following.

LEMMA 3.2. *Under Assumptions* 2.1, 3.1 *and* 3.2, *the processes N and M are independent $(P, \mathbb{G})$-Brownian motions.*

PROOF. By Lemma 3.1, N and M are $\mathbb{G}$-adapted. Moreover, we have $\langle N^i, N^j\rangle = \langle M^i, M^j\rangle = \delta_{ij}t$, and $\langle N^i, M^j\rangle = 0$, where δ_{ij} is the Kronecker notation. By the law of iterated conditional expectation, it is easy to check that N and M are $\mathbb{G}$-martingales. We then conclude by Lévy's characterization theorem on Brownian motions [see, e.g., Karatzas and Shreve (1991), Theorem 3.3.16]. $\square$

REMARK 3.2. Notice that (3.4) and (3.5) correspond to the decomposition of the $(P, \mathbb{F})$ special semimartingale $(\widetilde{W}, \widetilde{B})$ where (λ, α) satisfies the integrability condition (3.1). By Lemma 3.2, the unique decomposition of the $(P, \mathbb{G})$

special semimartingale $(\widetilde{W}, \widetilde{B})$ is given by

$$\widetilde{W}_t = N_t + \int_0^t \tilde{\lambda}_u \, du,$$

$$\widetilde{B}_t = M_t + \int_0^t \tilde{\alpha}_u \, du,$$

where $(\tilde{\lambda}, \tilde{\alpha})$ satisfies the integrability condition

$$\int_0^T |\tilde{\lambda}_t|^2 + |\tilde{\alpha}_t|^2 dt < \infty \quad \text{a.s.} \tag{3.11}$$

[see Stricker (1983)].

Denote by Λ the $(\widetilde{P}, \mathbb{F})$-martingale given by $\Lambda = 1/L$. We then have

$$\left.\frac{dP}{d\widetilde{P}}\right|_{\mathscr{F}_t} = \Lambda_t, \qquad 0 \le t \le T. \tag{3.12}$$

Computations of $\tilde{\lambda}$ and $\tilde{\alpha}$ are obtained by the so-called Kallianpur–Striebel formula, which is related to Bayes formula: for all $t \in [0, T]$ and $\zeta \in L^1(\Omega, \mathscr{F}_t, P)$, one has

$$E[\zeta | \mathscr{G}_t] = \frac{E^{\widetilde{P}}[\zeta \Lambda_t | \mathscr{G}_t]}{\tilde{\Lambda}_t}, \tag{3.13}$$

where $\widetilde{\Lambda}$ is the $(\widetilde{P}, \mathbb{G})$-martingale, given by

$$\widetilde{\Lambda}_t := E^{\widetilde{P}}[\Lambda_t | \mathscr{G}_t]. \tag{3.14}$$

Note that $\widetilde{\Lambda}$ is a continuous process since $\mathbb{G}$ is the augmented filtration generated by the $(\widetilde{P}, \mathscr{F})$- Brownian motion $(\widetilde{W}, \widetilde{B})$.

Let ξ be the optional projection of the P-martingale L to $\mathbb{G}$, so

$$\xi_t := E[L_t | \mathscr{G}_t].$$

By applying relation (3.13) to $\zeta = L_t$, we immediately obtain

$$\xi_t = 1/\widetilde{\Lambda}_t.$$

We have the following result for the representation of $\widetilde{\Lambda}$ and ξ.

PROPOSITION 3.1. *Under Assumptions* 2.1, 3.1 *and* 3.2, *we have*

$$\xi_t = \exp\left(-\int_0^t \tilde{\lambda}'_u \, dN_u - \int_0^t \tilde{\alpha}'_u dM_u - \tfrac{1}{2}\int_0^t |\tilde{\lambda}_u|^2 + |\tilde{\alpha}_u|^2 \, du\right), \tag{3.15}$$

$$\widetilde{\Lambda}_t = \exp\left(\int_0^t \tilde{\lambda}'_u \, d\widetilde{W}_u + \int_0^t \tilde{\alpha}'_u d\widetilde{B}_u - \tfrac{1}{2}\int_0^t |\tilde{\lambda}_u|^2 + |\tilde{\alpha}_u|^2 \, du\right). \tag{3.16}$$

PROOF. The arguments are very similar to those used by Lakner (1998) in the proof of his Theorem 3.1. By definition of Λ, we have

$$\Lambda_t = 1 + \int_0^t \Lambda_u \lambda'_u \, d\widetilde{W}_u + \int_0^t \Lambda_u \alpha'_u \, d\widetilde{B}_u. \tag{3.17}$$

Now, one can check that

$$E^{\widetilde{P}}[\Lambda_t|\lambda_t|] + E^{\widetilde{P}}[\Lambda_t|\alpha_t|] < \infty, \qquad 0 \le t \le T, \tag{3.18}$$

$$\int_0^T (E^{\widetilde{P}}[\Lambda_t\lambda_t|\mathscr{G}_t])^2 + (E^{\widetilde{P}}[\Lambda_t\alpha_t|\mathscr{G}_t])^2 dt < \infty \qquad \text{a.s.} \tag{3.19}$$

Indeed, (3.18) follows from Assumption 3.2 since

$$E^{\widetilde{P}}[\Lambda_t|\lambda_t|] + E^{\widetilde{P}}[\Lambda_t|\alpha_t|] = E[|\lambda_t|] + E[|\alpha_t|] < \infty, \qquad 0 \le t \le T.$$

Moreover, (3.13) applied to λ and α gives the following relation:

$$\begin{aligned}&\int_0^T (E^{\widetilde{P}}[\Lambda_t\lambda_t|\mathscr{G}_t])^2 + (E^{\widetilde{P}}[\Lambda_t\alpha_t|\mathscr{G}_t])^2 \, dt \\ &\qquad = \int_0^T \widetilde{\Lambda}_t^2(\tilde{\lambda}_t^2 + \tilde{\alpha}_t^2) \, dt < \infty \quad \text{a.s.}\end{aligned}$$

Note that the finiteness of this integral follows from the continuity of $\tilde{\Lambda}_t$, and from (3.11). Therefore, by Theorem 5.14 in Liptser and Shiryaev (1977), we have, by taking conditional expectation with respect to $\mathscr{G}_t$ in (3.17),

$$\begin{aligned}\widetilde{\Lambda}_t &= 1 + \int_0^t E^{\widetilde{P}}[\Lambda_u\lambda_u|\mathscr{G}_u]' \, d\widetilde{W}_u + \int_0^t E^{\widetilde{P}}[\Lambda_u\alpha_u|\mathscr{G}_u]' \, d\widetilde{B}_u \\ &= 1 + \int_0^t \widetilde{\Lambda}_u \tilde{\lambda}'_u d\widetilde{W}_u + \int_0^t \widetilde{\Lambda}_u \tilde{\alpha}'_u d\widetilde{B}_u,\end{aligned} \tag{3.20}$$

where the last relation follows from (3.13) applied to λ and α. Relation (3.20) shows that $\widetilde{\Lambda}$ is expressed as in (3.16). Finally, relation (3.15) follows from (3.16) and definition of N and M. □

By means of innovation processes, we can describe the dynamics of the partially observed stochastic volatility model within a framework of a complete observation model,

$$dS_t = \tilde{\mu}_t dt + \sigma(t, S_t, Y_t) dN_t, \tag{3.21}$$

$$dY_t = \tilde{\eta}_t dt + \rho(t, S_t, Y_t) dN_t + \gamma(t, S_t, Y_t) dM_t, \tag{3.22}$$

where $\tilde{\mu}$ and $\tilde{\eta}$ are $\mathbb{G}$-adapted processes defined by

$$\begin{aligned}\tilde{\mu}_t &= \sigma(t, S_t, Y_t)\tilde{\lambda}_t, \\ \tilde{\eta}_t &= \rho(t, S_t, Y_t)\tilde{\lambda}_t + \gamma(t, S_t, Y_t)\tilde{\alpha}_t.\end{aligned}$$

Hence, the operations of filtering and control can be put in sequence and thus separated.

REMARK 3.3. Notice that in general, $\mathbb{G}$ is strictly larger than the augmented filtration generated by the $(P, \mathbb{G})$-Brownian motions N and M. We shall even see later that formal substitution of $\tilde{\mu}$ for μ and $\tilde{\eta}$ for η in the formula of optimal portfolio in the full information case does not always yield the correct formula for the optimal portfolio in the partial information case (see in Sections 5 and 6 the examples of power utility functions).

4. Martingale dual approach under partial observation. The aim of this section is to give a dual formulation of the optimization problem (2.5) in terms of a suitable family of $(P, \mathbb{G})$-local martingales. A key lemma is to state a martingale representation theorem for $(P, \mathbb{G})$-local martingales with respect to N and M. Notice that it cannot be directly derived from usual martingale representation theorem since $\mathbb{G}$ is not equal to the filtration generated by N and M.

Assumptions 2.1, 3.1 and 3.2 stand in the rest of this section.

LEMMA 4.1. *Let m be any $(P, \mathbb{G})$-local martingale with $m_0 = 0$. Then, there exist an $\mathbb{R}^n$-valued process ϕ and an $\mathbb{R}^d$-valued process ψ which are $\mathbb{G}$-adapted processes, P-a.s. square-integrable and such that*

$$m_t = \int_0^t \phi'_u \, dN_u + \int_0^t \psi'_u \, dM_u, \qquad 0 \le t \le T.$$

PROOF. Let m be a $(P, \mathbb{G})$-local martingale. From Bayes' rule, the process $\tilde{m}$ given by

$$\tilde{m}_t = m_t \xi_t^{-1}, \qquad 0 \le t \le T$$

is a $(\widetilde{P}, \mathbb{G})$-local martingale (recall that $\xi_t = d\widetilde{P}/dP|_{\mathscr{G}_t}$). From Lemma 3.1 and the martingale representation theorem, there exist an $\mathbb{R}^n$-valued process $\tilde{\phi}$ and an $\mathbb{R}^d$-valued process $\tilde{\psi}$ which are $\mathbb{G}$-adapted processes P-a.s. square-integrable such that

$$\tilde{m}_t = \int_0^t \tilde{\phi}'_u \, d\widetilde{W}_u + \int_0^t \tilde{\psi}'_u \, d\tilde{B}_u, \qquad 0 \le t \le T.$$

By Itô's formula applied to $m_t = \tilde{m}_t \xi_t$, (3.15) and the definition of N and M, we obtain that

$$m_t = \int_0^t \phi'_u \, dN_u + \int_0^t \psi'_u \, dM_u, \qquad 0 \le t \le T,$$

with $\phi_t = \xi_t(\tilde{\phi}_t - \tilde{m}_t \tilde{\lambda}_t)$ and $\psi_t = \xi_t(\tilde{\psi}_t - \tilde{m}_t \tilde{\alpha}_t)$. $\square$

REMARK 4.1. The proof of this last lemma is quite similar to that of Proposition 5.8.6 in Karatzas and Shreve (1991) which states that in a complete market with $\mathbb{F}$ equal to the filtration generated by W, each $(\widetilde{P}, \mathbb{F})$-local martingale can be represented as a stochastic integral with respect to $\widetilde{W}$.

As is standard now in financial mathematics, the martingale approach to the utility maximization problem requires a dual representation formula for the superhedging price of any contingent claim, whose definition is recalled now.

DEFINITION 4.1. Let H be a contingent claim, that is, a nonnegative, $\mathscr{G}_T$-measurable random variable. The superhedging price of H is defined by

$$u_0 = \inf\left\{x \geq 0\colon \exists \theta \in \mathscr{A}(x),\ X_T^{x,\theta} \geq H \text{ a.s.}\right\},$$

with the convention that $\inf \varnothing = \infty$.

For any $\mathbb{G}$-adapted, $\mathbb{R}^d$-valued process $\nu = \{\nu_t, 0 \leq t \leq T\}$, which satisfies $\int_0^T |\nu_t|^2 dt < \infty$, we introduce the $(P, \mathbb{G})$-local martingale

$$Z_t^\nu = \exp\biggl(-\int_0^t \tilde{\lambda}'_u dN_u - \int_0^t \nu'_u dM_u - \tfrac{1}{2}\textstyle\int_0^t |\tilde{\lambda}_u|^2 + |\nu_u|^2 du\biggr).$$

REMARK 4.2. By Lemma 4.1, it is easily checked that the family of local martingales Z^ν correspond to the so called *equivalent local martingale measures*, defined as $(P, \mathbb{G})$-local martingales strictly positive Z with $Z_0 = 1$ such that the process ZS is a $(P, \mathbb{G})$-local martingale.

In what follows, we denote by $\mathscr{H}$ the Hilbert space of $\mathbb{G}$-adapted, $\mathbb{R}^d$-valued processes ν such that $E[\int_0^T |\nu_t|^2\, dt] < \infty$. We now show that the dual formulation of the superhedging price stated by El Karoui and Quenez (1995) in the case of complete information still holds in the case of partial information.

THEOREM 4.1. *Let H be a contingent claim. Then,*

$$u_0 = J_0 := \sup_{\nu \in \mathscr{H}} E\left[Z_T^\nu H\right] \tag{4.1}$$

and when $J_0 < \infty$, there exists $\theta^* \in \mathscr{A}(J_0)$ such that $X_T^{J_0, \theta^*} \geq H$. Moreover, in this case, for any $\nu^* \in \mathscr{H}$, the following conditions are equivalent:

(i) ν^* *achieves the supremum in* (4.1).

(ii) *H is attainable: there exists $\theta \in \mathscr{A}(J_0)$ s.t. $X_T^{J_0,\theta} = H$, and the process $Z^{\nu^*} X^{J_0,\theta}$ is a $(P, \mathbb{G})$-martingale.*

For the proof, see Appendix A.

REMARK 4.3. Notice that for a given contingent claim H, the superhedging price $J_t^{\mathbb{F}}$ of H, at time t, corresponding to the complete information case is different from $J_t^{\mathbb{G}}$, the superhedging price of H, at time t, corresponding to the partial information case. Furthermore, if $\mathscr{F}_0 = \mathscr{G}_0$, then $J_0^{\mathbb{F}} = J_0^{\mathbb{G}}$, but the associated portfolios $(\theta^{\mathbb{F}})^*$ and $(\theta^{\mathbb{G}})^*$ do not coincide.

We shall denote by I the (continuous, strictly decreasing) inverse of the function U'; this function maps $(0,\infty)$ onto itself, and satisfies $I(0^+)=\infty$, $I(\infty)=0$. Let $\widetilde{U}$ be the polar function of U,

$$\widetilde{U}(y)=\max_{x>0}\bigl[U(x)-xy\bigr]=U(I(y))-yI(y),\qquad 0<y<\infty. \tag{4.2}$$

As in Karatzas, Lehoczky, Shreve and Xu (1991) [see also Cvitanić (1997)], we make the following assumptions on the utility function.

ASSUMPTION 4.1. (i) $c\mapsto cU'(c)$ is nondecreasing on $(0,\infty)$

(ii) There exist $\alpha\in(0,1)$ and $\gamma\in(1,\infty)$, such that $\alpha U'(x)\ge U'(\gamma x)$, $\forall\, x\in(0,\infty)$.

(iii) For all $y\in(0,\infty)$, there exists $\nu\in\mathscr{H}$ such that $E[\widetilde{U}(yZ_T^\nu)]<\infty$.

Let us now introduce the dual problem of (2.5),

$$\widetilde{V}(z)=\inf_{\nu\in\mathscr{H}}E\bigl[\widetilde{U}(zZ_T^\nu)\bigr],\qquad z>0. \tag{4.3}$$

REMARK 4.4. Denote by $\mathscr{D}$ the subset of $\mathscr{H}$ consisting of all bounded processes. It is easily checked that

$$\widetilde{V}(z)=\inf_{\nu\in\mathscr{D}}E\bigl[\widetilde{U}(zZ_T^\nu)\bigr],\qquad z>0.$$

By same arguments as in Theorem 12.1 in Karatzas, Lehoczky, Shreve and Xu (1991), we have existence to the dual problem (4.3).

PROPOSITION 4.1. *Under Assumption* 4.1, *for all* $z>0$, *the dual problem* (4.3) *admits a solution,* $\nu^*(z)\in\mathscr{H}$.

The primal utility maximization problem (2.5) is then solved as follows.

THEOREM 4.2. *Assume that Assumption* 4.1 *holds. Then, for all* $x>0$, *there exists an optimal portfolio* θ^* *for the utility maximization problem* (2.5) *and the associated optimal wealth process* X^* *is given by*

$$X_t^*=E\biggl[\frac{Z_T^{\nu^*}}{Z_t^{\nu^*}}I(z_xZ_T^{\nu^*})\bigg|\mathscr{G}_t\biggr],$$

where $\nu^*=\nu^*(z_x)$ *and* $z_x>0$ *is such that* $E\bigl[Z_T^{\nu^*(z_x)}I\bigl(z_xZ_T^{\nu^*(z_x)}\bigr)\bigr]=x$.

SKETCH OF THE PROOF. The proof is similar to that in Theorem 11.6 of Karatzas, Lehoczky, Shreve and Xu (1991); see also Cvitanić (1997). For sake of completeness, we briefly recall the main ideas. By Proposition 4.1, the dual problem (4.3) admits a solution $\nu^*(z)$ in the set $\mathscr{H}$, for all $z>0$. The maximum principle applied to the dual problem [see Karatzas, Lehoczky, Shreve and Xu (1991) or Cvitanic (1997), Lemma 11.12, for a simpler proof] provides

$$E\Bigl[Z_T^\nu I\Bigl(zZ_T^{\nu^*(z)}\Bigr)\Bigr]\le E\Bigl[Z_T^{\nu^*(z)}I\Bigl(zZ_T^{\nu^*(z)}\Bigr)\Bigr]:=x_z\qquad\forall\,\nu\in\mathscr{H}.$$

By Theorem 4.1, this implies that the contingent claim given by $I(zZ_T^{\nu^*(z)})$ is attainable; that is, there exists $\theta_z^* \in \mathscr{A}(x_z)$ s.t. $X_T^{x_z,\theta_z^*} = I(zZ_T^{\nu^*(z)})$, and $Z^{\nu^*(z)}X^{x_z,\theta_z^*}$ is a $(P,\mathbb{G})$-martingale. Hence,

$$X_t^{x_z,\theta_z^*} = E\biggl[\frac{Z_T^{\nu^*(z)}}{Z_t^{\nu^*(z)}} I(zZ_T^{\nu^*(z)})\bigg|\mathscr{G}_t\biggr].$$

Let us now show that θ_z^* is optimal for the primal problem (2.5) associated with initial wealth x_z. By (4.2), we have $U(I(y)) - U(x) \geq y[I(y) - x]$, for all $x > 0$, $y > 0$. It follows that for each $\theta \in \mathscr{A}_0(x_z)$,

$$\begin{aligned} &E\Bigl[U\Bigl(I\Bigl(zZ_T^{\nu^*(z)}\Bigr)\Bigr)\Bigr] - E\Bigl[U\Bigl(X_T^{x_z,\theta}\Bigr)\Bigr] \\ &\quad \geq z\Bigl[E\Bigl[Z_T^{\nu^*(z)} I\Bigl(zZ_T^{\nu^*(z)}\Bigr)\Bigr] - E\Bigl[Z_T^{\nu^*(z)} X_T^{x_z,\theta}\Bigr]\Bigr]. \end{aligned}$$

By the supermartingale property of $Z^{\nu^*(z)}X^{x_z,\theta}$, the second member of the previous inequality is nonnegative. It follows that $I(zZ_T^{\nu^*(z)}) = X_T^{x_z,\theta_z^*}$ corresponds to the optimal terminal wealth and hence that θ_z^* is optimal.

The last step is to choose $z = z_x$ such that the constraint $x_z = x$ is satisfied; that is,

$$E\Bigl[Z_T^{\nu^*(z_x)} I\Bigl(z_x Z_T^{\nu^*(z_x)}\Bigr)\Bigr] = x.$$

This is done by choosing $z_x \in \operatorname{argmin}_{z>0}\{\widetilde{V}(z) + xz\}$ or $z_x = V'(x)$ [see Cvitanić and Karatzas (1992) or Kramkov and Schachermayer (1999)]. □

4.1. *Application* 1. *Logarithmic utility function* $U(x) = \ln(x)$. We have $I(y) = 1/y$ and $\widetilde{U}(y) = -(1 + \ln(y))$. For all $z > 0$, the dual problem (4.3) admits the solution $\nu^*(z) = 0$ and the Lagrange multiplier is $z_x = 1/x$. The optimal wealth is given by

$$X_t^* = E\biggl[\frac{Z_T^0}{Z_t^0}\frac{1}{z_x Z_T^0}\bigg|\mathscr{G}_t\biggr] = \frac{x}{Z_t^0}, \tag{4.4}$$

and the optimal portfolio is in the feedback form

$$\theta_t^* = \bigl(\sigma(t, S_t, Y_t)^{-1}\bigr)'\tilde{\lambda}_t X_t^*. \tag{4.5}$$

Recall that in the complete information case, the optimal portfolio is given by the feedback form

$$(\theta_t^c)^* = \bigl(\sigma(t, S_t, Y_t)^{-1}\bigr)'\lambda_t (X_t^c)^*, \tag{4.6}$$

where $(X^c)^*$ denotes the optimal wealth in the complete case. Therefore, in the case of partial information, the optimal portfolio can be formally derived from the full information case by replacing the unobservable risk-premium λ_t by its best estimate $\tilde{\lambda}_t$. This property corresponds to the so-called separation principle. It is actually proved in Kuwana (1995) that certainty equivalence holds if and only if the utilities functions are logarithmic.

4.2. *Application* 2. *Power utility function* $U(x) = x^p/p$ *with* $0 < p < 1$. We have $I(y) = y^{1/(p-1)}$ and $\widetilde{U}(z) = z^{-r}/r$, with $r = p/(1-p)$. By Theorem 4.2, the optimal wealth is strictly positive; that is,

$$P(X_t^* > 0 \qquad \forall\, t \in [0, T]) = 1.$$

Therefore, the value function $V(x)$ coincides with the utility maximization problem with a strictly positive constraint on the wealth. In other words, we can make the following change of variable: $\pi_t = \theta_t / X_t$, and $\pi_t^i S_t^i$ is interpreted as the proportion of wealth invested in the ith stock at time t. We define then a proportion portfolio as an $\mathbb{R}^n$-valued $\mathbb{G}$-adapted process π with $\int_0^T \left|\sigma(t, S_t, Y_t)'\pi_t\right|^2 dt < \infty$ a.s. and we denote by $\mathscr{A}$ the set of all proportion portfolios. Given an initial wealth $x \geq 0$, the wealth process corresponding to a proportion portfolio π is defined by $X_0^{x,\pi} = x$ and $dX_t^{x,\pi} = X_t^{x,\pi}\pi_t' dS_t$ so that

$$X_t^{x,\pi} = x X_t^{\pi},$$

where

$$X_t^{\pi} = \exp\left(\int_0^t \pi_u'\, dS_u - \frac{1}{2}\int_0^t |\pi_u'\sigma(u, S_u, Y_u)|^2\, du\right).$$

The utility maximization problem (2.5) can be written equivalently in terms of proportion portfolios,

$$V(x) = \sup_{\pi \in \mathscr{A}} E\big[U\big(X_T^{x,\pi}\big)\big], \qquad x > 0. \tag{4.7}$$

We now show that in the case of power utility functions, an explicit formula for the optimal proportion portfolio can be obtained directly from the primal problem (4.7) without using the duality relation of Theorem 4.2. Let us consider the RCLL process $\{Q_t,\ 0 \leq t \leq T\}$ (which exists) such that

$$Q_t = \operatorname*{ess\,sup}_{\pi \in \mathscr{A}} E\left[\left(\frac{X_T^{\pi}}{X_t^{\pi}}\right)^p \middle| \mathscr{G}_t\right], \qquad 0 \leq t \leq T,\ P\text{-a.s.}$$

Notice that $V(x) = (x^p/p)Q_0$, $x > 0$ and so the solution π^* of the problem (4.7) (which exists by Theorem 4.2) does not depend on $x > 0$. We now state a decomposition of the semimartingale Q, from which we derive a characterization of π^*.

PROPOSITION 4.2. *The process Q admits the following decomposition:*

$$dQ_t = -\frac{p}{2(1-p)} Q_t \left|\tilde{\lambda}_t + Q_t^{-1}\phi_t\right|^2 dt + \phi_t' dN_t + \psi_t' dM_t, \tag{4.8}$$

where ϕ and ψ are P-a.s. square integrable $\mathbb{G}$-adapted processes. Moreover, for all $x > 0$, the solution of the problem (4.7) *is given by*

$$\pi_t^* = \frac{(\sigma(t, S_t, Y_t)^{-1})'[\tilde{\lambda}_t + Q_t^{-1}\phi_t]}{1-p}. \tag{4.9}$$

For the proof, see Appendix B.

REMARK 4.5. Under some integrable and smooth conditions, we expect the processes ϕ to coincide with the Malliavin derivative of Q. More precisely, let $D = (D^W, D^b)$ be the Malliavin derivative acting on the subset of the class of functionnals of $\widetilde{W}, \tilde{b}$ called $D_{1,1}$ [for the definition of $D_{1,1}$ and D, one is referred to Ocone and Karatzas (1991)]. In this case, $D_t^W Q_t = \phi_t \; dt \otimes dP$-a.s. Hence, (4.9) can be written

$$\pi_t^* = \frac{\big(\sigma(t, S_t, Y_t)^{-1}\big)'[\tilde{\lambda}_t + Q_t^{-1} D_t^W Q_t]}{1-p}. \tag{4.10}$$

We state a verification Theorem which gives a sufficient condition for a process to coincide with the value function process Q.

PROPOSITION 4.3. *Let Q^0 be an adapted process satisfying*

$$dQ_t^0 = -\frac{p}{2(1-p)} Q_t^0 |\tilde{\lambda}_t + (Q_t^0)^{-1}\phi_t^0|^2 dt + (\phi_t^0)' dN_t + (\psi_t^0)' dM_t,$$
$$Q_T^0 = 1,$$

where ϕ^0 and ψ^0 are square integrable $\mathbb{G}$-adapted processes, with $Q^0 > 0$, $dP \otimes dt$-a.s., and also $E[\int_0^T Q_t^2 |\sigma(t, S_t, Y_t)|^2 dt] < \infty$. Then,

$$Q_t^0 \geq Q_t, \quad 0 \leq t \leq T, \; dP \otimes dt\text{-a.s.}$$

Moreover, if the process π^* given by

$$\pi_t^* = \frac{\big(\sigma(t, S_t, Y_t)^{-1}\big)'[\tilde{\lambda}_t + (Q_t^0)^{-1}\phi_t^0]}{1-p},$$

is such that the local martingale $(X^{\pi^*})^p Q^0$ is a martingale, then

$$Q_t^0 = Q_t, \quad 0 \leq t \leq T, \; dP \otimes dt\text{-a.s.}$$

For the proof, see Appendix B.

REMARK 4.6. In the case of full information, one can show that the corresponding value ($\mathbb{F}$-adapted) process Q admits a decomposition,

$$dQ_t = -\frac{p}{2(1-p)} Q_t \big|\lambda_t + Q_t^{-1}\phi_t\big|^2 dt + \phi_t' dW_t + dj_t,$$

where ϕ is a P-a.s. square integrable $\mathbb{F}$-adapted process and j is a $(P, \mathbb{F})$-local martingale orthogonal to W. Moreover, for all $x > 0$, the optimal proportion portfolio is given by

$$\pi_t^* = \frac{\big(\sigma(t, S_t, Y_t)^{-1}\big)'[\lambda_t + Q_t^{-1}\phi_t]}{1-p}.$$

5. The Bayesian framework. In this section, we assume that (λ, α) are $\mathscr{F}_0$-measurable, with prior known distribution $\kappa(dl, da)$ on $\mathbb{R}^n \times \mathbb{R}^d$. A similar framework is studied in Karatzas and Zhao (1998) in a complete market context. Notice that Assumption 3.1 is satisfied since

$$\begin{aligned} E[L_T] &= E\Big[e^{-((|\lambda|^2+|\alpha|^2)T/2)}E\big[e^{-\lambda' W_T-\alpha' B_T}\big|\mathscr{F}_0\big]\Big] \\ &= E\Big[e^{-((|\lambda|^2+|\alpha|^2)T/2)}E\big[e^{-l' W_T-\alpha' B_T}\big]\Big|_{l=\lambda,\, a=\alpha}\Big] \\ &= 1. \end{aligned} \tag{5.1}$$

Assumption 3.2 is equivalent to assuming that

$$\int |l| + |a|\kappa(dl, da) < \infty. \tag{5.2}$$

By noting that the pair of random variables (λ, α) is independent of the Brownian motion $(\widetilde{W}, \widetilde{B})$ under $\widetilde{P}$, and recalling that $\mathbb{G}$ is the augmented filtration of $(\widetilde{W}, \widetilde{B})$, we have the following explicit formula for the process $\widetilde{\Lambda}$ defined in (3.14):

$$\begin{aligned} \widetilde{\Lambda}_t &= E^{\widetilde{P}}\Big[\exp\Big(\lambda'\widetilde{W}_t + \alpha'\widetilde{B}_t - \tfrac{1}{2}(|\lambda|^2+|\alpha|^2)t\Big)\Big|\mathscr{G}_t\Big] \\ &= F\Big(t, \widetilde{W}_t, \widetilde{B}_t\Big), \end{aligned}$$

where

$$F(t, w, b) := \int \exp\Big(l'w + a'b - \tfrac{1}{2}(|l|^2+|a|^2)t\Big)\kappa(dl, da),$$

for all $(t, w, b) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^d$.

By (3.13), we then deduce explicit computations of $\tilde{\lambda}$ and $\tilde{\alpha}$,

$$\begin{aligned} \tilde{\lambda}_t &= \frac{E^{\widetilde{P}}\Big[\lambda\exp\Big(\lambda'\widetilde{W}_t + \alpha'\widetilde{B}_t - \tfrac{1}{2}(|\lambda|^2+|\alpha|^2)t\Big)|\mathscr{G}_t\Big]}{\widetilde{\Lambda}_t} \\ &= G\Big(t, \widetilde{W}_t, \widetilde{B}_t\Big), \end{aligned}$$

where

$$G(t, w, b) := \frac{\nabla_w F(t, w, b)}{F(t, w, b)}, \qquad (t, w, b) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^d$$

and

$$\begin{aligned} \tilde{\alpha}_t &= \frac{E^{\widetilde{P}}\Big[\alpha\exp\Big(\lambda'\widetilde{W}_t + \alpha'\widetilde{B}_t - \tfrac{1}{2}(|\lambda|^2+|\alpha|^2)t\Big)\Big|\mathscr{G}_t\Big]}{\widetilde{\Lambda}_t} \\ &= H\Big(t, \widetilde{W}_t, \widetilde{B}_t\Big), \end{aligned}$$

where

$$H(t,w,b) := \frac{\nabla_b F(t,w,b)}{F(t,w,b)}, \qquad (t,w,b) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^d.$$

The case λ and α independent. In this paragraph, we assume that λ and α are independent and so

$$\kappa(dl, da) = \kappa_1(dl) \otimes \kappa_2(da). \tag{5.3}$$

We deduce then that F is written in the form

$$F(t,w,b) = F_1(t,w) F_2(t,b),$$

with

$$F_1(t,w) := \int \exp\left(l'w - \tfrac{1}{2}|l|^2 t\right) \kappa_1(dl), \qquad (t,w) \in [0,T] \times \mathbb{R}^n,$$

$$F_2(t,b) := \int \exp\left(a'b - \tfrac{1}{2}|a|^2 t\right) \kappa_2(da), \qquad (t,b) \in [0,T] \times \mathbb{R}^d.$$

It follows that function G (resp. H) does not depend on b (resp. w) and

$$G(t,w) = \frac{\nabla F_1(t,w)}{F_1(t,w)}, \qquad (t,w) \in [0,T] \times \mathbb{R}^n,$$

$$H(t,b) = \frac{\nabla F_2(t,b)}{F_2(t,b)}, \qquad (t,b) \in [0,T] \times \mathbb{R}^d.$$

LEMMA 5.1. *For all $t \in [0,T]$ and for all measurable function k on $\mathbb{R}_+$, such that $Z_T^0 k(Z_T^0) \in L^1(P)$, we have $k(Z_T^0) \in L^1(\widetilde{P})$ and*

$$E^{\widetilde{P}}[k(Z_T^0)|\mathscr{G}_t] = E\left[\frac{Z_T^0}{Z_t^0} k(Z_T^0) \middle| \mathscr{G}_t\right]. \tag{5.4}$$

PROOF. From Proposition 3.1, we have

$$\left.\frac{d\widetilde{P}}{dP}\right|_{\mathscr{G}_t} = \xi_t = Z_t^0 \zeta_t,$$

where

$$Z_t^0 = \exp\left(-\int_0^t \tilde{\lambda}'_u \, dN_u - \tfrac{1}{2}\textstyle\int_0^t |\tilde{\lambda}_u|^2 du\right),$$

$$\zeta_t = \exp\left(-\int_0^t \tilde{\alpha}'_u \, dM_u - \tfrac{1}{2}\textstyle\int_0^t |\tilde{\alpha}_u|^2 du\right).$$

Since $\tilde{\lambda}_t = G(t, \widetilde{W}_t) = G(t, W_t + \lambda t)$ and $\tilde{\alpha}_t = H(t, \widetilde{B}_t) = H(t, B_t + \alpha t)$, we deduce from the definition of N and M that Z_T^0/Z_t^0 and ζ_T/ζ_t are independent under P, and also under P conditionally to $\mathscr{G}_t$ (because they are independent of $\mathscr{G}_t$). Recall that ξ is a $(P, \mathbb{G})$-martingale so that $1 = E[\xi_T] = E[Z_T^0 \zeta_T] = E[Z_T^0] E[\zeta_T]$. Since we already know that $E[Z_T^0] \le 1$ and $E[\zeta_T^0] \le 1$ by the supermartingale property of Z^0 and ζ under P, we deduce that $E[Z_T^0] = E[\zeta_T] = 1$, hence Z^0 and ζ are martingale under P. Since $E^{\widetilde{P}}[|k(Z_T^0)|] = E[Z_T^0 \zeta_T |k(Z_T^0)|] = E[Z_T^0 |k(Z_T^0)|]E[\zeta_T] = E[Z_T^0 |k(Z_T^0)|] < \infty$, the first assertion of the lemma is proved. Now, using Bayes' formula, we have

$$\begin{aligned} E^{\widetilde{P}}\left[k(Z_T^0)\middle|\mathscr{G}_t\right] &= E\left[\frac{Z_T^0}{Z_t^0}\frac{\zeta_T}{\zeta_t}k\left(\frac{Z_T^0}{Z_t^0}Z_t^0\right)\middle|\mathscr{G}_t\right] \\ &= E\left[\frac{Z_T^0}{Z_t^0}k\left(\frac{Z_T^0}{Z_t^0}Z_t^0\right)\middle|\mathscr{G}_t\right]E\left[\frac{\zeta_T}{\zeta_t}\middle|\mathscr{G}_t\right] \\ &= E\left[\frac{Z_T^0}{Z_t^0}k(Z_T^0)\middle|\mathscr{G}_t\right]. \qquad \square \end{aligned}$$

PROPOSITION 5.1. *The solution of the dual problem* (4.3) *is equal to* $\nu^* = 0$. *The associated process* Z_t^0 *is given by*

$$Z_t^0 = \frac{1}{F_1(t, \widetilde{W}_t)}. \tag{5.5}$$

Moreover, we have

$$\widetilde{V}(z) = \int F_1(T, w)\widetilde{U}\left(\frac{z}{F_1(T, w)}\right)\varphi_T(w)\,dw, \tag{5.6}$$

where

$$\varphi_t(w) := \frac{1}{(2\pi t)^{n/2}}\exp\left(-\frac{|w|^2}{2t}\right), \qquad t \in (0, T],\ w \in \mathbb{R}^n$$

is the Gaussian density function on $\mathbb{R}^n$.

PROOF. First, let us show that the optimal control for the dual problem (4.3) is equal to $\nu^* = 0$, for all $z > 0$. By Remark 4.4, it suffices to show that

$$\inf_{\nu \in \mathscr{D}} E\left[\widetilde{U}\left(zZ_T^\nu\right)\right] = E\left[\widetilde{U}\left(zZ_T^0\right)\right] \qquad \forall\, z > 0.$$

Denote by $\{\mathscr{F}_t^{\widetilde{W}}, 0 \le t \le T\}$ the augmented filtration generated by $\widetilde{W}$. Since the function $\widetilde{U}$ is convex, we have by Jensen's inequality

$$\widetilde{U}\left(zE[Z_T^\nu|\mathscr{F}_T^{\widetilde{W}}]\right) \le E\left[\widetilde{U}(zZ_T^\nu)|\mathscr{F}_T^{\widetilde{W}}\right].$$

It follows that

$$\inf_{\nu \in \mathscr{D}} E\left[\widetilde{U}\left(zE[Z_T^\nu|\mathscr{F}_T^{\widetilde{W}}]\right)\right] \le \inf_{\nu \in \mathscr{D}} E\left[\widetilde{U}(zZ_T^\nu)\right]. \tag{5.7}$$

Since Z_T^0 is $\mathscr{F}_T^{\widetilde{W}}$-measurable, we have

$$E[Z_T^\nu|\mathscr{F}_T^{\widetilde{W}}] = Z_T^0 E\left[\exp\left(-\int_0^T \nu_u' \, dM_u - \tfrac{1}{2}\int_0^T |\nu_u|^2\, du\right)\middle|\mathscr{F}_T^{\widetilde{W}}\right]. \tag{5.8}$$

According to (5.7) and (5.8), in order to prove that $\nu \equiv 0$ is solution to the dual problem, it remains to show that $E[\exp(-\int_0^T \nu'_u dM_u - \frac{1}{2}\int_0^T |\nu_u|^2 du)|\mathscr{F}_T^{\widetilde{W}}] = 1$. It is then sufficient to prove that for each positive borelian function h, for each $t_1, t_2, \dots, t_p \in [0, T]$,

$$E\left[\exp\left(-\int_0^t \nu'_u dM_u - \tfrac{1}{2}\int_0^T |\nu_u|^2 du\right) h(\widetilde{W}_{t_1}, \dots, \widetilde{W}_{t_p})\right] \tag{5.9}$$
$$= E[h(\widetilde{W}_{t_1}, \dots, \widetilde{W}_{t_p})].$$

Since ν is a bounded $\mathbb{G}$-adapted process, we can define a probability measure P^ν equivalent to P on $\mathscr{G}_T$ by

$$\frac{dP^\nu}{dP} = \exp\left(-\int_0^T \nu'_u dM_u - \tfrac{1}{2}\int_0^T |\nu_u|^2\, du\right).$$

By Girsanov's theorem, the process N is a $\mathbb{G}$-Brownian motion under P^ν. Thus, since the dynamics of $\widetilde{W}$ is given by

$$d\widetilde{W}_t = dN_t + G(t, \widetilde{W}_t)\, dt,$$

it follows that the law of $\widetilde{W}$ remains the same under P and P^ν. Hence, $E^{P^\nu}[h(\widetilde{W}_{t_1}, \dots, \widetilde{W}_{t_p})] = E[h(\widetilde{W}_{t_1}, \dots, \widetilde{W}_{t_p})]$ and so (5.9).

By definition of N, we have

$$Z_t^0 = \exp\left(-\int_0^t G(u, \widetilde{W}_u)'\, d\widetilde{W}_u + \tfrac{1}{2}\int_0^t |G(u, \widetilde{W}_u)|^2\, du\right).$$

Now, we have $Z_t^0 = E[1/\widetilde{\Lambda}_t|\mathscr{F}_T^{\widetilde{W}}]$, and since $\widetilde{\Lambda}_t = F_1(t, \widetilde{W}_t)F_2(t, \widetilde{B}_t)$, it follows that

$$E\left[\frac{1}{\widetilde{\Lambda}_t}\middle|\mathscr{F}_T^{\widetilde{W}}\right] = \frac{1}{F_1(t, \widetilde{W}_t)} E[F_2(t, \widetilde{B}_t)] = \frac{1}{F_1(t, \widetilde{W}_t)}.$$

Hence, $Z_t^0 = 1/F_1(t, \widetilde{W}_t)$.

From Lemma 5.1, the value function of the dual problem is given by

$$\begin{aligned}\widetilde{V}(z) &= E\left[\widetilde{U}(zZ_T^0)\right] \\ &= E^{\widetilde{P}}\left[\frac{1}{Z_T^0}\widetilde{U}(zZ_T^0)\right] \\ &= E^{\widetilde{P}}\left[F_1(T, \widetilde{W}_T)\widetilde{U}\left(z\frac{1}{F_1(T, \widetilde{W}_T)}\right)\right],\end{aligned}$$

and so (5.6). $\square$

We now assume that the function

$$(t, z, w) \longmapsto \int I\left(\frac{z}{F_1(T, w+v)}\right)\varphi_{T-t}(v)\, dv$$

is finite for all $(t, z, w) \in [0, T) \times (0, \infty) \times \mathbb{R}^n$, and has finite first derivatives with respect to (t, z, w) and second derivatives with respect to (z, w) on $[0, T) \times (0, \infty) \times \mathbb{R}^n$.

The Lagrange multiplier z_x is the unique solution $z > 0$ of $\widetilde{V}'(z) = -x$. Therefore, according to (5.6) and since $\widetilde{U}' = -I$, we have

$$\int I\left(\frac{z_x}{F_1(T, w)}\right)\varphi_T(w)\, dw = x. \tag{5.10}$$

The optimal wealth process is given by $X_t^* = E[Z_T^0/Z_t^0 I(z_x Z_T^0)|\mathscr{G}_t]$. Hence, by Lemma 5.1, we have

$$\begin{aligned} X_t^* &= E^{\widetilde{P}}\left[I\big(z_x Z_T^0\big)\middle|\mathscr{G}_t\right] \\ &= E^{\widetilde{P}}\left[I\left(\frac{z_x}{F_1(T, \widetilde{W}_T)}\right)\middle|\mathscr{G}_t\right] \\ &= \chi\big(t, \widetilde{W}_t\big), \end{aligned} \tag{5.11}$$

where

$$\chi(t, w) = \begin{cases} \displaystyle\int I\Big(\frac{z_x}{F_1(T, w + v)}\Big)\varphi_{T-t}(v)\, dv, & t \in [0, T), w \in \mathbb{R}^n, \\ I\Big(\dfrac{z_x}{F_1(T, w)}\Big), & t = T, w \in \mathbb{R}^n. \end{cases} \tag{5.12}$$

Since $dX_t^* = (\theta_t^*)'\sigma(t, S_t, Y_t)d\widetilde{W}_t$, we deduce that the optimal portfolio is expressed as

$$\theta_t^* = \sigma'(t, S_t, Y_t)^{-1}\nabla\chi(t, \widetilde{W}_t), \qquad t \in [0, T), \tag{5.13}$$

with

$$\nabla\chi(t, w) = -z_x \int \frac{G(T, w + v)}{F_1(T, w + v)} I'\left(\frac{z_x}{F_1(T, w + v)}\right)\varphi_{T-t}(v)\, dv. \tag{5.14}$$

By Lemma 5.1, the value function of problem (2.5) is given by

$$\begin{aligned} V(x) &= E\Big[U(X_T^*)\Big] = E\Big[U \circ I(z_x Z_T^0)\Big] \\ &= E^{\widetilde{P}}\left[\frac{1}{Z_T^0} U \circ I(z_x Z_T^0)\right] \\ &= E^{\widetilde{P}}\left[F_1(T, \widetilde{W}_T) U \circ I\left(\frac{z_x}{F_1(T, \widetilde{W}_T)}\right)\right] \\ &= \int F_1(T, w) U \circ I\left(\frac{z_x}{F_1(T, w)}\right)\varphi_T(w)\, dw. \end{aligned} \tag{5.15}$$

We summarize all of this as follows.

THEOREM 5.1. *For any $x \geq 0$, the control process θ^* expressed in* (5.13) *and* (5.14) *is optimal for problem* (2.5). *The associated optimal wealth process is given by* (5.11) *and* (5.12), *and the value function of* (2.5) *is expressed in* (5.15).

EXAMPLE [Power utility function $U(x) = x^p/p$, for $0 < p < 1$.] In this case, $I(z) = z^{-q}$ with $q = 1/(1-p)$. The Lagrange multiplier is given by

$$z_x = \left(\frac{\int F_1(T, w)^q \varphi_T(w)\, dw}{x}\right)^{1/q}.$$

Substituting back into (5.12) gives

$$\chi(t, w) = \begin{cases} x\dfrac{\int F_1(T, w+v)^q \varphi_{T-t}(v)\, dv}{\int F_1(T, v)^q \varphi_T(v)\, dv}, & t \in [0, T)\; w \in \mathbb{R}^n, \\ x\dfrac{F_1(T, w)^q}{\int F_1(T, v)^q \varphi_T(v)\, dv}, & t = T, w \in \mathbb{R}^n. \end{cases}$$

Hence,

$$\nabla\chi(t, w) = qx\frac{\int \nabla F_1(T, w+v) F_1(T, w+v)^{q-1} \pi_{T-t}(v)\, dv}{\int F_1(T, v)^q \varphi_T(v)\, dv},$$
$$t \in [0, T),\; w \in \mathbb{R}^n.$$

The optimal wealth process and the optimal portfolio are given by

$$X_t^* = \chi(t, \widetilde{W}_t),$$
$$\theta_t^* = \sigma'(t, S_t, Y_t)^{-1} \nabla\chi(t, \widetilde{W}_t),$$

and the optimal proportion portfolio is in the form

$$\pi_t^* = \sigma'(t, S_t, Y_t)^{-1} \frac{\nabla\chi}{\chi}(t, \widetilde{W}_t).$$

REMARK 5.1. In the special case $\kappa_1(dl) = \delta_{l_0}$, we have $\nabla\chi/\chi = ql_0$, so that the optimal proportion portfolio is given by

$$\pi_t^*(l_0) = \sigma'(t, S_t, Y_t)^{-1} q l_0.$$

Therefore, the certainty equivalence principle does not hold for power utility functions, since for a nondegenerate prior distribution κ_1, we typically have $\nabla\chi/\chi \neq qG(t, w)$. Formal substitution of $\tilde{\lambda} = G$ for l_0 in the expression $\pi^*(l_0)$ of the optimal proportion portfolio corresponding to $\kappa_1(dl) = \delta_{l_0}$ does not yield the correct expression for the optimal proportion portfolio in the nondegenerate prior distribution κ_1. This point has been observed by Karatzas and Zhao (1998) in a Bayesian complete model with power utility function.

REMARK 5.2. In the general case where λ and α are nonindependent, it is an open problem to derive an explicit characterization of the solution to the dual problem and then to obtain a more explicit form for the optimal portfolio.

6. The linear Gaussian case. In this section, the risk premia λ, α are supposed to be Gaussian processes modelled by a system of linear stochastic differential equations where the driving Brownian motions are independent from W and B. A similar framework is studied in Lakner (1998) in a complete market context.

More precisely, the process (λ, α) is supposed to satisfy the following dynamics:

$$d\begin{pmatrix}\lambda_t\\ \alpha_t\end{pmatrix} = \left(A_t\begin{pmatrix}\lambda_t\\ \alpha_t\end{pmatrix} + C_t\right)dt + k_t d\begin{pmatrix}W_t^1\\ W_t^2\end{pmatrix}, \qquad \begin{pmatrix}\lambda_0\\ \alpha_0\end{pmatrix} = \delta. \tag{6.1}$$

The functions A, C, k are bounded deterministic functions which are valued in $\mathbb{R}^{(n+d)\times(n+d)}$, $\mathbb{R}^{n+d}$, $\mathbb{R}^{(n+d)\times(n+d)}$, respectively. The variable δ is an $\mathbb{R}^{n+d}$-valued $\mathscr{F}_0$-measurable random variable with Gaussian probability law of mean m_0 and covariance matrix Δ_0. The processes W^1 and W^2 are independent $(P, \mathbb{F})$-Brownian motions valued, respectively, in $\mathbb{R}^n$ and $\mathbb{R}^d$ and independent from W and B. The matrix $k_t k_t^*$ is supposed to be uniformly positive definite. By similar arguments as in (5.1), it is easily checked that Assumption 3.1 is satisfied.

Recall that the vector $(\widetilde{W}, \widetilde{B})$ corresponds to the observation process and its dynamics can be written as:

$$d\begin{pmatrix}\widetilde{W}_t\\ \widetilde{B}_t\end{pmatrix} = d\begin{pmatrix}W_t\\ B_t\end{pmatrix} + \int_0^t \begin{pmatrix}\lambda_u\\ \alpha_u\end{pmatrix} du.$$

The vector process (N, M) corresponds to the innovation process and we are then in the framework of the classical Kalman–Bucy filter [see, e.g., Liptser and Shiryaev (1977), Theorem 10.3].

PROPOSITION 6.1. *The processes $\tilde{\lambda}$ and $\tilde{\alpha}$ are solutions of the linear s.d.e.*

$$d\begin{pmatrix}\tilde{\lambda}_t\\ \tilde{\alpha}_t\end{pmatrix} = \left(A_t\begin{pmatrix}\tilde{\lambda}_t\\ \tilde{\alpha}_t\end{pmatrix} + C_t\right)dt + \Gamma_t d\begin{pmatrix}N_t\\ M_t\end{pmatrix}, \qquad \begin{pmatrix}\tilde{\lambda}_0\\ \tilde{\alpha}_0\end{pmatrix} = m_0, \tag{6.2}$$

where the function Γ_t (which represents the covariance error) is solution of the Riccati equation

$$-\frac{d\Gamma}{dt} + A_t\Gamma_t + \Gamma_t A_t' - \Gamma_t^2 + k_t k_t' = 0, \qquad \Gamma_0 = \Delta_0. \tag{6.3}$$

In order to derive an explicit formula for the optimal portfolio, we will now consider the case where λ and α are independent.

The case λ and α independent. In this paragraph, the matrices A_t, k_t, $t \in [0, T]$ and Δ_0 are supposed to be diagonal:

$$A_t = \begin{pmatrix}A_t^1 & 0\\ 0 & A_t^2\end{pmatrix}, \qquad k_t = \begin{pmatrix}k_t^1 & 0\\ 0 & k_t^2\end{pmatrix}, \qquad \Delta_0 = \begin{pmatrix}\Delta_0^1 & 0\\ 0 & \Delta_0^2\end{pmatrix}. \tag{6.4}$$

It follows that λ and α are independent under P. Indeed, the problem can be separated in two parts. First, $\tilde{\lambda}$ is the Kalman–Bucy filter associated with the state process λ given by

$$d\lambda_t = \left(A_t^1\lambda_t + C_t^1\right)dt + k_t^1 dW_t^1, \qquad \lambda_0 = \delta^1 \tag{6.5}$$

and with the observation process $\widetilde{W}$. Second, $\tilde{\alpha}$ is the Kalman–Bucy filter associated with the state process α given by

$$d\alpha_t = \left(A_t^2\alpha_t + C_t^2\right)dt + k_t^2 dW_t^2, \qquad \alpha_0 = \delta^2 \tag{6.6}$$

and with the observation process $\widetilde{B}$, where $C = (C^1, C^2)$, $\delta = (\delta^1, \delta^2)$, $m_0 = (m_0^1, m_0^2)$. Therefore, $\tilde{\lambda}$ and $\tilde{\mu}$ satisfy

$$d\tilde{\lambda}_t = \left(A_t^1\tilde{\lambda}_t + C_t^1\right)dt + \Gamma_t^1 dN_t, \qquad \tilde{\lambda}_0 = m_0^1, \tag{6.7}$$

$$d\tilde{\alpha}_t = \left(A_t^2\tilde{\alpha}_t + C_t^2\right)dt + \Gamma_t^2 dM_t, \qquad \tilde{\alpha}_0 = m_0^2, \tag{6.8}$$

where the functions Γ^i for $i = 1, 2$ are solutions of the following deterministic Riccati equations:

$$-\frac{d\Gamma^i}{dt} + A_t^i\Gamma_t^i + \Gamma_t^i(A_t^i)' - (\Gamma_t^i)^2 + k_t^i(k_t^i)' 0, \qquad \Gamma_0 = \Delta_0^i. \tag{6.9}$$

It follows that $\tilde{\lambda}$ (respectively, $\tilde{\alpha}$) is adapted to the filtration generated by N (respectively, M) and also to the filtration generated by $\widetilde{W}$ (respectively, $\widetilde{B}$). As in the Bayesian case, we have the following result.

PROPOSITION 6.2. *The solution of the dual problem* (4.3) *is equal to* $\nu^* = 0$.

PROOF. The proof is very similar to the Bayesian case. Let us show that for each $z \geq 0$, we have

$$\inf_{\nu\in\mathscr{D}} E\left[\widetilde{U}(zZ_T^\nu)\right] = E\left[\widetilde{U}(zZ_T^0)\right].$$

Denote by $\{\mathscr{F}_t^N, 0 \leq t \leq T\}$ the augmented filtration generated by N. By Jensen's inequality,

$$\inf_{\nu\in\mathscr{D}} E\left[\widetilde{U}\left(zE[Z_T^\nu|\mathscr{F}_T^N]\right)\right] \leq \inf_{\nu\in\mathscr{D}} E\left[\widetilde{U}(zZ_T^\nu)\right].$$

Since Z_T^0 is $\mathscr{F}_T^N$-measurable, we have

$$E[Z_T^\nu|\mathscr{F}_T^N] = Z_T^0 E\left[\exp\left(-\int_0^T \nu_u'\, dM_u - \tfrac{1}{2}\int_0^T |\nu_u|^2\, du\right)\Big|\mathscr{F}_T^N\right]. \tag{6.10}$$

Similarly to the proof of Proposition 5.1, we have

$$E[\exp(-\int_0^T \nu_u'\, dM_u - \frac{1}{2}\int_0^T |\nu_u|^2\, du)|\mathscr{F}_T^N] = 1,$$

and so $E[Z_T^\nu|\mathscr{F}_T^N] = Z_T^0$ and the result follows. □

We now study a particular case for which an explicit formula for the optimal portfolio θ^* can be obtained. Lakner (1998) has studied a similar case in a complete market where the volatility σ is constant. In the following, we shall assume that for each t, $A_t^1 = A^1$, $C_t^1 = A^1C^1$, $k_t^1 = k^1$ where A^1, C^1, k^1 are fixed real matrices, and that

$$\operatorname{tr}(\Delta_0^1) + T\operatorname{tr}(k^1(k^1)') < K_1^1, \tag{6.11}$$

where $K_1^1 = 1/360TK^1$ with $K^1 = \max_{t\le T}\|e^{A^1 t}\|$. Then, by Lakner's results (1998) (see Lemma 4.1), $E^{\widetilde{P}}[(Z_T^0)^4 + (Z_T^0)^{-5}] < +\infty$. First, recall that the Riccati equation (6.9) (for $i = 1$) can be solved in the following way [see Lakner (1998)]. Let $\Phi: [0, T] \to \mathbb{R}^{n\times n}$ be the fundamental solution of the deterministic equation

$$\frac{d\Phi}{dt} = [A^1 - \Gamma_t^1]\Phi_t, \qquad \Phi_0 = I_n,$$

where I_n is the $n\times n$ identity matrix. Then, $\tilde{\lambda}$ is determined in terms of Γ^1 and Φ as

$$\tilde{\lambda}_t = \Phi_t\biggl(m_0^1 + \int_0^t \Phi_s^{-1}\Gamma_s^1\, d\widetilde{W}_s + \Bigl(\int_0^t \Phi_s^{-1}\, ds\Bigr)A^1C^1\biggr). \tag{6.12}$$

From Lakner's results stated in the case of a constant volatility σ (see his Theorem 4.3), we derive the following result.

THEOREM 6.1. *Suppose that U is twice continuously differentiable on* $(0, \infty)$ *and*

$$I(x) < K_2(1 + x^{-5}), \qquad -I'(x) < K_2(1 + x^{-2}) \tag{6.13}$$

for some $K_2 > 0$. Then, the optimal portfolio is

$$\begin{aligned}\theta_t^* &= \sigma'(t, S_t, Y_t)^{-1}\frac{z_x}{Z_t^0}\\ &\quad\times E\biggl[I'(z_xZ_T^0)(Z_T^0)^2\Bigl(-\Gamma_t^1(\Phi_t')^{-1}\textstyle\int_t^T \Phi_u' dN_u - \tilde{\lambda}_t\Bigr)\bigg|\mathscr{F}_t^{\widetilde{W}}\biggr],\end{aligned} \tag{6.14}$$

where $\{\mathscr{F}_t^{\widetilde{W}}, 0 \le t \le T\}$ is the augmented filtration generated by $\widetilde{W}$.

SKETCH OF THE PROOF. We give here some idea of the proof. For details, we refer to Lakner (1998). The main technique of the proof is the use of the Malliavin derivative D acting on the subset of the class of functionals of $\widetilde{W}$ called $D_{1,1}$ [for the definition of $D_{1,1}$ and D; see Ocone and Karatzas (1991)].

First, recall that the optimal wealth is given by $X_t^* = E[(Z_T^0/Z_t^0)\, I(z_xZ_T^0)\,|\mathscr{G}_t]$. Notice now that Z^0 is adapted with respect to the filtration generated by $\widetilde{W}$ since $\tilde{\lambda}$ depends only on $\widetilde{W}$. Hence,

$$X_t^* = E^{P^0}\bigl[I(z_xZ_T^0)|\mathscr{F}_t^{\widetilde{W}}\bigr],$$

where P^0 is the probability measure which admits Z_T^0 as density with respect to P on $\mathscr{F}_T^{\widetilde{W}}$. The aim is to compute θ^* such that $dX_t^* = (\theta_t^*)'\sigma(t, S_t, Y_t)d\widetilde{W}_t$. By results on Malliavin derivative, we have

$$\sigma'(t, S_t, Y_t)\theta_t^* = E^{P^0}\Big[D_t I(z_x Z_T^0)\Big|\mathscr{F}_t^{\widetilde{W}}\Big], \qquad dt \otimes dP\text{-a.s.}$$

See the generalized version of Clark's formula [Karatzas, Ocone and Li (1991)] which gives that for every variable $H \in D^{1,1}$, we have the stochastic representation,

$$E^{P^0}[H|\mathscr{F}_t^{\widetilde{W}}] = H_0 + \int_0^t \left(E^{P^0}[D_u H|\mathscr{F}_u^{\widetilde{W}}]\right)' d\widetilde{W}_u, \qquad 0 \le t \le T.$$

It remains to determine $D_t I(z_x Z_T^0)$. By standard calculation, we have

$$D_t I(z_x Z_T^0) = z_x I'(z_x Z_T^0) D_t Z_T^0,$$

with

$$D_t Z_T^0 = Z_T^0\, D_t\bigg(-\int_0^T \tilde{\lambda}_u'\, d\widetilde{W}_u + \tfrac{1}{2}\int_0^T |\tilde{\lambda}_u|^2\, du\bigg). \tag{6.15}$$

Now,

$$D_t\bigg(\tfrac{1}{2}\int_0^T |\tilde{\lambda}_u|^2\, du\bigg) = \int_t^T (D_t\tilde{\lambda}_u)\tilde{\lambda}_u\, du$$

and by Proposition 2.3 of Ocone and Karatzas (1991),

$$D_t\bigg(-\int_0^T \tilde{\lambda}_u' d\widetilde{W}_u\bigg) = -\tilde{\lambda}_t - \int_t^T D_t\tilde{\lambda}_u d\widetilde{W}_u.$$

The result follows by using the fact that from (6.12), we have

$$D_t\tilde{\lambda}_u = \Gamma_t^1(\Phi_t')^{-1}\Phi_u'\mathbf{1}_{\{t \le u\}}. \qquad \square$$

EXAMPLE [Power utility function $U(x) = x^p/p$, for $p < 1$]. If $p < 0$, then condition (6.13) is satisfied. Notice that if $p > 0$, then some Stronger condition than (6.11) has to be made on the coefficients to ensure that condition (6.13) holds [for details, see Lakner (1998), Proposition 4.6]. In this case, $I(y) = y^{-q}$ and $I'(y) = -qy^{-q-1}$, where $q = 1/1 - p$. From Theorem 6.1, the optimal portfolio is given by

$$\theta_t^* = q(\sigma(t, S_t, Y_t)^{-1})'\tilde{\lambda}_t X_t^* + G_t, \tag{6.16}$$

where

$$G_t = q(\sigma(t, S_t, Y_t)^{-1})' z_x^{-q}\Gamma_t^1(\Phi_t^{-1})' E\bigg[\frac{Z_T^0}{Z_t^0}(Z_T^0)^{-q}\int_t^T \Phi_u'\, dN_u\bigg|\mathscr{F}^{\widetilde{W}} t\bigg].$$

The optimal wealth is given by

$$X_t^* = E\bigg[\frac{Z_T^0}{Z_t^0}(Z_T^0)^{-q}\bigg|\mathscr{G}_t\bigg]\frac{x}{E[(Z_T^0)^{1-q}]}.$$

REMARK 6.1. Notice that under full information (in this Gaussian framework) λ and α are deterministic and hence, the optimal portfolio for power utility function is

$$(\theta^c_t)^* = q(\sigma(t, S_t, Y_t)^{-1})'\lambda_t(X^c)^*_t, \tag{6.17}$$

where $(X^c)^*$ is the optimal wealth [Ocone and Karatzas (1991), formula (4.22)]. Thus, the certainty equivalence principle does not hold in this case since our formula (6.16) cannot be derived from (6.17) by replacing the risk premium λ_t by $\tilde{\lambda}_t$ due to the additional term G_t. This point has been observed by Lakner (1998) in his complete market context.

APPENDIX A

Proof of Theorem 4.1. The techniques are similar to those in El Karoui-Quenez (1995) [more simple since here the filtration $\mathbb{G}$ is a Brownian filtration; see also Cvitanić and Karatzas (1993)]. We first want to show that $u_0 \geq J_0$. As usual, it is derived from the $(P, \mathbb{G})$-supermartingale property of $Z^\nu X^{x,\theta}$ for any $\nu \in \mathscr{H}$ and $\theta \in \mathscr{A}(x)$, $x \geq 0$.

We now show that $u_0 \leq J_0$, which is the most difficult part. Clearly, we may assume that $J_0 < \infty$. We now consider the RCLL process $\{J_t,\ 0 \leq t \leq T\}$ (which exists) such that

$$J_t = \operatorname*{ess\,sup}_{\nu \in \mathscr{H}} E\left[\frac{Z^\nu_T}{Z^\nu_t} H \,\middle|\, \mathscr{G}_t\right], \qquad 0 \leq t \leq T,\ P\text{-a.s.}$$

Suppose now that the following decomposition of J holds:

$$J_t = J_0 + \int_0^t (\theta^*_u)'\sigma(u, S_u, Y_u)\, d\widetilde{W}_u - C_t, \qquad 0 \leq t \leq T, \tag{A.1}$$

where $\theta^* \in \mathscr{A}(J_0)$ and $\{C_t, 0 \leq t \leq T\}$ is an increasing $\mathbb{G}$-predictable process with $C_0 = 0$. Then, it follows that $H \leq X^{J_0,\theta^*}_T$ a.s. and hence

$$u_0 \leq J_0.$$

It remains to show that J admits the decomposition (A.1). Classical stochastic control results give the following dynamic characterization of J.

LEMMA A.1. *The process J is characterized as the smallest RCLL process equal to H at time T such that for each $\nu \in \mathscr{H}$, the process $\{Z^\nu_t J_t, 0 \leq t \leq T\}$ is a $(P, \mathbb{G})$-supermartingale.*

From Lemma A.1 applied to $\nu = 0$, the process $\widetilde{J}$ given by $\widetilde{J} = Z^0 J$, is a $(P, \mathbb{G})$-supermartingale. Hence, by the Doob–Meyer decomposition, we have

$$\widetilde{J}_t = J_0 + m_t - A_t, \qquad 0 \leq t \leq T,$$

where m is a $(P, \mathbb{G})$-local RCLL martingale with $m_0 = 0$ and A is an increasing RCLL $\mathbb{G}$-adapted process with $A_0 = 0$ and $E[A_T] < \infty$. By Lemma 4.1, there exist an $\mathbb{R}^n$-valued process ϕ and an $\mathbb{R}^d$-valued process ψ which are $\mathbb{G}$-adapted processes P-a.s. square-integrable such that P-a.s.,

$$m_t = \int_0^t \phi'_u \, dN_u + \int_0^t \psi'_u \, dM_u, \qquad 0 \le t \le T.$$

We now want to prove that

$$\psi = 0, \qquad dt \otimes dP\text{-a.s.} \tag{A.2}$$

For each $\nu \in \mathscr{D}$, set

$$\xi_t^\nu = \exp\biggl(-\int_0^t \nu'_u dM_u - \tfrac{1}{2}\int_0^t |\nu_u|^2 \, du\biggr). \tag{A.3}$$

From Lemma A.1, for each $\nu \in \mathscr{H}$, the process given by $\{\xi_t^\nu \widetilde{J}_t,\ 0 \le t \le T\}$ is a $(P, \mathbb{G})$-supermartingale. Now, by Itô's formula applied to $\xi_t^\nu \widetilde{J}_t$, we obtain that the P-a.s. finite variational process A^ν which appears in the decomposition of the semimartingale $\xi^\nu \widetilde{J}$ is given by

$$A_t^\nu = \int_0^t \xi_u^\nu (dA_u + \psi'_u \nu_u \, du). \tag{A.4}$$

Since the semimartingale $\xi^\nu \widetilde{J}$ is a supermartingale, it follows that A^ν must be an increasing process. Fix $\nu \in \mathscr{D}$ and define the set $F_\nu = \{(t, \omega) \in [0, T] \times \Omega / \psi'_t(\omega)\nu_t(\omega) < 0\}$. Let $\nu_t^n = \nu_t \mathbf{1}_{F_\nu^c} + n\nu_t \mathbf{1}_{F_\nu}$, $n \in \mathbb{N}$. Then, $\nu_t^n \in \mathscr{D}$ and assuming that (A.2) does not hold, we get for n large enough

$$E[A_T^{\nu^n}] = E\biggl[\int_0^T \xi_u^\nu \mathbf{1}_{F_\nu^c}(dA_u + \psi'_u \nu_u \, du)\biggr] + nE\biggl[\int_0^T \xi_u^\nu \mathbf{1}_{F_\nu} \, \psi'_u \nu_u \, du\biggr] < 0,$$

which leads to a contradiction. This implies that $\psi = 0 \, dt \otimes dP$-a.s. It follows that P-a.s,

$$\widetilde{J}_t = J_0 + \int_0^t \phi'_s \, dN_s - A_t, \qquad 0 \le t \le T.$$

Now, recall that $J_t = (Z_t^0)^{-1}\widetilde{J}_t$ with $(Z_t^0)^{-1} = \exp(\int_0^t \tilde{\lambda}'_u \, d\widetilde{W}_u - \frac{1}{2}\int_0^t |\tilde{\nu}_u|^2 \, du)$.

By Itô's formula,

$$dJ_t = (Z_t^0)^{-1}\phi'_t d\widetilde{W}_t + J_t \tilde{\lambda}'_t d\widetilde{W}_t - (Z_t^0)^{-1} dA_t.$$

Hence, equality (A.1) holds with

$$\theta_t^* = \bigl(\sigma'(t, S_t, Y_t)\bigr)^{-1}\bigl((Z_t^0)^{-1}\phi_t + J_t\tilde{\lambda}_t\bigr), \qquad C_t = \int_0^t (Z_s^0)^{-1} \, dA_s. \tag{A.5}$$

It remains now to show the second part of the theorem. Let (iii) be the condition defined by the process $\{Z_t^{\nu^*} J_t,\ 0 \le t \le T\}$ is a $(P, \mathbb{G})$-martingale.

We show that conditions (i), (ii) and (iii) are equivalent. The $(P, \mathbb{G})$-supermartingale $Z^{\nu^*}J$ is a $(P, \mathbb{G})$-martingale if and only if $J_0 = E[Z_T^{\nu^*} J_T] \iff J_0 = E[Z_T^{\nu^*} H] \iff$ (i).

On the other hand, (iii) implies $A^{\nu^*}_{\cdot} = 0$, and so from (A.4) and (A.5), $A_{\cdot} = C_{\cdot} = 0$ a.s. Hence, $J_{\cdot} = X_{\cdot}^{J_0, \theta^*}$ a.s. Thus, (ii) is satisfied with $\theta = \theta^*$. On the other hand, suppose that (ii) holds. Then, $J^0 = E[Z_T^{\nu^*} H]$ and (i) holds.

REMARK A.1. In the case where the process Z^0 is supposed to be a martingale, then the previous decomposition of J can be derived as a direct consequence of Theorem 2.1.2 of El Karoui-Quenez (1995). Indeed, let P^0 be the (risk-neutral) probability measure which admits Z_T^0 as density with respect to P on $\mathscr{G}_T$. Since $(P^0, \mathbb{G})$-local martingales are all continuous, the decomposition (A.1) holds.

APPENDIX B

Proofs of Propositions 4.2 and 4.3. As in the proof of the surreplication Theorem 4.1, the results can be obtained by using dynamic control tools. Classical stochastic control results give the following dynamic charaterization of Q.

LEMMA B.1. *The process Q is characterized as the smallest RCLL process equal to* 1 *at time T such that for each $\pi \in \mathscr{A}$, the process $\{(X_t^\pi)^p Q_t, 0 \le t \le T\}$ is a $(P, \mathbb{G})$-supermartingale. If π^* is solution of the problem* (4.7), *i.e. $Q_0 = E[(X_T^{\pi^*})^p]$, then the process $\{(X_t^{\pi^*})^p Q_t, 0 \le t \le T\}$ is a $(P, \mathbb{G})$-martingale.*

PROOF OF PROPOSITION 4.2. From Lemma B.1 applied to $\pi = 0$, the process Q is a $(P, \mathbb{G})$-supermartingale. The Doob–Meyer decomposition gives that P-a.s.,

$$Q_t = Q_0 + n_t - a_t, \qquad 0 \le t \le T,$$

where n is a $(P, \mathbb{G})$-local martingale with $n_0 = 0$ and a is an increasing P-a.s. integrable RCLL $\mathbb{G}$-adapted process with $a_0 = 0$. By Lemma 4.1, there exist an $\mathbb{R}^n$-valued process ϕ and an $\mathbb{R}^d$-valued process ψ which are P-a.s. square integrable $\mathbb{G}$-adapted processes such that P-a.s.,

$$n_t = \int_0^t \phi_u' \, dN_u + \int_0^t \psi_u' \, dM_u, \qquad 0 \le t \le T.$$

By Lemma B.1, for each $\pi \in \mathscr{A}$, the process given by $\{(X_t^\pi)^p Q_t, 0 \le t \le T\}$ is a $(P, \mathbb{G})$-supermartingale. Now, by Itô's formula applied to $(X_t^\pi)^p Q_t$, we obtain that the P-a.s. finite variational process which appears in the decomposition of the semimartingale $(X^\pi)^p Q$ is given by $-a^\pi$, where

$$a_t^\pi = \int_0^t (X_u^\pi)^p \left(da_u - Q_u \frac{p(p-1)}{2} |\sigma_u' \pi_u|^2 du - p \pi_u' \sigma_u (Q_u \tilde{\lambda}_u + \phi_u) du \right),$$

where, to simplify notation, σ_t denotes $\sigma(t, S_t, Y_t)$.

The supermartingale property of $(X_t^\pi)^p Q_t$ gives that a^π must be an increasing process. Moreover, by Theorem 4.2, there exists $\pi^* \in \mathscr{A}$ such that $Q_0 = E[(X_T^{\pi^*})^p]$. Thus, by Lemma B.1, the process $(X^{\pi^*})^p Q$ is a martingale and hence $a^{\pi^*} = 0$, that is,

$$a_t = \int_0^t \left(Q_u \frac{p(p-1)}{2} |\sigma_u' \pi_u^*|^2 du + p(\pi_u^*)' \sigma_u (Q_u \tilde{\lambda}_u + \phi_u) \right) du.$$

Hence, for each $\pi \in \mathscr{A}$, the increasing property of a^π gives that, $dt \otimes dP$-a.s,

$$\begin{aligned} &Q_u \frac{p(p-1)}{2} |\sigma_u' \pi_u^*|^2 du + p(\pi_u^*)' \sigma_u (Q_u \tilde{\lambda}_u + \phi_u) \\ &\quad \geq Q_u \frac{p(p-1)}{2} |\sigma_u' \pi_u|^2 du + p(\pi_u)' \sigma_u (Q_u \tilde{\lambda}_u + \phi_u). \end{aligned}$$

Since this inequality holds for each $\pi \in \mathscr{A}$, we have

$$\begin{aligned} &Q_u \frac{p(p-1)}{2} |\sigma_u' \pi_u^*|^2 du + p(\pi_u^*)' \sigma_u (Q_u \tilde{\lambda}_u + \phi_u) \\ &\quad = \operatorname*{ess\,sup}_{\pi \in \mathscr{A}} \left\{ Q_u \frac{p(p-1)}{2} |\sigma_u' \pi_u|^2 du + p(\pi_u)' \sigma_u (Q_u \tilde{\lambda}_u + \phi_u) \right\} \\ &\quad = \frac{p}{2(1-p)} Q_u |\tilde{\lambda}_u + Q_u^{-1} \phi_u|^2, \end{aligned}$$

and also $dt \otimes dP$-a.s.,

$$\pi_t^* = \frac{(\sigma_t')^{-1} [\tilde{\lambda}_t + Q_t^{-1} \phi_t]}{1-p}$$

(note that $dt \otimes dP$-a.s., $Q_t > 0$). It follows that

$$a_t = \int_0^t \frac{p}{2(1-p)} Q_u |\tilde{\lambda}_u + Q_u^{-1} \Phi_u|^2 \, du, \qquad 0 \leq t \leq T, P \text{ a.s.},$$

and hence Q is solution of the backward equation (4.8) with $Q_T = 1$. □

Proof of Proposition 4.3. Let $\mathscr{A}'$ be the set of processes $\pi \in \mathscr{A}$ satisfying

$$E \int_0^T (X_t^\pi)^{2p} (1 + |\pi_t|^2) \, dt < \infty.$$

For each $\pi \in \mathscr{A}'$, Itô's formula gives that $d(X_t^\pi)^p Q_t^0 = dM_t^\pi - dA_t^\pi$, where A^π is an increasing process and where M^π is a martingale since $\pi \in \mathscr{A}'$ and $E[\int_0^T Q_t^2 |\sigma_t|^2 dt] < \infty$. Hence, $(X^\pi)^p Q^0$ is a supermartingale for each $\pi \in \mathscr{A}'$. By Lemma B.1 (which still holds with $\mathscr{A}'$ instead of $\mathscr{A}$), it follows that $Q^0 \geq Q$.

Furthermore, if $(X^{\pi^*})^p Q^0$ is a martingale, then we have

$$(X_t^{\pi^*})^p Q_t^0 = E\Big[(X_T^{\pi^*})^p \Big| \mathscr{G}_t \Big],$$

and hence, $Q^0 \leq Q$. The result follows. □

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