The Annals of Applied Probability
1991, Vol. 1, No. 4, 582–612

NONLINEAR REGRESSION OF STABLE RANDOM VARIABLES[1]

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Let (X_1, X_2) be an α-stable random vector, not necessarily symmetric, with $0 < \alpha < 2$. This article investigates the regression $E(X_2|X_1 = x)$ for all values of α. We give conditions for the existence of the conditional moment $E(|X_2|^p|X_1 = x)$ when $p \ge \alpha$, and we obtain an explicit form of the regression $E(X_2|X_1 = x)$ as a function of x. Although this regression is, in general, not linear, it can be linear even when the vector (X_1, X_2) is skewed. We give a necessary and sufficient condition for linearity and characterize the asymptotic behavior of the regression as $x \to \pm\infty$. The behavior of the regression functions is also illustrated graphically.

1. Introduction. The stable distributions, according to the central limit theorem, are the only limiting distributions of normalized sums of independent, identically distributed random variables; and perforce include the normal, or Gaussian, distributions as distinguished elements. Gaussian distributions and processes have long been well understood, and their utility as both stochastic modeling constructs and analytical tools is well accepted. The much richer class of non-Gaussian stable distributions and processes is the subject of a great deal of recent research, and holds much promise for use in modeling and analysis as well.

Stable distributions are defined in Section 2. They are indexed by a parameter $0 < \alpha \le 2$. The distribution is Gaussian when $\alpha = 2$ and is non-Gaussian when $0 < \alpha < 2$. A good reference for univariate stable distributions is Feller [3] and the more recent monograph of Zolotarev [17]. For multivariate distributions, see Cambanis and Miller [2] and Hardin [5].

The central limit argument often used to justify the use of a Gaussian model in applications may also be applied to support the choice of a non-Gaussian stable model. That is, if the randomness observed is the result of summing many small effects, and those effects follow a heavy-tailed distribution, then a non-Gaussian stable model may be appropriate. An important distinction between Gaussian and non-Gaussian stable distributions is that the stable distributions are heavy-tailed, always with infinite variance, and in some

Received December 1989; revised August 1990.

[1]The first author was supported by AFOSR contract F49620-82-C-0009 of the Center for Stochastic Processes at the University of North Carolina. The two other authors were supported by NSF Grant DMS-88-05627, by AFOSR Grant 89-0115 and by ONR Grant 90-J-1287 at Boston University.

AMS 1980 *subject classifications*. Primary 60E07, 62J02; secondary 60E10.

Key words and phrases. Stable random vectors, linear regression, symmetric α-stable.

cases with infinite first moment. Another distinction is that they admit asymmetry, or skewness, while a Gaussian distribution is necessarily symmetric about its mean. In certain applications, then, where an asymmetric or heavy-tailed model is called for, a stable model may be a viable candidate. In any case, the non-Gaussian stable distributions furnish tractable examples of non-Gaussian behavior and provide points of comparison with the Gaussian case, highlighting the special nature of Gaussian distributions and processes. We refer the reader to Cambanis [1], Weron [16] and Zolotarev [17], for surveys of applications. Mandelbrot and Wallis [11] use the term "Noah effect" to describe random situations characterized by high variability, where a stable model may be applicable (the biblical figure Noah lived through an unusually severe flood).

A fundamental and crucial step toward understanding stable distributions and eventually employing them in applications is to recognize their behavior under conditioning. Perhaps the first question to be answered regarding conditional behavior concerns the conditional expectation, or regression, of one stable variable given the observation of others.

In the Gaussian case, the bivariate regression is always linear and is determined solely by the first two moments:

$$E(X_2|X_1 = x) = \mu_2 + \kappa(x - \mu_1),$$

where μ_i is the mean of X_i and

$$\kappa = \frac{\operatorname{Cov}(X_2, X_1)}{\operatorname{Var} X_1}.$$

In the stable case, Kanter [8] shows that the same relation holds for symmetric stable distributions with first moments, where κ is defined to be the normalized covariation of X_2 on X_1, which is the stable analog of the normalized covariance. Also, Samorodnitsky and Taqqu [14] show that the first moment requirement may be relaxed; that is, regressions involving variables without first moments can be legitimately defined under appropriate conditions and are linear.

A distinction between Gaussian and stable distributions in the symmetric case is seen, however, in the case of regression on more than one variate. Although general multivariate regressions in the Gaussian case are always linear, the papers of Miller [12], Cambanis and Miller [2] and Hardin [4] show that multivariate regressions involving symmetric stable variates are not always linear, and they illustrate some of the complexities involved.

This article gives a complete picture of bivariate regression behavior in the general (possibly asymmetric) stable case. We show that when skewness is present, regressions can be either linear or nonlinear. We determine the form of the regression, when it exists, and give necessary and sufficient conditions for its linearity. The sometimes exotic behavior of the regression functions in the nonlinear case is illustrated graphically. Interestingly, these regressions are always asymptotically linear. We make no moment assumptions on the stable variates, assuming only a weaker condition assuring that the regression is defined.

Since we want to study $E(X_2|X_1 = x)$ for all $0 < \alpha < 2$, we must determine when the conditional expectation is defined. It is always defined for $\alpha > 1$ because the mean exists in that case. We show in Section 2 that when $\alpha \le 1$, the condition for existence of the conditional expectation, given in [14] for the symmetric stable case, is also sufficient here. Therefore the applications to moving averages, sub-Gaussian, sub-stable and harmonizable processes, discussed in [14], apply under exactly the same conditions when the vector (X_1, X_2) is skewed. The regression, however, will typically be nonlinear.

Explicit formulas for the regression involve the ratio of two integrals [see (3.9)], neither of which can be computed analytically. (The integral in the denominator is proportional to the univariate stable density function.) To make matters worse, these integrals have features which make them unamenable to the use of standard numerical integration software packages. In the past, the lack of *usable* formulas has been an impediment to the application of stable distributions to the real-life phenomena. In Hardin, Samorodnitsky and Taqqu [6], we develop efficient algorithms for computing the regression formulas and we present them in a form useful to practitioners. These numerical procedures can also be used to evaluate other integrals which appear in the context of stable distributions. The article by Hardin, Samorodnitsky and Taqqu [6], moreover, lists the complete source code, written in the C language, for computing stable density functions and regression functions. That code is used in this article to obtain the figures of stable density functions and regression functions displayed.

This article is structured as follows. Basic definitions and conditions for the finiteness of the conditional moment $E(|X_2|^p|X_1 = x)$, $p \ge \alpha$, are given in Section 2. Explicit formulas for $E(X_2|X_1 = x)$ are given in Section 3 and established in Section 4. Section 5 displays graphs of various regression functions.

2. Definitions and existence of conditional expectations. A random variable X is said to have a stable distribution if for any $A > 0$, $B > 0$, there exist a $C > 0$ and a $D \in \mathbb{R}^1$ such that

$$AX^{(1)} + BX^{(2)} =_d CX + D,$$

where $X^{(1)}$ and $X^{(2)}$ are independent copies of X. Necessarily, $C = (A^\alpha + B^\alpha)^{1/\alpha}$ for some $0 < \alpha \le 2$. The characteristic function of X, $\phi_X(t) = E\exp\{itX\}$, has the form

$$\phi_X(t) = \begin{cases} \exp\{-\sigma^\alpha|t|^\alpha + ia\beta\sigma^\alpha t^{\langle\alpha\rangle} + i\mu t\}, & \text{if } \alpha \ne 1, \\ \exp\left\{-\sigma|t| - i\dfrac{2}{\pi}\beta\sigma t\ln|t| + i\mu t\right\}, & \text{if } \alpha = 1, \end{cases} \tag{2.1}$$

where $\sigma \ge 0$, $a = \tan(\pi\alpha/2)$, $\beta \in [-1, 1]$, $\mu \in \mathbb{R}$ and where $u^{\langle v\rangle} = |u|^v \operatorname{sign} u$ for any reals u and v. Thus, in addition to the *index of stability* $\alpha \in [0, 2]$, the distribution of X is characterized by a *scale parameter* $\sigma \geq 0$, a *skewness parameter* $\beta \in [-1, 1]$ and a *shift parameter* $\mu \in \mathbb{R}$.

When $\alpha = 2$, $\phi_X(t)$ reduces to $\exp\{-\sigma^2 t^2 + i\mu t\}$, the characteristic function of a Gaussian random variable with mean μ and standard deviation $\sqrt{2}\,\sigma$. The random variable X is Cauchy when $\alpha = 1$. Figures 2 and 3 illustrate the shape of various density functions of X when $0 < \alpha < 2$. These density functions are not generally known in closed form. Their tails are much fatter than the Gaussian tails, and

$$P(|X| > \lambda) \sim \text{const}\, \lambda^{-\alpha} \quad \text{as } \lambda \to \infty. \tag{2.2}$$

The lower the α, the slower the decay of the probability tails. Because of (2.2), no α-stable random variable has finite second moment when $\alpha < 2$ and even the first moment does not exist when $\alpha \leq 1$.

The regression problem involves a pair of random variables X_1 and X_2. The definition of the univariate stable distribution generalizes readily to more than one variable. A random vector $\mathbf{X} = (X_1, X_2)$ is said to have a stable distribution if for any scalars $A > 0$, $B > 0$, there exists a $C > 0$ and a $\mathbf{D} \in \mathbb{R}^2$ such that

$$A\mathbf{X}^{(1)} + B\mathbf{X}^{(2)} =_d C\mathbf{X} + \mathbf{D},$$

where $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ are independent copies of $\mathbf{X}$. Again, $C = (A^\alpha + B^\alpha)^{1/\alpha}$ for some $0 < \alpha \leq 2$. The vector $\mathbf{X}$ is called α-stable. The vector $\mathbf{X}$ is called symmetric α-stable ($S\alpha S$) if it is α-stable and $\mathbf{X} =_d -\mathbf{X}$. Every symmetric α-stable vector $\mathbf{X}$ has characteristic function $\phi_{\mathbf{X}}(t, r) = E \exp i(tX_1 + rX_2)$ of the form

$$\exp\left\{-\int_{S_2} |ts_1 + rs_2|^\alpha \Gamma(d\mathbf{s})\right\}, \tag{2.3}$$

where Γ is a finite symmetric measure on the Borel subsets of the unit circle S_2 in $\mathbb{R}^2$. Changing Γ changes the characteristic function (2.3) of (X_1, X_2). When $\alpha = 2$, the characteristic function reduces to the usual Gaussian one, since (2.3) equals

$$\exp\left\{-\left[t^2 \int_{S_2} s_1^2 \Gamma(d\mathbf{s}) + 2tr \int_{S_2} s_1 s_2 \Gamma(d\mathbf{s}) + r^2 \int_{S_2} s_2^2 \Gamma(d\mathbf{s})\right]\right\},$$

so that $\operatorname{Var} X_1 = 2\int_{S_2} s_1^2 \Gamma(d\mathbf{s})$, $\operatorname{Var} X_2 = 2\int_{S_2} s_2^2 \Gamma(d\mathbf{s})$ and $\operatorname{Cov}(X_1, X_2) = 2\int_{S_2} s_1 s_2 \Gamma(d\mathbf{s})$.

For convenience, unless stated otherwise, we suppose from now on that $0 < \alpha < 2$; that is, we exclude the Gaussian case $\alpha = 2$.

An α-stable random vector $\mathbf{X} = (X_1, X_2)$ with $0 < \alpha < 2$ is not necessarily symmetric. The general form of its characteristic function is

$$\phi_{\mathbf{X}}(t, r) = \exp\left\{-\int_{S_2} |ts_1 + rs_2|^\alpha (1 - ia \operatorname{sign}(ts_1 + rs_2)) \Gamma(d\mathbf{s}) + i(t\mu_1^0 + r\mu_2^0)\right\}$$

if $\alpha \neq 1$, and

$$\phi_{\mathbf{X}}(t,r) = \exp\left\{ -\int_{S_2} |ts_1 + rs_2| \left(1 + i\frac{2}{\pi}\operatorname{sign}(ts_1 + rs_2)\ln|ts_1 + rs_2| \right) \Gamma(d\mathbf{s}) + i\left(t\mu_1^0 + r\mu_2^0\right) \right\}$$

if $\alpha = 1$. Here $a = \tan(\pi\alpha/2)$, Γ is as before but not necessarily symmetric, and $\boldsymbol{\mu}^0 = (\mu_1^0, \mu_2^0)$ is a vector in $\mathbb{R}^2$. The measure Γ is called the *spectral measure* and the pair $(\Gamma, \boldsymbol{\mu}^0)$ is called the *spectral representation* of the random vector $\mathbf{X}$. The spectral representation is unique when $0 < \alpha < 2$ (see Kuelbs [9], for example, for details).

The components X_1 and X_2 of $\mathbf{X}$ have a marginal α-stable distribution. In fact, X_1 has characteristic function (2.1) with $\sigma = \sigma_1$, $\beta = \beta_1$ and $\mu = \mu_1$, where

$$\sigma_1 = \left(\int_{S_2} |s_1|^\alpha \Gamma(d\mathbf{s}) \right)^{1/\alpha} \tag{2.4}$$

is the *scale parameter* of X_1,

$$\beta_1 = \frac{1}{\sigma_1^\alpha} \int_{S_2} s_1^{\langle\alpha\rangle} \Gamma(d\mathbf{s}) \in [-1, 1] \tag{2.5}$$

is the *skewness parameter* of X_1 and

$$\mu_1 = \begin{cases} \mu_1^0, & \text{if } \alpha \neq 1, \\ \mu_1^0 - \dfrac{2}{\pi}\displaystyle\int_{S_2} s_1 \ln|s_1| \Gamma(d\mathbf{s}), & \text{if } \alpha = 1, \end{cases} \tag{2.6}$$

is the shift parameter of X_1.

Kanter [8] proved that if (X_1, X_2) is $S\alpha S$ and $\alpha > 1$, then the regression $E(X_2 | X_1 = x)$ is linear and, for almost every x,

$$E(X_2 | X_1 = x) = \kappa x, \tag{2.7}$$

where

$$\kappa = \frac{1}{\sigma_1^\alpha} \int_{S_2} s_2 s_1^{\langle\alpha-1\rangle} \Gamma(d\mathbf{s}).$$

The constant $\int_{S_2} s_2 s_1^{\langle\alpha-1\rangle} \Gamma(d\mathbf{s})$ is called the *covariation* of X_2 on X_1 and is often denoted $[X_2, X_1]_\alpha$. Samorodnitsky and Taqqu [14] showed that (2.7) may still hold when (X_1, X_2) is a $S\alpha S$ vector with $\alpha \leq 1$.

The purpose of this article is to obtain the regression $E(X_2 | X_1 = x)$ when (X_1, X_2) is a general, possibly skewed α-stable random vector with $0 < \alpha < 2$. Since we want to study $E(X_2 | X_1 = x)$ for all $0 < \alpha < 2$, we must determine when the conditional expectation is defined.

The following theorem was proved in [14] in the symmetric α-stable case. We extend it here to an arbitrary α-stable vector $\mathbf{X}$. It provides conditions for the existence of conditional moments of order greater or equal to α.

THEOREM 2.1. *Let* $\mathbf{X} = (X_1, X_2)$ *have representation* $(\Gamma, \boldsymbol{\mu}^0)$ *and suppose that*

$$\int_{S_2} \frac{\Gamma(d\mathbf{s})}{|s_1|^{\nu}} < \infty \tag{2.8}$$

for some $\nu \geq 0$. *Then* $E(|X_2|^p | X_1 = x) < \infty$ *for almost every* x *if*

$$p < \begin{cases} \alpha + \nu, & \text{if } \nu < 1, \\ \alpha + 1, & \text{if } \nu \geq 1, \end{cases}$$

when $0 < \alpha < 1$, *and*

$$p < \begin{cases} \alpha + \nu, & \text{if } \nu < 2 - \alpha, \\ 2, & \text{if } \nu \geq 2 - \alpha, \end{cases}$$

when $1 \leq \alpha \leq 2$. *In the latter case, if* $\nu \geq 2 - \alpha$, *then* $E(X_2^2 | X_1 = x) < \infty$ *for a.e.* x.

PROOF. Let ν and p be as in the theorem and let (Y_1, Y_2) be an independent copy of (X_1, X_2). Then $(Z_1, Z_2) = (X_1, X_2) - (Y_1, Y_2)$ is a symmetric α-stable vector with spectral measure $\tilde{\Gamma}$, defined by $\tilde{\Gamma}(A) = \Gamma(A) + \Gamma(-A)$ for every Borel set A of S_2. Since $\int_{S_2} |s_1|^{-\nu} \tilde{\Gamma}(d\mathbf{s}) = 2\int_{S_2} |s_1|^{-\nu} \Gamma(d\mathbf{s}) < \infty$, we have

$$E\big(E(|Z_2|^p | X_1, Y_1, Y_2) | Z_1\big) = E\big(|Z_2|^p | Z_1\big) < \infty \quad \text{a.s.}$$

by [14], and consequently $E(|Z_2|^p | X_1, Y_1, Y_2) < \infty$ a.s. by Fubini's theorem. Since (Y_1, Y_2) is independent of (X_1, X_2),

$$\begin{aligned} E\big(|X_2|^p | X_1\big) &= E\big(|X_2 - Y_2 + Y_2|^p | X_1, Y_1, Y_2\big) \\ &\leq 2^p\big(E\big(|X_2 - Y_2|^p | X_1, Y_1, Y_2\big) + E\big(|Y_2|^p | X_1, Y_1, Y_2\big)\big) \\ &= 2^p\big(E\big(|Z_2|^p | X_1, Y_1, Y_2\big) + |Y_2|^p\big) \end{aligned}$$

is a.s. finite. □

REMARKS. "For almost every x" means for all x in the support of the probability density function of X_1. (When $\alpha < 1$ and $\beta_1 = 1$ for example, the support of the probability density function of X_1 is the nonnegative real line.)

Relation (2.8) is always satisfied if $\nu = 0$ because Γ is a finite measure. In this case Theorem 2.1 merely tells us what we already know, that the (conditional) moment of order p exists for $p < \alpha$. If relation (2.8) is satisfied for some $\nu > 0$, then the conditional moment $E(|X_2|^p | X_1 = x)$ may exist with $p \geq \alpha$, even though the unconditional moment $E|X_2|^p$ is infinite. For an alternative expression of relation (2.8), see Proposition 3.1 below.

To understand the significance of condition (2.8), consider the following extreme situation where Γ concentrates its mass on the points $(0, 1)$ and

$(0, -1)$ of the unit circle S_2. Then X_1 is constant, $E(|X_2|^p | X_1 = x) = E|X_2|^p$ a.s. and hence the conditional moment is finite if and only if $p < \alpha$. Theorem 2.1 states that, up to some limit, the lower the density of Γ around the points $(0, 1)$ and $(0, -1)$, the higher the admissible conditional moments. Refer to Samorodnitsky and Taqqu [14] for more details.

3. Analytic representations of the nonlinear regression functions. Let (X_1, X_2) be α-stable, $0 < \alpha < 2$, with spectral representation $(\Gamma, \boldsymbol{\mu}^0)$. In order to study the regression $E(X_2|X_1 = x)$, we must ensure that it is well defined. Clearly,

$$\alpha > 1 \Rightarrow E|X_2| < \infty \Rightarrow E\big(|X_2|\,|X_1 = x\big) < \infty \quad \text{for a.e. } x.$$

When $\alpha \leq 1$, we make the following assumption:

STANDARD ASSUMPTION. *If (X_1, X_2) is α-stable with spectral measure Γ and $\alpha \leq 1$, then there is a number $\nu > 1 - \alpha$ such that*

$$\int_{S_2} \frac{\Gamma(d\mathbf{s})}{|s_1|^{\nu}} < \infty. \tag{3.1}$$

Theorem 2.1 then ensures that $E(|X_2|\,|X_1 = x) < \infty$ for a.e. x. Observe that the choice $\nu > 0$ is adequate when $\alpha = 1$ and the choice $\nu = 1$ is adequate for all $\alpha \leq 1$.

Since we are interested in the regression $E(X_2|X_1 = x)$ as a function of x, we assume $\sigma_1 > 0$, because $\sigma_1 = 0$ implies X_1 degenerate and hence $E(X_2|X_1) = EX_2$. Because of (2.4), $\sigma_1 > 0$ is equivalent to

$$\Gamma(S_2 \setminus \{(0,1) \cup (0,-1)\}) > 0. \tag{3.2}$$

We also assume, without loss of generality, $\boldsymbol{\mu}^0 = (\mu_1^0, \mu_2^0) = \mathbf{0}$; that is, (X_1, X_2) has representation $(\Gamma, 0)$, because if $\boldsymbol{\mu}^0 \neq \mathbf{0}$, then setting $X_1 = \tilde{X}_1 + \mu_1^0$ and $X_2 = \tilde{X}_2 + \mu_2^0$ yields

$$E(X_2|X_1 = x) = E\big(\tilde{X}_2|\tilde{X}_1 = x - \mu_1^0\big) + \mu_2^0,$$

with $(\tilde{X}_1, \tilde{X}_2)$ having representation $(\tilde{\Gamma}, \mathbf{0})$.

When $\boldsymbol{\mu}^0 = \mathbf{0}$, the density function of X_1, $f_{X_1}(x) = (1/2\pi)\int_{-\infty}^{+\infty} e^{-itx}\phi_{X_1}(t)\,dt$, equals

$$\begin{aligned} f_{X_1}(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx} \exp\{-\sigma_1^{\alpha}|t|^{\alpha} + ia\beta_1\sigma_1^{\alpha}t^{\langle\alpha\rangle}\}\,dt \\ &= \frac{1}{\pi}\int_0^{\infty} e^{-\sigma_1^{\alpha}t^{\alpha}} \cos(tx - a\beta_1\sigma_1^{\alpha}t^{\alpha})\,dt \end{aligned} \tag{3.3}$$

if $\alpha \neq 1$, and

$$(3.4)\qquad \begin{aligned} f_{X_1}(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx}\exp\left\{-\sigma_1|t| - i\beta_1\sigma_1\frac{2}{\pi}t\ln|t| + i\mu_1 t\right\}dt \\ &= \frac{1}{\pi}\int_0^{\infty} e^{-\sigma_1 t}\cos\left(t(x-\mu_1) + \frac{2}{\pi}\beta_1\sigma_1 t\ln t\right)dt \end{aligned}$$

if $\alpha = 1$, where $\mu_1 = -(2/\pi)\int_{S_2} s_1 \ln|s_1|\Gamma(d\mathbf{s})$.

The following theorem provides an explicit formula for the regression in the case $\alpha \neq 1$.

THEOREM 3.1 (Case $\alpha \neq 1$). *Let (X_1, X_2) be α-stable, $\alpha \neq 1$, with spectral representation $(\Gamma, \mathbf{0})$. If $0 < \alpha < 1$, let (X_1, X_2) satisfy the standard assumption.*

Then, for almost every x,

$$(3.5)\qquad E(X_2|X_1 = x) = \kappa x + \frac{a(\lambda - \beta_1\kappa)}{1 + a^2\beta_1^2}\left[a\beta_1 x + \frac{1 - xH(x)}{\pi f_{X_1}(x)}\right],$$

where

$$(3.6)\qquad H(x) = \int_0^{\infty} e^{-\sigma_1^\alpha t^\alpha}\sin(tx - a\beta_1\sigma_1^\alpha t^\alpha)\,dt,$$

$$(3.7)\qquad \kappa = \frac{[X_2, X_1]_\alpha}{\sigma_1^\alpha} = \frac{\int_{S_2} s_2 s_1^{\langle\alpha-1\rangle}\Gamma(d\mathbf{s})}{\sigma_1^\alpha},$$

$$(3.8)\qquad \lambda = \frac{\int_{S_2} s_2|s_1|^{\alpha-1}\Gamma(d\mathbf{s})}{\sigma_1^\alpha},$$

and where $a = \tan(\pi\alpha/2)$, and σ_1, β_1 and f_{X_1} are respectively the scale parameter (2.4), the skewness parameter (2.5) and the probability density function (3.3) of the random variable X_1.

If $\alpha < 1$ and $\beta_1 = 1$, relation (3.5) makes sense only for $x \geq 0$, and if $\alpha < 1$ and $\beta_1 = -1$, it makes sense only for $x \leq 0$.

REMARKS.

1. To understand the reason for the last statement in the theorem, recall that when $\alpha < 1$ and $\beta_1 = 1$, the random variable X_1 is totally skewed to the right and when $\alpha < 1$ and $\beta_1 = -1$, X_1 is totally skewed to the left. The density function $f_{X_1}(x)$ vanishes for $x < 0$ when $\beta_1 = 1$ and it vanishes for $x > 0$ when $\beta_1 = -1$. Therefore conditioning with respect to $X_1 = x$ makes no sense when $x < 0$ if $\beta_1 = 1$ or when $x > 0$ if $\beta_1 = -1$. When either $\alpha < 1$, $\beta_1 \neq \pm 1$ or $\alpha \geq 1$, the support of the density $f_{X_1}(x)$ is the whole real line.

2. As can easily be seen from the proof of the theorem, the following expression is equivalent to (3.5):

$$E(X_2|X_1 = x) = \kappa x + \alpha\sigma_1^\alpha a(\lambda - \beta_1\kappa) \times \frac{\int_0^\infty e^{-\sigma_1^\alpha t^\alpha} t^{\alpha-1}\cos(xt - a\beta_1\sigma_1^\alpha t^\alpha)\,dt}{\int_0^\infty e^{-\sigma_1^\alpha t^\alpha}\cos(xt - a\beta_1\sigma_1^\alpha t^\alpha)\,dt}. \tag{3.9}$$

   Relation (3.9) is particularly useful for evaluating $E(X_2|X_1 = x)$ numerically.
3. The constants κ and λ in the theorem are finite when $\alpha \le 1$, because, by (3.1),

$$\left|\int_{S_2} s_2|s_1|^{\alpha-1}\Gamma(d\mathbf{s})\right| \le \int_{S_2} |s_2|\,|s_1|^{\alpha-1+\nu}\frac{\Gamma(d\mathbf{s})}{|s_1|^\nu} \le \int_{S_2}\frac{\Gamma(d\mathbf{s})}{|s_1|^\nu} < \infty,$$

   since $|s_1| \le 1$, $|s_2| \le 1$ and $\alpha - 1 + \nu > 0$.
4. If $\mathbf{X}$ is symmetric α-stable, $\alpha \ne 1$, we have Γ symmetric, $\beta_1 = 0$ and $\lambda = 0$ and hence

$$E(X_2|X_1 = x) = \kappa x \quad \text{for a.e. } x.$$

   We thus recover the result of [8] in the case $\alpha > 1$ and that of [14] in the case $\alpha \le 1$.
5. If X_1 is (marginally) symmetric, then $\beta_1 = 0$ and

$$E(X_2|X_1 = x) = \kappa x + \left(\tan\frac{\pi\alpha}{2}\right)\lambda\frac{1}{\pi f_{X_1}(x)}\left(1 - x\int_0^\infty e^{-\sigma_1^\alpha t^\alpha}\sin tx\,dt\right).$$

We now turn to the case $\alpha = 1$.

THEOREM 3.2 (Case $\alpha = 1$). *Let (X_1, X_2) be α-stable with $\alpha = 1$ and spectral representation $(\Gamma, \mathbf{0})$ satisfying the standard assumption. Then, for almost every x,*

$$E(X_2|X_1 = x) = -\frac{2\sigma_1}{\pi}k_0 + \kappa(x - \mu_1) + \frac{\lambda - \beta_1\kappa}{\beta_1}\left[(x - \mu_1) - \sigma_1\frac{U(x)}{xf_{X_1}(x)}\right] \tag{3.10}$$

if $\beta_1 \ne 0$, and

$$E(X_2|X_1 = x) = -\frac{2\sigma_1}{\pi}k_0 + \kappa(x - \mu_1) - \frac{2\sigma_1}{\pi}\lambda\frac{V(x)}{\pi f_{X_1}(x)} \tag{3.11}$$

if $\beta_1 = 0$. *Here*

$$(3.12)\quad U(x) = \int_0^\infty e^{-\sigma_1 t} \sin\left(t(x-\mu_1) + \frac{2}{\pi}\beta_1\sigma_1 t \ln t\right) dt,$$

$$(3.13)\quad V(x) = \int_0^\infty e^{-\sigma_1 t}(1+\ln t)(\cos t(x-\mu_1))\, dt,$$

$$k_0 = \frac{1}{\sigma_1}\int_{S_2} s_2 \ln|s_1|\Gamma(d\mathbf{s}),$$

$$\kappa = \frac{[X_2, X_1]_1}{\sigma_1} = \frac{1}{\sigma_1}\int_{S_2} s_2 s_1^{\langle 0\rangle}\Gamma(d\mathbf{s}) = \frac{1}{\sigma_1}\int_{S_2} s_2 \operatorname{sign} s_1 \Gamma(d\mathbf{s}),$$

$$\lambda = \frac{1}{\sigma_1}\int_{S_2} s_2\Gamma(d\mathbf{s})$$

and

$$(3.14)\qquad \mu_1 = -\frac{2}{\pi}\int_{S_2} s_1 \ln|s_1|\Gamma(d\mathbf{s}).$$

Remarks.

1. The shift parameter μ_1 also appears in the expression (3.4) of $f_{X_1}(x)$.
2. The constants κ and λ are defined as in Theorem 3.1. When $\alpha = 1$, the constant λ is proportional to the skewness parameter of X_2.
3. If $\beta_1 \neq 0$,
$$\frac{U(x)}{\pi f_{X_1}(x)} = \frac{\operatorname{Im} \int_0^\infty e^{-itx}\phi_{X_1}(t)\, dt}{\operatorname{Re} \int_0^\infty e^{-itx}\phi_{X_1}(t)\, dt}.$$
4. If $\mathbf{X} = (X_1, X_2)$ is symmetric, then Γ is symmetric and $\mu_1 = \beta_1 = k_0 = \lambda = 0$. Hence by (3.11)
$$E(X_2|X_1 = x) = \kappa x \quad \text{for a.e. } x,$$
as established in [14].

The following corollary shows that the regression is linear when X_1 is totally skewed to the right ($\beta_1 = 1$) or when it is totally skewed to the left ($\beta_1 = -1$).

Corollary 3.1 (Case $\beta_1 = \pm 1$). *Suppose that the conditions of Theorems* 3.1 *or* 3.2 *hold. If* $\beta_1 = \pm 1$, *then for almost every* x,

$$(3.15)\qquad E(X_2|X_1 = x) = \begin{cases} \kappa x, & \text{if } \alpha \neq 1, \\ -\dfrac{2\sigma_1}{\pi}k_0 - \kappa\mu_1 + \kappa x, & \text{if } \alpha = 1. \end{cases}$$

In the case $\alpha < 1$, *this relation makes sense only for* $x \geq 0$ *when* $\beta_1 = 1$, *and for* $x \leq 0$ *when* $\beta_1 = -1$.

PROOF. $\beta_1 = 1$ implies $s_1 \geq 0$ a.e. Γ by (2.4) and (2.5), and therefore $\lambda = \kappa$ by (3.7) and (3.8). Similarly, $\beta_1 = -1$ implies $s_1 \leq 0$ a.e. Γ and $\kappa = -\lambda$. In both cases, $\lambda - \beta_1\kappa = 0$, and the corollary follows from Theorems 3.1 and 3.2. $\square$

The next result shows that the regression $E(X_2|X_1 = x)$ is always asymptotically linear as $x \to \pm\infty$. We know from Corollary 3.1 that the regression is linear when $\beta_1 = \pm 1$. For other values of β_1, one has the following corollary.

COROLLARY 3.2 (Asymptotic relations). *Let (X_1, X_2) be α-stable, $0 < \alpha < 2$, with spectral representation $(\Gamma, \mathbf{0})$ and suppose that the standard assumption holds if $0 < \alpha \leq 1$. Then for $\beta_1 \neq \pm 1$,*

$$E(X_2|X_1 = x) \sim \frac{\kappa + \lambda}{1 + \beta_1}x, \qquad x \to \infty, \tag{3.16}$$

and

$$E(X_2|X_1 = x) \sim \frac{\kappa - \lambda}{1 - \beta_1}x, \qquad x \to -\infty. \tag{3.17}$$

The following corollary gives a necessary and sufficient condition for linearity of the regression.

COROLLARY 3.3 (Linearity). *Let (X_1, X_2) be α-stable, $0 < \alpha < 2$, with spectral representation $(\Gamma, \mathbf{0})$ and suppose that the standard assumption holds if $0 < \alpha \leq 1$. Then the regression $E(X_2|X_1 = x)$ is linear if and only if*

$$\lambda = \beta_1\kappa. \tag{3.18}$$

If $\lambda = \beta_1\kappa$, then $E(X_2|X_1 = x)$ is given by (3.15) for a.e. x.

Examples.

EXAMPLE 1. The regression can be linear even though (X_1, X_2) is not symmetric. For example, suppose $\alpha > 1$ and let $\mathbf{X} = (X_1, X_2)$ have spectral measure

$$\Gamma = \delta\left(\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right) + \delta\left(\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right) + \delta((0, -1)),$$

where $\delta((x_1, x_2))$ puts a unit mass at the point (x_1, x_2). The vector $\mathbf{X}$ is not symmetric because Γ is not symmetric. However, $\beta_1 = 0$ (X_1 is $S\alpha S$) and $\lambda = 0$. Since (3.18) is satisfied in this case, the regression is linear. The slope is $\kappa = 1$ because both σ_1^α and the covariation $\int_{S_2} s_2 s_1^{\langle\alpha-1\rangle}\Gamma(d\mathbf{s})$ equal $2(1/\sqrt{2})^\alpha$.

EXAMPLE 2. The regression may be nonlinear even though both components X_1 and X_2 are $S\alpha S$. For example, let $\alpha > 1$ and $\mathbf{X} = (X_1, X_2)$ have spectral measure

$$\Gamma = \delta((1,0)) + \delta((0,1)) + 2^{\alpha/2}\delta\left(\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right).$$

In this case, $\mathbf{X}$ is not symmetric, X_1 and X_2 are $S\alpha S$ and the regression is not linear because $\beta_1 = 0$ but

$$\sigma_1^\alpha \lambda = \int_{S_2} s_2 |s_1|^\alpha \Gamma(d\mathbf{s}) = -2^{\alpha/2}\left(\frac{1}{\sqrt{2}}\right)^{\alpha+1} \neq 0.$$

Integral representation. In applications, (X_1, X_2) is often given through its integral representation

$$(X_1, X_2) =_d \left(\int_E f_1(x) M(dx), \int_E f_2(x) M(dx)\right), \tag{3.19}$$

where M is an α-stable random measure on the measurable space $(E, \mathscr{E})$ with control measure m and skewness intensity $\beta(\cdot)$: $E \to [-1, 1]$. Moreover, f_j: $E \to \mathbb{R}^1$, $j = 1, 2$, satisfies

$$\int_E |f_j(x)|^\alpha m(dx) < \infty,$$

and it also satisfies the condition

$$\int_E |f_j(x)\beta(x) \ln|f_j(x)|\,| m(dx) < \infty$$

if $\alpha = 1$ (see [5] and [13]). The following proposition expresses condition (3.1) and the constants of the regression in terms of f_1, f_2, $\beta(\cdot)$ and $m(dx)$.

PROPOSITION 3.1.

$$\int_{S_2} \frac{\Gamma(d\mathbf{s})}{|s_1|^\nu} < \infty \Rightarrow \int_{E_+} \frac{|f_2(x)|^{\alpha+\nu}}{|f_1(x)|^\nu} m(dx) < \infty, \tag{3.20}$$

where $E_+ = \{x \in E: f_1^2(x) + f_2^2(x) \neq 0\}$. *Moreover,*

$$\sigma_1^\alpha = \int_E |f_1(x)|^\alpha m(dx),$$

$$\beta_1 = \frac{1}{\sigma_1^\alpha} \int_E f_1(x)^{\langle\alpha\rangle} \beta(x) m(dx),$$

$$\mu_1 = -\frac{2}{\pi} \int_{E_+} f_1(x) \left(\ln \frac{|f_1(x)|}{\sqrt{f_1^2(x) + f_2^2(x)}}\right) \beta(x) m(dx) \qquad (\textit{case } \alpha = 1),$$

$$k_0 = \frac{1}{\sigma_1} \int_{E_+} f_2(x) \left(\ln \frac{|f_1(x)|}{\sqrt{f_1^2(x) + f_2^2(x)}}\right) \beta(x) m(dx) \qquad (\textit{case } \alpha = 1),$$

$$\kappa = \frac{[X_2, X_1]_\alpha}{\sigma_1^\alpha} = \frac{1}{\sigma_1^\alpha} \int_{E_+} f_2(x) f_1(x)^{\langle\alpha-1\rangle} m(dx),$$

$$\lambda = \frac{1}{\sigma_1^\alpha} \int_{E_+} f_2(x) |f_1(x)|^{\alpha-1} \beta(x) m(dx).$$

Any α-stable vector with spectral representation $(\Gamma, \mathbf{0})$ has an integral representation (3.19) with $\beta(\cdot) \equiv 1$. In fact, if $\alpha \neq 1$, it has one with $(E, \mathscr{E}) = ([0,1], \mathscr{B})$, $\beta(\cdot) \equiv 1$ and $m(dx) = dx$ (see [5] and [13]). See Section 5 for an example.

4. Proofs.

Proof of Theorem 3.1. Since (3.2) holds, the characteristic function $\phi_{\mathbf{X}}(t, r)$ of $\mathbf{X} = (X_1, X_2)$ is absolutely integrable with respect to t and therefore the conditional characteristic function $\phi_{X_2|x}(r)$ of X_2 given $X_1 = x$ equals

$$\phi_{X_2|x}(r) = 1 + \frac{1}{2\pi f_{X_1}(x)} \int_{-\infty}^{+\infty} e^{-itx}\big(\phi_{\mathbf{X}}(t,r) - \phi_{X_1}(t)\big)\, dt.$$

(See Section 2 of [14].) Hence, for almost any x,

$$\begin{aligned} E(X_2|X_1 = x) &= -i\phi'_{X_2|x}(0) \\ &= -\frac{i}{2\pi f_{X_1}(x)}\left(\frac{\partial}{\partial r}\int_{-\infty}^{+\infty} e^{-itx}\phi_{\mathbf{X}}(t,r)\, dt\bigg|_{r=0}\right). \end{aligned} \tag{4.1}$$

We start with the evaluation of $(\partial/\partial r)\phi_{\mathbf{X}}(t,r)|_{r=0}$. For any $t \neq 0$,

$$\frac{\partial}{\partial r}\phi_{\mathbf{X}}(t,r)\bigg|_{r=0} = \phi_{\mathbf{X}}(t,0)\lim_{r\to 0}\frac{1}{r}\left(\frac{\phi_{\mathbf{X}}(t,r)}{\phi_{\mathbf{X}}(t,0)} - 1\right), \tag{4.2}$$

where $\phi_{\mathbf{X}}(t,r)/\phi_{\mathbf{X}}(t,0) = e^{-u(t,r)}$ with

$$\begin{aligned} u(t,r) = \int_{S_2} \big[&|ts_1 + rs_2|^\alpha(1 - i a\,\mathrm{sign}(ts_1 + rs_2)) \\ &- |ts_1|^\alpha(1 - ia\,\mathrm{sign}\, ts_1)\big]\Gamma(d\mathbf{s}). \end{aligned}$$

Since $u(t,r) \to 0$ as $r \to 0$, we get

$$\begin{aligned} \lim_{r\to 0}\frac{1}{r}\left(\frac{\phi_{\mathbf{X}}(t,r)}{\phi_{\mathbf{X}}(t,0)} - 1\right) &= \lim_{r\to 0}\frac{\exp\{-u(t,r)\} - 1}{u(t,r)}\lim_{r\to 0}\frac{u(t,r)}{r} \\ &= -\lim_{r\to 0}\frac{u(t,r)}{r} \\ &= -\left[\lim_{r\to 0}\int_{S_2}\frac{1}{r}\big(|ts_1 + rs_2|^\alpha - |ts_1|^\alpha\big)\Gamma(d\mathbf{s})\right. \\ &\qquad \left. - ia\lim_{r\to 0}\int_{S_2}\frac{1}{r}\big((ts_1 + rs_2)^{\langle\alpha\rangle} - (ts_1)^{\langle\alpha\rangle}\big)\Gamma(d\mathbf{s})\right] \\ &:= -[L_1 - iaL_2]. \end{aligned}$$

Clearly,

$$L_1 = \alpha t^{\langle \alpha-1\rangle} \int_{S_2} s_2 s_1^{\langle \alpha-1\rangle} \Gamma(d\mathbf{s}) = \alpha \sigma_1^\alpha \kappa t^{\langle \alpha-1\rangle}$$

by (3.7). To evaluate L_2, we express the corresponding integral as a sum of two integrals $Q_1(t,r)$ and $Q_2(t,r)$, the first over $S_2 \cap \{\mathbf{s}\colon |ts_1| \geq 2|r|\}$ and the second over $S_2 \cap \{\mathbf{s}\colon |ts_1| < 2|r|\}$. The mean value theorem gives

$$(ts_1 + rs_2)^{\langle \alpha\rangle} - (ts_1)^{\langle \alpha\rangle} = \alpha s_2 r u^{\alpha-1},$$

where $u \in (|ts_1| \wedge |ts_1 + rs_2|, |ts_1| \vee |ts_1 + rs_2|)$, and so, for any $s_1 \neq 0$, the integrand of Q_1 converges to $s_2|ts_1|^{\alpha-1}$. The integrand is dominated by an integrable function, because $|ts_1| \geq 2|r|$ implies $u \in [|ts_1|/2, 2|t|]$ and therefore $|u|^{\alpha-1} \leq |ts_1/2|^{\alpha-1} + |2t|^{\alpha-1}$. This is certainly integrable with respect to Γ if $\alpha > 1$. If $\alpha < 1$, on S_2,

$$|u|^{\alpha-1} \leq C|t|^{\alpha-1}\left(|s_1|^{\alpha-1+\nu}|s_1|^{-\nu} + 1\right) \leq C|t|^{\alpha-1}\left(|s_1|^{-\nu} + 1\right),$$

which, by (3.1), is integrable with respect to Γ. Therefore, by the dominated convergence theorem,

$$\lim_{r\to 0} Q_1(t,r) = \alpha|t|^{\alpha-1} \int_{S_2} s_2|s_1|^{\alpha-1}\Gamma(d\mathbf{s}).$$

Now, consider

$$Q_2(t,r) = \int_{S_2 \cap \{(s_1,s_2)\colon |ts_1|<2|r|\}} \frac{1}{r}\left((ts_1 + rs_2)^{\langle \alpha\rangle} - (ts_1)^{\langle \alpha\rangle}\right)\Gamma(d\mathbf{s}).$$

Suppose first $\alpha < 1$. When $|ts_1| < 2|r|$ and $(s_1, s_2) \in S_2$, the integrand is majorized by

$$|r|^{-1}C|rs_2|^\alpha \leq C|r|^{\alpha-1}|s_1|^\nu|s_1|^{-\nu} \leq C'|r|^{\alpha+\nu-1}|t|^{-\nu}|s_1|^{-\nu},$$

which is integrable with respect to Γ. Since $\alpha + \nu > 1$, we get

$$\lim_{r\to 0} Q_2(t,r) = 0.$$

The case $\alpha > 1$ is similar. Therefore, by (3.8),

$$L_2 = \alpha|t|^{\alpha-1}\int_{S_2} s_2|s_1|^{\alpha-1}\Gamma(d\mathbf{s}) = \alpha\sigma_1^\alpha \lambda|t|^{\alpha-1},$$

and we conclude by (4.2),

$$\left.\frac{\partial \phi_{\mathbf{X}}(t,r)}{\partial r}\right|_{r=0} = e^{-\sigma_1^\alpha|t|^\alpha} e^{i\alpha\beta_1\sigma_1^\alpha t^{\langle\alpha\rangle}} \alpha\sigma_1^\alpha\left(\kappa t^{\langle \alpha-1\rangle} - i\alpha\lambda|t|^{\alpha-1}\right) \tag{4.3}$$

for almost all t.

A similar argument applied to (4.1) yields

$$E(X_2|X_1 = x) = -\frac{i}{2\pi f_{X_1}(x)}\int_{-\infty}^{+\infty} e^{-itx}\left(\left.\frac{\partial}{\partial r}\phi_{\mathbf{X}}(t,r)\right|_{r=0}\right)dt. \tag{4.4}$$

Substituting (4.3) in (4.4), we get

$$
\begin{aligned}
E(X_2|X_1 = x) & \\
&= \frac{i\alpha\sigma_1^\alpha}{2\pi f_{X_1}(x)} \int_{-\infty}^{+\infty} e^{-itx} e^{ia\beta_1\sigma_1^\alpha t^{\langle\alpha\rangle}} e^{-\sigma_1^\alpha |t|^\alpha} \left(\kappa t^{\langle\alpha-1\rangle} - ia\lambda |t|^{\alpha-1}\right) dt \qquad (4.5)\\
&= \frac{\alpha\sigma_1^\alpha}{\pi f_{X_1}(x)} \left[\kappa(I_{11} + I_{12}) + a\lambda(I_{21} + I_{22})\right],
\end{aligned}
$$

where

$$
\begin{aligned}
I_{11} &= \int_0^\infty \sin tx \cos(a\beta_1\sigma_1^\alpha t^\alpha) e^{-\sigma_1^\alpha t^\alpha} t^{\alpha-1}\, dt,\\
I_{12} &= -\int_0^\infty \cos tx \sin(a\beta_1\sigma_1^\alpha t^\alpha) e^{-\sigma_1^\alpha t^\alpha} t^{\alpha-1}\, dt,\\
I_{21} &= \int_0^\infty \cos tx \cos(a\beta_1\sigma_1^\alpha t^\alpha) e^{-\sigma_1^\alpha t^\alpha} t^{\alpha-1}\, dt,\\
I_{22} &= \int_0^\infty \sin tx \sin(a\beta_1\sigma_1^\alpha t^\alpha) e^{-\sigma_1^\alpha t^\alpha} t^{\alpha-1}\, dt.
\end{aligned}
$$

After integrating by parts,

$$
\begin{aligned}
I_{11} &= -\frac{x}{K}\int_0^\infty e^{-\sigma_1^\alpha t^\alpha}\left[a\beta_1 \sin(a\beta_1\sigma_1^\alpha t^\alpha) - \cos(a\beta_1\sigma_1^\alpha t^\alpha)\right]\cos tx\, dt,\\
I_{12} &= -\frac{a\beta_1}{K} + \frac{x}{K}\int_0^\infty e^{-\sigma_1^\alpha t^\alpha}\left[a\beta_1 \cos(a\beta_1\sigma_1^\alpha t^\alpha) + \sin(a\beta_1\sigma_1^\alpha t^\alpha)\right]\sin tx\, dt,\\
I_{21} &= \frac{1}{K} + \frac{x}{K}\int_0^\infty e^{-\sigma_1^\alpha t^\alpha}\left[a\beta_1 \sin(a\beta_1\sigma_1^\alpha t^\alpha) - \cos(a\beta_1\sigma_1^\alpha t^\alpha)\right]\sin tx\, dt,\\
I_{22} &= -\frac{x}{K}\int_0^\infty e^{-\sigma_1^\alpha t^\alpha}\left[a\beta_1 \cos(a\beta_1\sigma_1^\alpha t^\alpha) + \sin(a\beta_1\sigma_1^\alpha t^\alpha)\right]\cos tx\, dt,
\end{aligned}
$$

where $K = \alpha\sigma_1^\alpha(1 + a^2\beta_1^2)$. Therefore, by (3.3) and (3.6),

$$
\begin{aligned}
I_{11} + I_{12} &= \frac{1}{\alpha\sigma_1^\alpha(1 + a^2\beta_1^2)}\left[-a\beta_1 + x\pi f_{X_1}(x) + a\beta_1 x H(x)\right],\\
I_{21} + I_{22} &= \frac{1}{\alpha\sigma_1^\alpha(1 + a^2\beta_1^2)}\left[1 + a\beta_1 x\pi f_{X_1}(x) - xH(x)\right].
\end{aligned}
$$

Substituting these expressions in (4.5) yields

$$
\begin{aligned}
E(X_2|X_1 = x) &= \frac{\kappa + a^2\beta_1\lambda}{1 + a^2\beta_1^2}x + \frac{a(\lambda - \beta_1\kappa)}{1 + a^2\beta_1^2}\left[\frac{1}{\pi f_{X_1}(x)} - \frac{xH(X)}{\pi f_{X_1}(x)}\right]\\
&= \kappa x + \frac{a(\lambda - \beta_1\kappa)}{1 + a^2\beta_1^2}\left[a\beta_1 x + \frac{1 - xH(x)}{\pi f_{X_1}(x)}\right]. \qquad \square
\end{aligned}
$$

PROOF OF THEOREM 3.2. We have, as in (4.2),

$$\frac{\partial \phi_X(t,r)}{\partial r}\bigg|_{t=0} = \phi_{\mathbf{X}}(t,0) \lim_{r\to 0} \frac{1}{r}\left(\frac{\phi_{\mathbf{X}}(t,r)}{\phi_{\mathbf{X}}(t,0)} - 1\right)$$

$$:= -\phi_{\mathbf{X}}(t,0)\left[L_1 + i\frac{2}{\pi}L_2\right], \tag{4.6}$$

where

$$L_1 = \lim_{r\to 0}\int_{S_2} \frac{1}{r}\left[|ts_1 + rs_2| - |ts_1|\right]\Gamma(d\mathbf{s}),$$

$$L_2 = \lim_{r\to 0}\int_{S_2} \frac{1}{r}\left[(ts_1 + rs_2)\ln|ts_1 + rs_2| - ts_1 \ln|ts_1|\right]\Gamma(d\mathbf{s}).$$

Clearly,

$$L_1 = \operatorname{sign} t \int_{S_2} s_2 \operatorname{sign} s_1 \Gamma(d\mathbf{s}) = \sigma_1 \kappa t^{\langle 0\rangle}. \tag{4.7}$$

As in the case $\alpha \neq 1$, to evaluate L_2, we write $L_2 = \lim_{r\to 0}(Q_1(t,r) + Q_2(t,r))$, where Q_1 involves integration over $S_2 \cap \{\mathbf{s}\colon |ts_1| \geq 2|r|\}$ and Q_2 over $S_2 \cap \{\mathbf{s}\colon |ts_1| < 2|r|\}$, and we apply the dominated convergence theorem.

We give details only for Q_2. Let $f\colon [0,\infty) \to [0,\infty)$ be defined by $f(r) = r|\ln r|$, $f(0) = 0$. For $|r|$ small enough ($0 < |r| < e^{-1}$), f is monotone increasing, and therefore, when $|ts_1| < 2|r|$ and $0 < |r| < (3e)^{-1}$, one has

$$\begin{aligned}|r|^{-1}\big(f(|ts_1 + rs_2|) + f(|ts_1|)\big) &\leq |r|^{-1}\big(f(3|r|) + f(2|r|)\big)\\ &\leq 2|r|^{-1}f(3|r|) \leq 6|\ln 3|r|\,| \leq 6|\ln 1/(\tfrac{3}{2}|ts_1|)|\\ &\leq 6\big(|\ln(\tfrac{2}{3}|t|)| + |s_1|^{-\nu/2}\big),\end{aligned}$$

which is integrable by (3.1). Applying the dominated convergence theorem, we get $\lim_{r\to 0} Q_2(t,r) = 0$. Therefore,

$$\begin{aligned}L_2 &= \lim_{r\to 0} Q_1(t,r) + \lim_{r\to 0} Q_2(t,r)\\ &= \int_{S_2} s_2(1 + \ln|ts_1|)\Gamma(d\mathbf{s})\\ &= (1 + \ln|t|)\int_{S_2} s_2\Gamma(d\mathbf{s}) + \int_{S_2} s_2 \ln|s_1|\Gamma(d\mathbf{s})\\ &= \sigma_1\lambda(1 + \ln|t|) + \sigma_1 k_0.\end{aligned} \tag{4.8}$$

Substituting (4.7) and (4.8) in (4.6) yields

$$\left.\frac{\partial \phi_{\mathbf{X}}(t,r)}{\partial r}\right|_{r=0} = -\phi_{\mathbf{X}}(t,0)\sigma_1\left[\kappa t^{\langle 0\rangle} + i\frac{2}{\pi}\lambda(1+\ln|t|) + i\frac{2}{\pi}k_0\right].$$

The dominated convergence theorem applied to (4.1), with μ_1 as in (3.14), gives

$$\begin{aligned} E(X_2|X_1=x) &= -\frac{i}{2\pi f_{X_1}(x)}\int_{-\infty}^{+\infty} e^{-itx}\left(\left.\frac{\partial\phi_{\mathbf{X}}(t,r)}{\partial r}\right|_{r=0}\right)dt \\ &= \frac{i\sigma_1}{2\pi f_{X_1}(x)}\int_{-\infty}^{+\infty} e^{-it(x-\mu_1)}e^{-i(2/\pi)\beta_1\sigma_1 t\ln|t|}e^{-\sigma_1|t|} \\ &\quad\times\left[\kappa t^{\langle 0\rangle} + i\frac{2}{\pi}\lambda(1+\ln|t|) + i\frac{2}{\pi}k_0\right]dt \\ &= \frac{\sigma_1}{\pi f_{X_1}(x)}\left[\kappa U(x) - \frac{2}{\pi}\lambda(I_{21}+I_{22}) - \frac{2}{\pi}k_0\pi f_{X_1}(x)\right], \end{aligned} \tag{4.9}$$

where

$$I_{21} = \int_0^\infty e^{-\sigma_1 t}\cos t(x-\mu_1)(1+\ln t)\cos\left(\frac{2}{\pi}\beta_1\sigma_1 t\ln t\right)dt,$$

$$I_{22} = -\int_0^\infty e^{-\sigma_1 t}\sin t(x-\mu_1)(1+\ln t)\sin\left(\frac{2}{\pi}\beta_1\sigma_1 t\ln t\right)dt.$$

(i) Case $\beta_1 \neq 0$: After integrating by parts,

$$\begin{aligned} I_{21} &= \frac{\pi}{2\beta_1}\int_0^\infty e^{-\sigma_1 t}\cos t(x-\mu_1)\sin\left(\frac{2}{\pi}\beta_1\sigma_1 t\ln t\right)dt \\ &\quad+\frac{\pi}{2\beta_1\sigma_1}(x-\mu_1)\int_0^\infty e^{-\sigma_1 t}\sin t(x-\mu_1)\sin\left(\frac{2}{\pi}\beta_1\sigma_1 t\ln t\right)dt, \\ I_{22} &= \frac{\pi}{2\beta_1}\int_0^\infty e^{-\sigma_1 t}\sin t(x-\mu_1)\cos\left(\frac{2}{\pi}\beta_1\sigma_1 t\ln t\right)dt \\ &\quad-\frac{\pi}{2\beta_1\sigma_1}(x-\mu_1)\int_0^\infty e^{-\sigma_1 t}\cos t(x-\mu_1)\cos\left(\frac{2}{\pi}\beta_1\sigma_1 t\ln t\right)dt, \end{aligned}$$

so that

$$I_{21}+I_{22} = -\frac{\pi}{2\beta_1\sigma_1}(x-\mu_1)\pi f_{X_1}(x) + \frac{\pi}{2\beta_1}U(x).$$

Substituting this expression in (4.9) and rearranging the terms, we get

$$E(X_2|X_1=x) = -\frac{2\sigma_1}{\pi}k_0 + \frac{\lambda}{\beta_1}(x-\mu_1) - \frac{\sigma_1}{\beta_1}(\lambda-\beta_1\kappa)\frac{U(x)}{\pi f_{X_1}(x)}$$

$$= -\frac{2\sigma_1}{\pi}k_0 + \kappa(x-\mu_1) + \frac{\lambda-\beta_1\kappa}{\beta_1}\left[(x-\mu_1) - \sigma_1\frac{U(x)}{\pi f_{X_1}(x)}\right].$$

(ii) Case $\beta_1 = 0$: After integrating by parts,

$$U(x) = \int_0^\infty e^{-\sigma_1 t}\sin t(x-\mu_1)\,dt = \frac{\pi f_{X_1}(x)}{\sigma_1}(x-\mu_1).$$

Since

$$I_{21} + I_{22} = \int_0^\infty e^{-\sigma_1 t}\cos t(x-\mu_1)(1+\ln t)\,dt = V(x),$$

relation (4.9) becomes

$$E(X_2|X_1=x) = \frac{\sigma_1}{\pi f_{X_1}(x)}\left[\kappa\frac{\pi f_{X_1}(x)}{\sigma_1}(x-\mu_1) - \frac{2}{\pi}\lambda V(x) - \frac{2}{\pi}k_0\pi f_{X_1}(x)\right]$$

$$= -\frac{2\sigma_1}{\pi}k_0 + \kappa(x-\mu_1) - \frac{2\sigma_1}{\pi}\lambda\frac{V(x)}{\pi f_{X_1}(x)}. \qquad \square$$

PROOF OF COROLLARY 3.2. It is sufficient to focus on $x \to \infty$. Indeed, consider the vector $(-X_1, X_2)$ whose parameters are $\hat\beta_1 = -\beta_1$, $\hat\kappa = -\kappa$, and $\hat\lambda = \lambda$. Since $E(X_2|X_1 = x) = E(X_2| - X_1 = -x)$, one can obtain the asymptotic behavior of $E(X_2|X_1 = x)$ as $x \to -\infty$ from that of $E(X_2|X_1 = x)$ as $x \to \infty$ by replacing β_1 by $-\beta_1$, κ by $-\kappa$, and x by $-x = |x|$. Suppose $\beta_1 \neq \pm 1$.

We study first the cases with $\alpha \neq 1$ and then those with $\alpha = 1$.

(a) Case $\alpha \neq 1$, $x \to \infty$: To obtain the asymptotic behavior of $E(X_2|X_1 = x)$, we use its expression in terms of I_{ij}, $i, j = 1, 2$, as given in (4.5). By [15], Theorem 126, as $x \to \infty$,

$$\begin{aligned} I_{11} &\sim \Gamma(\alpha)\left(\sin\frac{\pi\alpha}{2}\right)x^{-\alpha}, & I_{12} &= o(x^{-\alpha}),\\ I_{21} &\sim \Gamma(\alpha)\left(\cos\frac{\pi\alpha}{2}\right)x^{-\alpha}, & I_{22} &= o(x^{-\alpha}), \end{aligned} \tag{4.10}$$

when $0 < \alpha < 1$ and also when $1 < \alpha < 2$. (If $1 < \alpha < 2$, use integration by parts to transform the terms $t^{\alpha-1}$ in I_{ij}, $i, j = 1, 2$, into $t^{\alpha-2}$.) Moreover,

$$f_{X_1}(x) \sim \frac{1}{\pi}(1+\beta_1)\left(\sin\frac{\pi\alpha}{2}\right)\sigma_1^\alpha\Gamma(\alpha+1)x^{-\alpha-1}$$

(see [7], Theorem 2.4.2). Substituting in (4.5) and using $a = \tan(\pi\alpha/2)$, we get

$$E[X_2|X_1 = x] \sim \frac{\alpha\sigma_1^\alpha}{\pi} \frac{\pi x^{\alpha+1}}{(1+\beta_1)(\sin(\pi\alpha/2))\sigma_1^\alpha\Gamma(1+\alpha)}$$

$$\times\left(\kappa\Gamma(\alpha)\left(\sin\frac{\pi\alpha}{2}\right)x^{-\alpha} + a\lambda\Gamma(\alpha)\left(\cos\frac{\pi\alpha}{2}\right)x^{-\alpha}\right)$$

$$\sim \frac{\kappa+\lambda}{1+\beta_1}x.$$

(b) Case $\alpha = 1$, $\beta_1 \neq 0$, $x \to \infty$: Since $x - \mu_1 \sim x$ as $x \to \infty$, we may assume $\mu_1 = 0$. Then

$$U(x) = \int_0^\infty e^{-\sigma_1 t}\sin\left(tx + \frac{2}{\pi}\beta_1\sigma_1 t\ln t\right)dt = U_1(x) + U_2(x),$$

where

$$U_1(x) = \int_0^\infty e^{-\sigma_1 t}\sin\left(\frac{2}{\pi}\beta_1\sigma_1 t\ln t\right)\cos tx\,dt,$$

$$U_2(x) = \int_0^\infty e^{-\sigma_1 t}\cos\left(\frac{2}{\pi}\beta_1\sigma_1 t\ln t\right)\sin tx\,dt.$$

After integrating by parts,

$$U_1(x) = -\frac{1}{x}\int_0^\infty e^{-\sigma_1 t}\left[-\sigma_1\sin\left(\frac{2}{\pi}\sigma_1\beta_1 t\ln t\right)\right.$$
$$\left. + \frac{2}{\pi}\sigma_1\beta_1(1+\ln t)\cos\left(\frac{2}{\pi}\sigma_1\beta_1 t\ln t\right)\right]\sin tx\,dt,$$

$$U_2(x) = \frac{1}{x} + \frac{1}{x}\int_0^\infty e^{-\sigma_1 t}\left[-\sigma_1\cos\left(\frac{2}{\pi}\sigma_1\beta_1 t\ln t\right)\right.$$
$$\left. + \frac{2}{\pi}\sigma_1\beta_1(1+\ln t)\sin\left(\frac{2}{\pi}\sigma_1\beta_1 t\ln t\right)\right]\cos tx\,dt.$$

Since the factors of $\sin tx$ and $\cos tx$ are integrable, we get $U_1(x) = o(x^{-1})$ and $U_2(x) \sim x^{-1}$ as $x \to \infty$ by the Riemann–Lebesgue lemma ([15], Theorem 1). Therefore

$$U(x) \sim \frac{1}{x}. \tag{4.11}$$

Since, as is well known,

$$f_{X_1}(x) \sim \frac{\sigma_1(1+\beta_1)}{\pi}x^{-2}, \tag{4.12}$$

we get

$$\frac{U(x)}{\pi f_{X_1}(x)} \sim \frac{1}{\sigma_1(1+\beta_1)} x,$$

which, substituted in (3.10), gives

$$E(X_2|X_1 = x) \sim \kappa x + \frac{\lambda - \beta_1\kappa}{\beta_1}\left[x - \frac{x}{1+\beta_1}\right] = \frac{\kappa+\lambda}{1+\beta_1} x.$$

This is the same result as in the case $\alpha \neq 1$.

(c) Case $\alpha = 1$, $\beta_1 = 0$, $x \to \infty$: If Y is $S\alpha S$ with $\alpha = 1$ and scaling parameter σ_1, then by the inversion formula ([10], Theorem 3.2.1),

$$P(0 < Y \leq x) = \frac{1}{\pi}\int_0^\infty e^{-\sigma_1 t} t^{-1} \sin tx\, dt.$$

On the other hand, because of the symmetry, $\lim_{x\to\infty} P(0 < Y \leq x) = \frac{1}{2}$. Therefore

$$\lim_{x\to\infty} \int_0^\infty e^{-\sigma_1 t} t^{-1} \sin tx\, dt = \frac{\pi}{2},$$

and hence

$$\begin{aligned} V(x) &= \int_0^\infty e^{-\sigma_1 t}(1 + \ln t)\cos tx\, dt \\ &= \frac{\sigma_1}{x}\int_0^\infty e^{-\sigma_1 t}(1+\ln t)\sin tx\, dt - \frac{1}{x}\int_0^\infty e^{-\sigma_1 t} t^{-1}\sin tx\, dt \qquad (4.13) \\ &\sim -\frac{\pi}{2}\frac{1}{x}, \end{aligned}$$

since the first integral is $o(1)$ by the Riemann–Lebesgue lemma. Substituting (4.13) and (4.12) in (3.11) yields $E(X_2|X_1 = x) \sim (\kappa + \lambda)x$, that is, (3.16) with $\beta_1 = 0$. □

PROOF OF COROLLARY 3.3. If $\lambda = \beta_1\kappa$, then $E(X_2|X_1 = x) = \kappa x$ for a.e. x by Theorem 3.1 when $\alpha \neq 1$, and $E(X_2|X_1 = x) = -(2\sigma_1/\pi)k_0 + \kappa(x - \mu_1)$ for a.e. x by Theorem 3.2 when $\alpha = 1$.

Suppose now $E(X_2|X_1 = x) = Ax + B$ for a.e. x and also, *ad absurdum*, $\lambda \neq \beta_1\kappa$. In view of Corollary 3.1 this contradicts either (3.5) or (3.10) if $\beta_1 = \pm 1$. If $\beta_1 \neq \pm 1$, then by linearity,

$$\lim_{x\to\infty} \frac{E(X_2|X_1 = x)}{x} = \lim_{x\to-\infty} \frac{E(X_2|X_1 = x)}{x}.$$

In view of (3.16) and (3.17), this implies

$$\frac{\kappa + \lambda}{1 + \beta_1} = \frac{\kappa - \lambda}{1 - \beta_1},$$

and hence $\lambda = \beta_1 \kappa$, a contradiction. □

PROOF OF PROPOSITION 3.1. Suppose first $\alpha \neq 1$ and set $|\mathbf{f}| = \sqrt{f_1^2 + f_2^2}$. The characteristic function of (X_1, X_2) is

$$E \exp\left\{ i \sum_{j=1}^{2} \theta_j \int_E f_j M(dx) \right\}$$

$$= \exp\left\{ \int_E \left[-\left| \sum_{j=1}^{2} \theta_j f_j \right|^{\alpha} + ia \left(\sum_{j=1}^{2} \theta_j f_j \right)^{\langle \alpha \rangle} \beta(x) \right] m(dx) \right\}$$

$$= \exp\left\{ \int_{E_+} \left[-\left| \sum_{j=1}^{2} \theta_j \frac{f_j}{|\mathbf{f}|} \right|^{\alpha} + ia \left(\sum_{j=1}^{2} \theta_j \frac{f_j}{|\mathbf{f}|} \right)^{\langle \alpha \rangle} \right] \frac{1 + \beta(x)}{2} |\mathbf{f}|^{\alpha} m(dx) \right.$$

$$+ \int_{E_+} \left[-\left| \sum_{j=1}^{2} \theta_j \frac{(-f_j)}{|\mathbf{f}|} \right|^{\alpha} + ia \left(\sum_{j=1}^{2} \theta_j \frac{(-f_j)}{|\mathbf{f}|} \right)^{\langle \alpha \rangle} \right]$$

$$\left. \times \frac{1 - \beta(x)}{2} |\mathbf{f}|^{\alpha} m(dx) \right\}$$

$$= \exp\left\{ \int_{S_2} \left[-\left| \sum_{j=1}^{2} \theta_j s_j \right|^{\alpha} + ia \left(\sum_{j=1}^{2} \theta_j s_j \right)^{\langle \alpha \rangle} \right] \Gamma(d\mathbf{s}), \right.$$

by making the change of variables

$$T: E_+ \to S_2, \qquad T: x \mapsto (s_1, s_2) = \left(\frac{f_1(x)}{|\mathbf{f}(x)|}, \frac{f_2(x)}{|\mathbf{f}(x)|} \right)$$

and setting $\Gamma(d\mathbf{s}) = \Gamma_+(d\mathbf{s}) + \Gamma_-(-d\mathbf{s})$ with

$$\Gamma_{\pm}(d\mathbf{s}) = \frac{1 \pm \beta(x)}{2} |\mathbf{f}(x)|^{\alpha} m(dx), \qquad x = T^{-1}(\mathbf{s}).$$

Hence, in order to transform an integral involving s_1, s_2 and Γ into one involving f_1, f_2 and m, one expresses it as a sum of two terms: The first is obtained by replacing each s_j by $f_j(x)/|\mathbf{f}(x)|$, $j = 1, 2$, and $\Gamma(d\mathbf{s})$ by $\frac{1}{2}(1 + \beta(x))|\mathbf{f}(x)|^{\alpha} m(dx)$, and the second is obtained by replacing each s_j by $-f_j(x)/|\mathbf{f}(x)|$, $j = 1, 2$, and $\Gamma(d\mathbf{s})$ by $\frac{1}{2}(1 - \beta(x))|\mathbf{f}(x)|^{\alpha} m(dx)$. (The same rule applies when $\alpha = 1$.) For example, when $\alpha = 1$,

$$\begin{aligned} k_0 &= \int_{S_2} s_1 \ln|s_2|\Gamma(d\mathbf{s}) \\ &= \int_{E_+} \frac{f_1}{|\mathbf{f}|}\left(\ln\frac{|f_2|}{|\mathbf{f}|}\right)\frac{1+\beta(x)}{2}|\mathbf{f}|m(dx) \\ &\quad + \int_{E_+} \frac{-f_1}{|\mathbf{f}|}\left(\ln\frac{|-f_2|}{|\mathbf{f}|}\right)\frac{1-\beta(x)}{2}|\mathbf{f}|m(dx) \\ &= \int_{E_+} f_1\left(\ln\frac{|f_2|}{|\mathbf{f}|}\right)\beta(x)m(dx). \end{aligned}$$

Similarly,

$$\begin{aligned} \int_{S_2} \frac{\Gamma(d\mathbf{s})}{|s_1|^\nu} &= \int_{E_+}\left(\frac{|\mathbf{f}|}{|f_1|}\right)^\nu \left\{\frac{1+\beta(x)}{2} + \frac{1-\beta(x)}{2}\right\}|\mathbf{f}|^\alpha m(dx) \\ &\in \left[\int_{E_+}\frac{|f_2|^{\alpha+\nu}}{|f_1|^\nu}m(dx), c\left(\int_{E_+} m(dx) + \int_{E_+}\frac{|f_2|^{\alpha+\nu}}{|f_1|^\nu}m(dx)\right)\right]. \qquad \square \end{aligned}$$

5. Graphical representations. In this section we present graphical representations of the regression functions given analytically in Theorems 3.1 and 3.2. In the case $\alpha \neq 1$, the five parameters α, β_1, κ, λ, and σ_1 are required to describe the regression function completely. In the case $\alpha = 1$, these five parameters, with the addition of k_0 and μ_1, are required to describe the regression function completely. Thus, there are a myriad of possible choices for these parameters which are consistent with their definitions, giving rise to an unmanageably large family of regression functions. In producing the figures that follow, we have chosen to restrict the parameter space considerably, but in a way which we hope will give the reader some feeling for the general character of these functions. Clearly though, the following examples do not illustrate all possible behavior.

It is convenient to use the stochastic integral representation (3.19) for (X_1, X_2). Here, the random measure M is taken to be totally right-skewed [i.e., $\beta(\cdot) \equiv 1$], with domain $E = [0, 1]$, and Lebesgue control measure, m. The functions f_j, illustrated in Figure 1, are restricted to those of the form

$$f_j(t) = 1_{[0, c_j)}(t) - 1_{[c_j, 1]}(t),$$

where 1_A is the indicator function of the set A. With these restrictions, the three parameters α, c_1 and c_2 determine the regression function.

This choice forces all scale parameters to be unity, and ensures that the standard assumption is always satisfied. Since $E(a_2X_2|a_1X_1 = x) = a_2E(X_2|X_1 = x/a_1)$ for nonzero a_i, other related regressions may be inferred from those strictly in this class. Regressions for (X_1, X_2) in this class may be interpreted as regressions involving three independent stable variables as

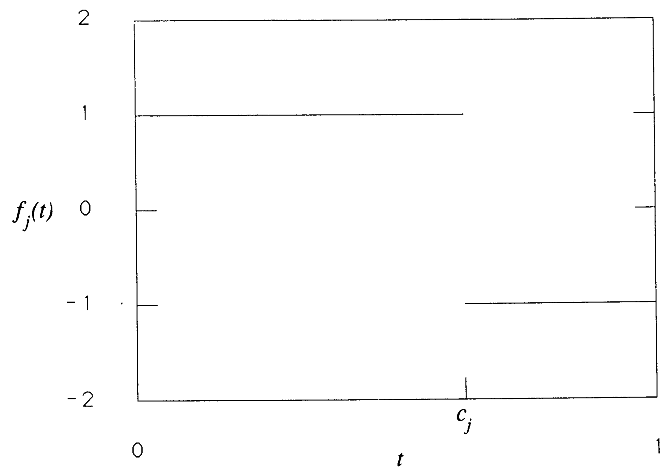

FIG. 1. *The function $f_j(t)$.*

follows. Define $c_{\min} = \min(c_1, c_2)$ and $c_{\max} = \max(c_1, c_2)$, and let Z_1, Z_2 and Z_3 be independent identically distributed totally right-skewed ($\beta = 1$) α-stable random variables with unity scale parameter. Then (X_1, X_2) is distributed as (Y_1, Y_2), where

$$Y_j = c_{\min}^{1/\alpha} Z_1 \pm (-1)^{j+1} (c_{\max} - c_{\min})^{1/\alpha} Z_2 - (1 - c_{\max})^{1/\alpha} Z_3$$

and the upper sign prevails if $c_{\max} = c_1$, and the lower sign prevails otherwise. For example, if $c_1 = 0.5$ and $c_2 = 1$, then the regression of X_2 on X_1 is the regression of $k(Z_1 + Z_2)$ on $k(Z_1 - Z_2)$ for the appropriate constant k.

From Proposition 3.1, the parameters defining the regressions in Theorem 3.2 are

$$\sigma_1 = 1,$$

$$\beta_1 = 2c_1 - 1,$$

$$\mu_1 = \frac{2}{\pi} \ln(\sqrt{2}) \beta_1,$$

$$k_0 = -\ln(\sqrt{2}) \lambda,$$

$$\kappa = 1 - 2|c_1 - c_2|$$

and

$$\lambda = 2c_2 - 1.$$

We shall refer to the skewness parameter for X_2 as β_2. For the present class of distributions, β_2 is equal to λ.

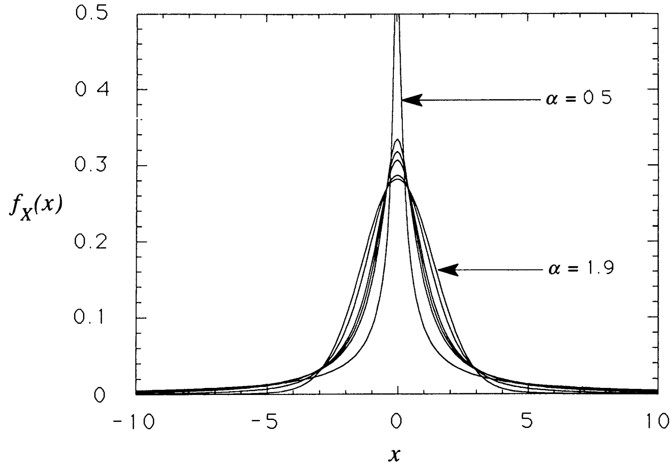

FIG. 2. *Symmetric stable density functions, $\beta = 0$.*

The asymptotic results of Corollary 3.2 are illustrated in the graphs, and translated to the present parameterization as follows:

$$\lim_{x \to +\infty} \frac{E(X_2|X_1 = x)}{x} = \begin{cases} 1, & \text{if } c_2 \geq c_1, \\ 2(c_2/c_1) - 1, & \text{if } c_1 > c_2, \end{cases}$$

and

$$\lim_{x \to -\infty} \frac{E(X_2|X_1 = x)}{x} = \begin{cases} 1 - 2\dfrac{c_2 - c_1}{1 - c_1}, & \text{if } c_2 \geq c_1, \\ 1, & \text{if } c_1 > c_2. \end{cases}$$

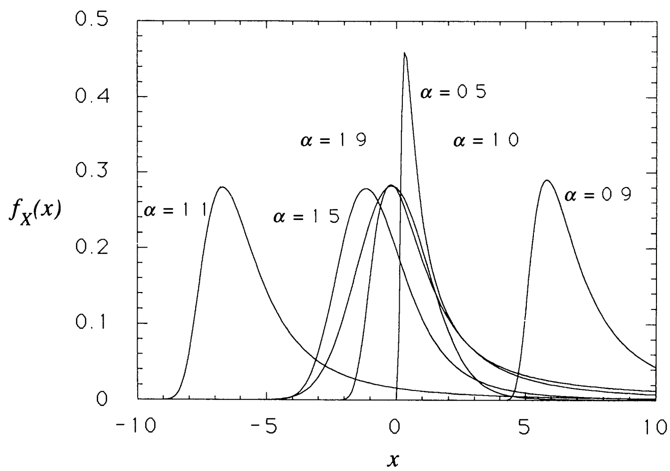

FIG. 3. *Skewed stable density functions, $\beta = 1$.*

The limits here do not depend on the value of α. Moreover, by Corollary 3.3, the regression is linear when $c_1 = c_2$ (this corresponds here to $X_1 = X_2$).

The regression functions are computed according to Theorems 3.1 and 3.2. Unfortunately, the functions $H(x)$, $U(x)$ and $V(x)$ and the density function $f_{X_1}(x)$ do not in general have representations in terms of elementary functions, and thus their values are computed via numerical integrations. See Hardin, Samorodnitsky and Taqqu [6] for a detailed discussion of the numerical procedures. For reference, a few density functions for selected values of α are plotted in Figures 2 and 3.

Figure 2 illustrates the symmetric densities for the values $\alpha = 0.5, 0.9, 1.0, 1.1, 1.5$ and 1.9. There is a natural ordering in that for smaller α, the value of the density at the mode is higher, the tails are larger, and the width of the modal peak is narrower.

Figure 3 shows totally right-skewed densities for the same values of α. Here, the "ordering" is not continuous across the value $\alpha = 1$. For $\alpha > 1$, the mode is negative, while the tail is heavier on the positive axis, resulting in a zero mean density. As α approaches 1 from above, the mode becomes more negative, and the right tail becomes heavier. For $\alpha < 1$, the densities are nonzero only on the positive axis, with increasing mode as α approaches 1 from below. For $\alpha = 1$, the mode is negative, although the modal peak has more mass to the right of 0 than to the left. In no discernible way, however, do the skewed densities approach the skewed $\alpha = 1$ density as α approaches 1, as is true in the symmetric case. This is due to the choice of parameters in the (marginal) characteristic function. There is a different choice which makes the characteristic function continuous as $\alpha \to 1$ (see Zolotarev [17]).

In the graphs that follow, two of the parameters α, c_1 and c_2 are held constant while the third varies. Although not all possible variations are

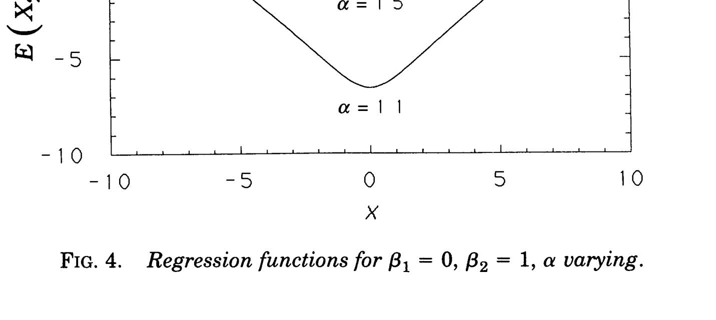

FIG. 4. *Regression functions for $\beta_1 = 0$, $\beta_2 = 1$, α varying.*

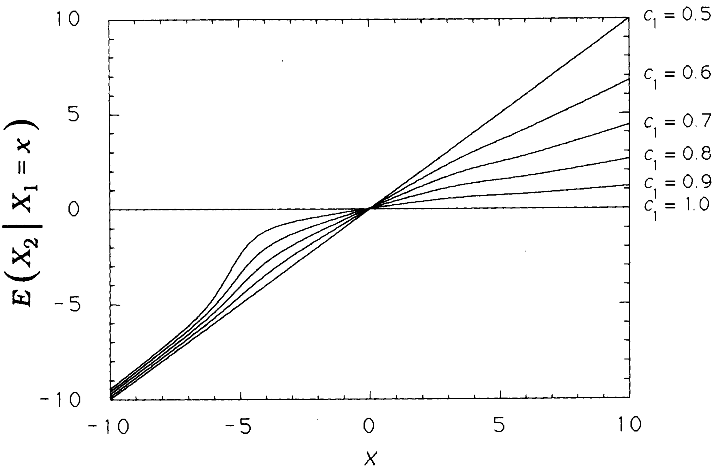

FIG. 5. *Regression functions for* $\alpha = 1.9$, $\beta_1 = 2c_1 - 1$ *and* $\beta_2 = 0$.

illustrated, much of the behavior for parameter choices not illustrated can be correctly inferred from the graphs.

Figure 4 corresponds to $c_1 = 0.5$ and $c_2 = 1$ and hence to $\beta_1 = 0$ and $\beta_2 = \lambda = 1$. It shows the regression of a totally right-skewed variable X_2 upon a symmetric variable X_1 for selected α, or equivalently, the regression of $k(Z_1 + Z_2)$ on $k(Z_1 - Z_2)$ as mentioned above. When $\alpha \neq 1$, the value of these regression functions at the origin has the same sign as $a = \tan(\pi\alpha/2)$ and hence is positive for $\alpha < 1$ and negative for $\alpha > 1$.

Figure 5 corresponds to $c_2 = 0.5$ and various values of c_1. It represents a regression of a symmetric variable upon X_2 upon variables X_1 with skewness

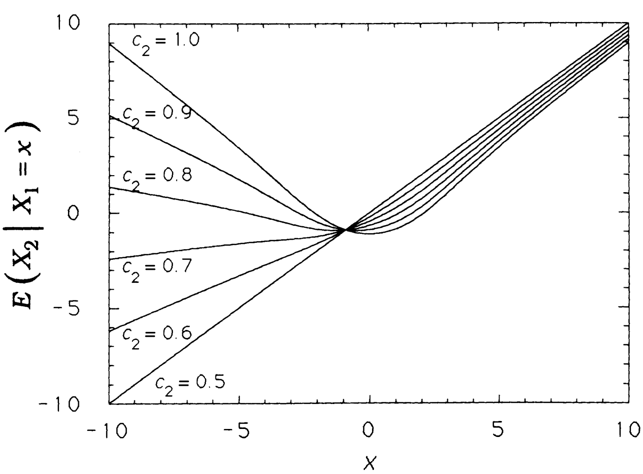

FIG. 6. *Regression functions for* $\alpha = 1.5$, $\beta_1 = 0$ *and* $\beta_2 = 2c_2 - 1$.

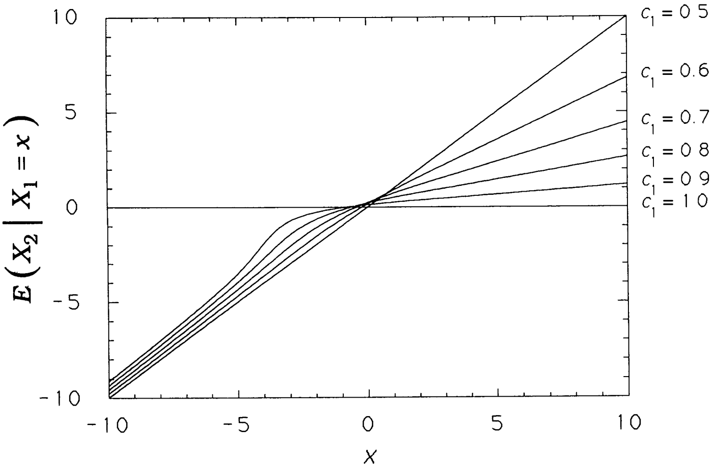

FIG. 7. *Regression functions for* $\alpha = 1.5$, $\beta_1 = 2c_1 - 1$ *and* $\beta_2 = 0$.

β_1 varying from 0 to 1, for the case $\alpha = 1.9$. The value $c_1 = 0.5$ corresponds to X_1 symmetric, in which case $X_1 = X_2$ and the regression is linear. When $c_1 = 1$, X_1 is totally right-skewed and the regression is linear at 0.

In Figure 6, X_2 of varying skewness is regressed on a symmetric X_1 for the value $\alpha = 1.5$. Here $c_1 = 0.5$. When $c_2 = 0.5$, the regression is linear since $X_2 = X_1$. The curious nonzero intersection of the regression lines occurs for all fixed values of $\alpha \geq 1$ at an x value depending on α, but does not occur for values of $\alpha < 1$ (see Figure 12). Figure 7 also corresponds to $\alpha = 1.5$, but this time it is X_2 that is symmetric ($\beta_2 = 0$).

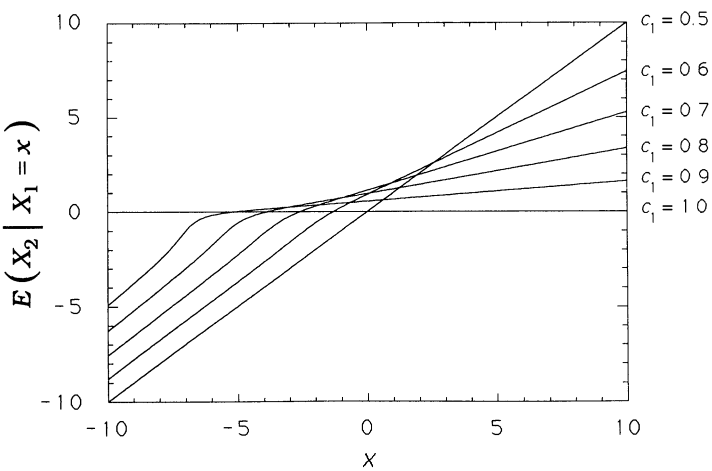

FIG. 8. *Regression functions for* $\alpha = 1.1$, $\beta_1 = 2c_1 - 1$ *and* $\beta_2 = 0$.

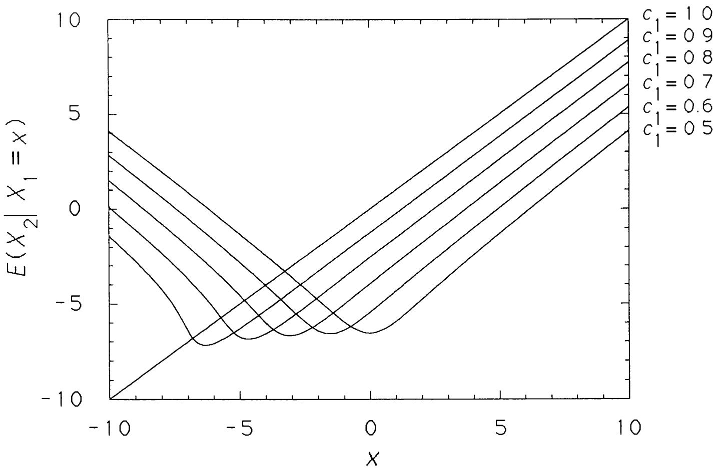

FIG. 9. *Regression functions for* $\alpha = 1.1$, $\beta_1 = 2c_1 - 1$ *and* $\beta_2 = 1$.

In Figures 8 and 9, α equals 1.1. Figure 8 should be compared to Figure 7 because they both illustrate the regression of a symmetric random variable X_2 on random variables X_1 of varying skewness. Figure 9 shows the regression of a totally right-skewed X_2 upon X_1 of varying skewness. Here $c_2 = 1$ and hence $\beta_2 = \lambda = 1$. As the skewness of X_1 approaches that of X_2, the regression function approaches the identity, yet the left asymptote always has slope -1.

In Figure 10, $\alpha = 1$. The parameter c_1 is chosen to be 0.9, so that X_1 has skewness 0.8. The skewness of X_2 varies from -0.8 to 1. (For X_2 of skewness

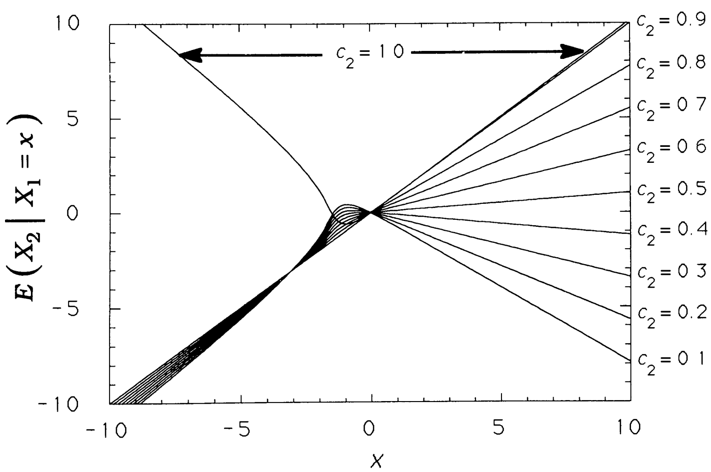

FIG. 10. *Regression functions for* $\alpha = 1.0$, $\beta_1 = 0.8$ *and* $\beta_2 = 2c_2 - 1$.

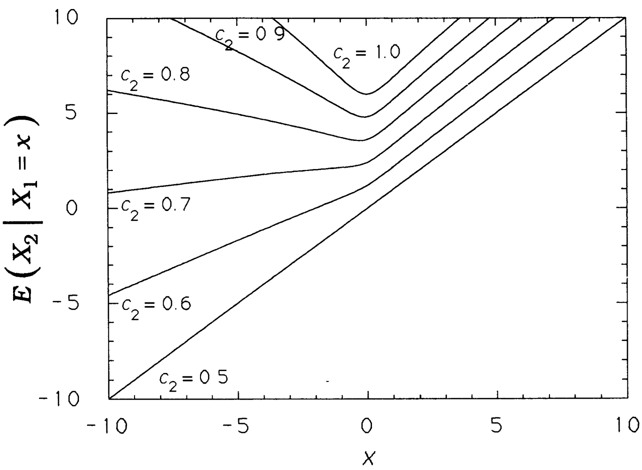

FIG. 11. *Regression functions for* $\alpha = 0.9$, $\beta_1 = 0$ *and* $\beta_2 = 2c_2 - 1$.

-1, the regression is the negative of that for X_2 of skewness 1.) The value $c_2 = 0.9$ corresponds to $X_2 = X_1$, in which case the regression is linear. This figure shows that a small change in skewness can result in a large change in the global shape of the regression function.

Figure 11 represents the regressions of variables of varying skewness upon a symmetric variable for the value $\alpha = 0.9$. The value $c_2 = 0.5$ corresponds to $X_2 = X_1$, in which case the regression is linear. This plot should be compared with the case $\alpha = 1.5$ illustrated in Figure 6.

Figures 12 and 13 both correspond to $\alpha = 0.5$, but $\beta_1 = 0$ in Figure 12 whereas $\beta_2 = 0$ in Figure 13. Observe that Figures 6 and 12, where $\beta_1 = 0$,

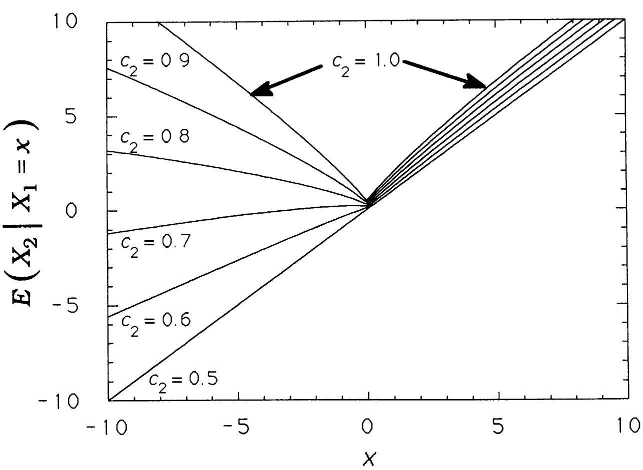

FIG. 12. *Regression functions for* $\alpha = 0.5$, $\beta_1 = 0$ *and* $\beta_2 = 2c_2 - 1$.

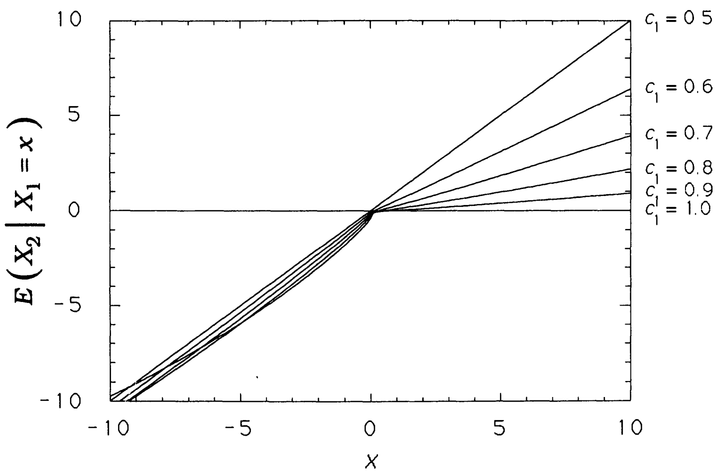

FIG. 13. *Regression functions for* $\alpha = 0.5$, $\beta_1 = 2c_1 - 1$ *and* $\beta_2 = 0$.

have roughly the same shape. So do Figures 7, 8 and 13, where $\beta_2 = 0$. In Figure 13 the regression for $\beta_1 = 1$ ($c_1 = 1$) is defined only for $x \geq 0$, because the density of X_1 has support only on $[0, \infty]$. It is linear with slope $\kappa = 0$.

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