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GREEDY LATTICE ANIMALS II: LINEAR GROWTH

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Let $\{X_v: v \in \mathbb{Z}^d\}$ be i.i.d. positive random variables and define $M_n = \max\left\{\sum_{v \in \pi} X_v: \pi \text{ a self-avoiding path of length } n \text{ starting at the origin}\right\}$, $N_n = \max\left\{\sum_{v \in \xi} X_v: \xi \text{ a lattice animal of size } n \text{ containing the origin}\right\}$. In a preceding paper it was shown that if $E\{X_0^d(\log^+ X_\mathbf{0})^{d+a}\} < \infty$ for some $a > 0$, then there exists some constant C such that w.p.1, $0 \le M_n \le N_n \le Cn$ for all large n. In this part we improve this result by showing that, in fact, there exist constants $M, N < \infty$ such that w.p.1, $M_n/n \to M$ and $N_n/n \to N$.

1. Outline. Since the description and the motivation of our model were already given in part I [Cox, Gandolfi, Griffin and Kesten (1993)], we just remind the reader of some definitions and jump right in with a statement of our present results. Definitions are exactly as in Cox, Gandolfi, Griffin and Kesten (1993). In particular, we consider *self-avoiding paths* π of length n and *lattice animals* ξ of size n, the former being sequences $(v_0, v_1, \dots, v_{n-1})$ of n vertices of $\mathbb{Z}^d$ such that v_{i+1} and v_i are adjacent, $0 \le i \le n-2$, and $v_i \ne v_j$ for $i \ne j$; the latter being connected subsets $\xi \subseteq \mathbb{Z}^d$ of cardinality n. For a path π or lattice animal ξ we set

$$S(\pi) = \sum_{v \in \pi} X_v, \qquad S(\xi) = \sum_{v \in \xi} X_v.$$

We indicate by $|\pi|$ and $|\xi|$ the *length* of a path π and the *cardinality* of a lattice animal ξ, respectively. With M_n and N_n defined as in the abstract, the *greedy lattice animals* of size n are those lattice animals ξ, of size n and containing the origin $\mathbf{0} \in \mathbb{Z}^d$, whose *weight* $S(\xi)$ equals N_n; similarly, *greedy lattice paths* of length n are paths π such that $S(\pi) = M_n$. Here and in the following, paths will always be self-avoiding even if that is not stated explicitly.

Throughout $\{X_v: v \in \mathbb{Z}^d\}$ *is a family of positive i.i.d. random variables.* (*N.B.* "X_v positive" means $X_v \ge 0$.") The common distribution function of

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the X_v's is denoted by F. Finally, the *probability measure* governing the family $\{X_v\}$ is denoted by P, while *expectation* with respect to P is indicated by E. (In other words, $P = \prod_{v \in \mathbb{Z}^d} F$.)

The *variance* of a random variable Y is denoted by $\sigma^2(Y)$, while the *indicator function* of an event $\mathscr{A}$ is denoted by $\mathbb{1}[\mathscr{A}]$. Constants whose values are unimportant are indicated as C_i, $i \in \mathbb{N}$, and their values may change from appearance to appearance.

The main result of the present paper establishes the asymptotic linearity of M_n and N_n:

THEOREM 1. *If, for some $a > 0$,*

$$E\{X_0^d(\log^+ X_0)^{d+a}\} < \infty, \tag{1.1}$$

then there exist constants $M, N < \infty$ such that

$$\frac{M_n}{n} \to M \quad w.p.1 \text{ and in } L^1 \tag{1.2}$$

and

$$\frac{N_n}{n} \to N \quad w.p.1 \text{ and in } L^1. \tag{1.3}$$

Unfortunately, we have very little information on the dependence of M and N on the distribution of X_0. For $d = 1$ one easily sees from the strong law of large numbers that always $M = N = E\{X_0\}$. Lee (1993) has shown that in dimension $d \geq 2$ one usually has

$$N > M.$$

For $d \geq 2$ the same proof as for Theorem 7.4 in Smythe and Wierman (1978) shows that

$$N \geq M > E\{X_0\}$$

whenever F is not concentrated on one point.

Since Theorem 1 is immediate from the strong law of large numbers for $d = 1$, *we take $d \geq 2$ for the remainder of this paper.*

In Section 6 we discuss the relation between Theorem 1 and ρ-percolation as introduced by Menshikov and Zuyev (1992).

One's first reaction is that (1.2) and (1.3) should be provable by some version of the subadditive ergodic theorem. This is more or less the case for N_n and certain "directed" examples (see the Remark at the end of Section 3). However, we have been unable to find obvious subadditivity properties for M_n itself. As a consequence, we first introduce "cylinder" versions of M_n (and N_n), which are, for all intents and purposes, superadditive (see Lemma 4 for the precise statement), and then reduce the general case to the cylinder versions. To give a more detailed outline, we require some more notation.

Define $v(i)$ to be the ith *coordinate of* $v \in \mathbb{Z}^d$,

$$\|v\| = \max_{1 \le i \le d} |v(i)|,$$
$$H(x) = \{v: v(1) = \lfloor x \rfloor\}$$

(a *hyperplane* perpendicular to the first coordinate axis and located at the *integer part* $\lfloor x \rfloor$ of $x \in \mathbb{R}$). Also, let e_i be the ith coordinate vector. For two real numbers a, b, let $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$ and $a^+ = \max(a, 0)$. For integers k, m, $k|m$ means k divides m.

A *cylinder path* of length n is a path $\pi = (v_0, v_1, \dots, v_{n-1})$ such that the first coordinates of its vertices satisfy $v_0(1) \le v_i(1) \le v_{n-1}(1)$ for $1 \le i \le n-1$. A cylinder path starting at the origin is a cylinder path with $v_0 = \mathbf{0}$. For $0 \le \beta \le 1$ and $n \ge 1$, define:

$$R_n(\beta) = \max\{S(\pi)\colon \pi \text{ a cylinder path of length } n \text{ starting at the origin, with } v_{n-1}(1) = \lfloor \beta n \rfloor\}, \tag{1.4}$$

$$R_n = \max\{S(\pi)\colon \pi \text{ a cylinder path of length } n \text{ starting at the origin}\}, \tag{1.5}$$

$$V_n(\beta) = \max\{S(\xi)\colon \xi \text{ a lattice animal of size } n \text{ containing the origin, such that } 0 \le v(1) \le \lfloor \beta n \rfloor, \text{ for all } v \in \xi, \text{ and } v(1) = \lfloor \beta n \rfloor \text{ for at least one } v \in \xi\}, \tag{1.6}$$

$$V_n = \max\{S(\xi)\colon \xi \text{ a lattice animal of size } n \text{ containing the origin such that } v(1) \ge 0 \text{ for all } v \in \xi\}. \tag{1.7}$$

Notice that $R_n = \max_\beta R_n(\beta)$ and $V_n = \max_\beta V_n(\beta)$, where, as defined previously, $R_n(\beta)$ and $V_n(\beta)$ are the maxima over paths and lattice animals, respectively, which are of size n, lie between the hyperplanes $H(0)$ and $H(\lfloor \beta n \rfloor)$ and have "endpoints" on these hyperplanes.

For fixed β, $R_n(\beta)$ and $V_n(\beta)$ are (almost) superadditive, or more accurately, subconvolutive, and we could use Note 7 of Hammersley (1979) (which is closely related to the subadditive ergodic theorem) to conclude that

$$\frac{R_n(\beta)}{n} \to M(\beta), \qquad \frac{V_n(\beta)}{n} \to N(\beta) \quad \text{w.p.1} \tag{1.8}$$

for suitable $M(\beta)$ and $N(\beta)$. It would even be possible to base our entire proof only on Note 7 of Hammersley (1979). This note, however, does not give estimates for the speed of convergence in (1.8) and this method of proof becomes very cumbersome. Instead, we first use a martingale argument (see Section 2) to estimate the tail of the distribution of $R_n(\beta) - ER_n(\beta)$ and similar quantities. This estimate is of interest in its own right, because the proof seems very widely applicable; it is based on the "method of bounded differences," which has been used with great success recently in combinatorics and probabilistic graph theory [see, for instance, McDiarmid (1989) and

Kesten (1993)]. Once we have (1.8) with an estimate on the speed of convergence, a simple continuity argument shows that, in fact,

$$\frac{R_n}{n} \to M := \sup_{\beta \in (0,1)} M(\beta) \quad \text{w.p.1},$$
$$(1.9)\qquad \frac{V_n}{n} \to N := \sup_{\beta \in (0,1)} N(\beta) \quad \text{w.p.1}.$$

Basically this is all we need for lattice animals; (1.3) will follow from a slightly strengthend version of (1.9). In fact, any lattice animal is a translation of a lattice animal lying in the half space $\{w\colon w(1) \geq 0\}$, and we apply (1.9) (or rather, the more explicit estimates of Lemma 5) to the various translates (see Section 3). However, (1.9) is not sufficient to prove (1.2) directly, as only cylinder paths would be obtained by translating the paths used in defining R_n. We therefore study the deviations of $(1/n)R_n$ from its limit. Define, for integers n, h and real $\varepsilon > 0$,

$$(1.10)\qquad \begin{aligned} \mathscr{B}(n,h) &= \mathscr{B}(n,h,\varepsilon) \\ &= \big\{\text{there exists a cylinder path } \pi \text{ of length} \\ &\qquad |\pi| \in [h,n], \text{ with initial point in } [-n,n]^d, \\ &\qquad \text{and such that } S(\pi) \geq (M+\varepsilon)|\pi|\big\}. \end{aligned}$$

From the martingale estimates, it follows that w.p.1 $\mathscr{B}(n, n^{\delta})$ occurs only finitely often, for any fixed $\delta > 0$. We then prove (1.2), roughly speaking, by taking $\varepsilon > 0$ and breaking up every path π into cylinder paths ρ for which we know from the above that $S(\rho) \leq (M+\varepsilon)|\rho|$. This will show that also $S(\pi) \leq (M+\varepsilon)|\pi|$ for long paths π starting at the origin $\mathbf{0}$. Consequently,

$$\limsup \frac{M_n}{n} \leq M \quad \text{w.p.1},$$

whereas $M_n \geq R_n$ and (1.8) show that

$$\liminf \frac{M_n}{n} \geq M \quad \text{w.p.1}.$$

The somewhat complicated procedure for breaking up a path π into cylinder paths will be explained further in Section 5.

The more detailed analysis required for showing the existence of the limit in (1.2) for paths makes use of results from Cox, Gandolfi, Griffin and Kesten (1993), which we recall here for convenience of the reader.

Let $\Pi_0(n)$ be the collection of paths of length n starting at the origin. The main estimate we use is contained in Proposition 1 of Cox, Gandolfi, Griffin and Kesten (1993) and is as follows.

PROPOSITION A. *If* (1.1) *holds, then there exists a strictly increasing function* γ *such that* [*with* a *as in* (1.1)]

$$\frac{\gamma(x)(\log x)^{1+a/d}}{x} \to 0 \quad \textit{but} \tag{1.11}$$
$$\frac{\gamma(x)(\log x)^{1+a/d}\log\log x}{x} \to \infty \quad \textit{as } x \to \infty.$$

Moreover,

$$X_v < \gamma(\|v\|) \quad \textit{eventually } w.p.1. \tag{1.12}$$

Moreover, there exist constants $c_0, \bar{c}, c_6, c_7 < \infty$ *such that for all* $c \geq \bar{c}$ *and* $n \geq 2$,

$$P\left\{\max_{\substack{\|\xi\|=n\\ \mathbf{0}\in\xi}} \sum_{v\in\xi}(X_v \wedge \gamma(n)) \geq (1+cc_0)n\right\} \leq \exp\left(-\frac{c}{8}(\log n)^{1+a/d}\right) \tag{1.13}$$

and

$$P\left\{\max_{\substack{|\xi|=n\\ \mathbf{0}\in\xi}} \sum_{v\in\xi} X_v \mathbb{1}\left[\frac{1}{2d}\log n \leq X_v < \gamma(\|v\|)\right] \geq c_6 n\left[(\log\log n)^{-a/d} + \frac{\gamma(n)(\log n)^{1+a/d}}{n}\right]\right\} \leq \exp\left(-c_7(\log n)^{1+a/d}\right). \tag{1.14}$$

Here ξ stands for a connected set of vertices, or a lattice animal. Since each path π is connected, the probability estimate (1.13) also holds for

$$\max_{\pi\in\Pi_0(n)} \sum_{v\in\pi}(X_v \wedge \gamma(n)).$$

2. A martingale argument. In this section we basically give an estimate for the tail of the distribution of $|R_n - E(R_n)|$ and the related quantities $|Y_n(\beta) - E(Y_n(\beta))|$, $|Z_n(\beta) - E(Z_n(\beta))|$ defined in (2.7) and (2.8). This is done by means of the remarkably general "method of bounded differences." The present version is taken from Kesten (1993); see McDiarmid (1989) for a somewhat more restricted version with many applications.

In order to get a good bound, it is necessary to truncate the variables X_v first. Take

$$\hat{X}_v = X_v \wedge \log(\|v\|+1). \tag{2.1}$$

We decorate our various quantities by a caret when X_v is replaced by $\hat{X}_v$ in their definition; for example,

$$(2.2) \qquad \hat{S}(\pi) = \sum_{v \in \pi} \hat{X}_v, \qquad \hat{M}_n = \max_{\pi \in \Pi_0(n)} \{\hat{S}(\pi)\}, \ldots.$$

Before we get to our main estimate, we show that this truncation does not influence the asymptotic behavior of M_n, R_n, $R_n(\beta)$, N_n, V_n or $V_n(\beta)$.

LEMMA 1. *If* (1.1) *holds, then*

$$\frac{1}{n} \max_{\substack{|\xi|=n \\ \mathbf{0} \in \xi}} \big(S(\xi) - \hat{S}(\xi)\big) \to 0 \quad w.p.1.$$

PROOF. Let $\gamma(\cdot)$ be as in Proposition A. Clearly

$$\begin{aligned} 0 &\le \max_{\substack{|\xi|=n \\ \mathbf{0} \in \xi}} \big(S(\xi) - \hat{S}(\xi)\big) \\ (2.3) \qquad &\le \sum_v X_v \mathbb{1}[X_v \ge \gamma(\|v\|)] \\ &\quad + \max_{\substack{|\xi|=n \\ \mathbf{0} \in \xi}} \Big\{ \sum_{v \in \xi} X_v \mathbb{1}[\log(\|v\| + 1) \le X_v < \gamma(\|v\|)] \Big\}. \end{aligned}$$

The first term in the right-hand side is finite, and hence $o(n)$, w.p.1 by virtue of (1.12). In the second term, note that $\|v\| \le n$ whenever $v \in \xi$ for some ξ with $|\xi| = n$ and $\mathbf{0} \in \xi$. Therefore the second term on the right-hand side of (2.3) is at most

$$(2.4) \qquad \sum_{v : \|v\| < n^{1/(2d)}} X_v + \max_{\substack{|\xi|=n \\ \mathbf{0} \in \xi}} \left\{ \sum_{v \in \xi} X_v \mathbb{1}\left[\frac{1}{2d} \log n \le X_v < \gamma(n)\right]\right\}.$$

Since $|\{v: \|v\| \le n^{1/(2d)}\}| \le (2n^{1/(2d)} + 1)^d \le C_1 n^{1/2}$ and $E(X_0) \le \{E(X_0^d)\}^{1/d} < \infty$ by (1.1), the strong law of large numbers implies

$$\frac{1}{n} \sum_{\|v\| \le n^{1/(2d)}} X_v \to 0 \quad \text{w.p.1.}$$

In addition, it follows from (1.14) and the Borel–Cantelli lemma that, w.p.1, the second term in (2.4) is at most

$$c_6 n \left[(\log \log n)^{-a/d} + \frac{\gamma(n)(\log n)^{1+a/d}}{n} \right] = o(n)$$

for large n. Thus (2.4) is $o(n)$ w.p.1 and the lemma follows. □

Define

$$\hat{R}_n(v) = \max\{\hat{S}(\pi): \pi = (v_0 = v, v_1, \dots, v_{n-1}) \text{ a self-avoiding path of } n \text{ steps, starting at } v \text{ and with } v(1) \le v_i(1) \le v_{n-1}(1) \text{ for } 0 \le i \le n-1\}. \tag{2.5}$$

For fixed $0 < \beta < 1$, choose an integer $L = L(\beta) \ge 1$ such that

$$L - 1 > L\beta \ge 3 \tag{2.6}$$

and define

$$Y_n = Y_n(\beta) = \begin{cases} 0, & \text{if } n \le L, \\ R_{(n-L)}(\beta), & \text{if } n > L, \end{cases} \tag{2.7}$$

$$Z_n = Z_n(\beta) = \begin{cases} 0, & \text{if } n \le L, \\ V_{(n-L)}(\beta), & \text{if } n > L. \end{cases} \tag{2.8}$$

PROPOSITION 2. *For some constants* $0 < C_3 - C_5 < \infty$ *and for all*

$$0 < y \le C_3 h(\log(2n+1))^3, \qquad h \le n \quad \textit{and} \quad v \in [-n, n]^d, \tag{2.9}$$

it holds that

$$P\{|\hat{R}_h(v) - E\hat{R}_h(v)| \ge y\sqrt{h}\} \le C_4 \exp\left(-C_5(\log(2n+1))^{-1} y\right). \tag{2.10}$$

The estimate (2.10) *remains valid when* $\hat{R}_h(v) - E(\hat{R}_h(v))$ *is replaced by* $\hat{Y}_h(\beta) - E(\hat{Y}_h(\beta))$ *or* $\hat{Z}_h(\beta) - E(\hat{Z}_h(\beta))$.

PROOF. We only give the proof for $\hat{R}_h(v) - E\hat{R}_h(v)$; the proofs for the other cases are the same. We wish to apply the "method of bounded differences;" we shall represent $\hat{R}_h(v) - E(\hat{R}_h(v))$ as a sum of martingale differences, and use the martingale estimate in Theorem 3 of Kesten (1993). To this purpose, note that any path of length $h \le n$ and starting in $[-n, n]^d$ is contained in $[-2n, 2n]^d$. In particular, $\hat{R}_h(v)$ depends only the the X_w for $w \in [-2n, 2n]^d$, and for these w,

$$\hat{X}_w \le \log(2n+1).$$

Now order the vertices in $[-2n, 2n]^d$ in some way as $w^1, w^2, \dots, w^N$, with $N = (4n+1)^d$. Define $\mathscr{F}_k$ to be the σ-field generated by $X_{w^1}, \dots, X_{w^k}$. Note that

$$\hat{R}_h(v) - E(\hat{R}_h(v)) = \sum_{k=1}^{N} \left[E\{\hat{R}_h(v)|\mathscr{F}_k\} - E\{\hat{R}_h(v)|\mathscr{F}_{k-1}\}\right], \tag{2.11}$$

since $\hat{R}_h(v)$ is $\mathscr{F}_N$-measurable and $\mathscr{F}_0$ is trivial. This is the representation considered by the "method of bounded differences" as the summands in (2.11) are the successive differences of the martingale $E(\hat{R}_h(v)|\mathscr{F}_k)$, $0 \le k \le N$.

We shall now estimate the size of these differences in order to apply Theorem 3 of Kesten (1993). Observe that

$$\hat{R}_h(v) = f(X_{w^1}, \dots, X_{w^N})$$

for some Borel function f. If F is the common distribution of the X_w, then it follows from the independence of the X_w that

$$E\big(\hat{R}_h(v)\big|\mathscr{F}_k\big) = \int f\big(X_{w^1}, \dots, X_{w^k}, x^{k+1}, \dots, x^N\big) \prod_{i=k+1}^{N} F(dx^i)$$

and

$$\begin{aligned} \Delta_k &:= E\big(\hat{R}_h(v)\big|\mathscr{F}_k\big) - E\big(\hat{R}_h(v)\big|\mathscr{F}_{k-1}\big) \\ &= \int\big[f\big(X_{w^1}, \dots, X_{w^k}, x^{k+1}, \dots, x^N\big) \\ &\qquad -f\big(X_{w^1}, \dots, X_{w^{k-1}}, x^k, x^{k+1}, \dots, x^N\big)\big] \prod_{i=k}^{N} F(dx^i). \end{aligned} \tag{2.12}$$

Note that the arguments of the f's in the right-hand side only differ in the kth place, and that x^k does not appear in the first f; thus the integration over x^k has no effect for the first f. Due to a change from X_{w^k} to x^k, the value of $\hat{R}_h(v)$ cannot change more than

$$\big|X_{w^k} - x^k\big| \wedge \log(2n + 1).$$

However, we can say more. Let $\pi = \pi(h, v)$ be the (random) path of length h that starts at v and at which $\hat{R}_h(v)$ is achieved, that is, with

$$\hat{S}(\pi) = \hat{R}_h(v); \tag{2.13}$$

the path π can be chosen in a unique way by ordering all paths of length h in some way, and then taking the first permissible path in this ordering for which (2.13) holds. Then

$$\begin{aligned} &\big|f\big(X_{w^1}, \dots, X_{w^k}, x^{k+1}, \dots, x^N\big) \\ &\quad -f\big(X_{w^1}, \dots, X_{w^{k-1}} x^k, x^{k+1}, \dots, x^N\big)\big| \end{aligned} \tag{2.14}$$

vanishes, unless w^k belongs to the optimal path $\pi(h, v)$ in at least one of the configurations $(X_{w^1}, \dots, X_{w^k}, x^{k+1}, \dots, x^N)$ or $(X_{w^1}, \dots, X_{w^{k-1}}, x^k, x^{k+1}, \dots, x^N)$. Let $X = (X_{w^1}, X_{w^2}, \dots, X_{w^N})$ and $x = (x^1, x^2, \dots, x^N)$, and define

$$I_k(X, x) = \mathbb{1}\big[w^k \in \pi(h, v) \text{ in } \big(X_{w^1}, \dots, X_{w^k}, x^{k+1}, \dots, x^N\big)\big]$$

and

$$I_k^*(X, x) = \mathbb{1}\big[w^k \in \pi(h, v) \text{ in } \big(X_{w^1}, \dots, X_{w^{k-1}}, x^k, x^{k+1}, \dots, x^N\big)\big].$$

The foregoing considerations show that (2.14) is bounded by

$$\log(2n + 1)\{I_k(X, x) + I_k^*(X, x)\}.$$

Substituting this into (2.12) we find

$$
\begin{aligned}
|\Delta_k| &\le \log(2n+1)\int\{I_k(X,x)+I_k^*(X,x)\}\prod_{i=k}^{N}F(dx^i)\\
&= \log(2n+1)\big[P\{w^k\in\pi(h,v)|\mathscr{F}_k\}\\
&\qquad +P\{w^k\in\pi(h,v)|\mathscr{F}_{k-1}\}\big].
\end{aligned}
\tag{2.15}
$$

We are now ready to apply Theorem 3 in Kesten (1993). The σ-fields $\mathscr{F}_n$ are defined as before, and the random variable U_k are defined by

$$
U_k := 4(\log(2n+1))^2\mathbb{1}\big[w^k\in\pi(h,v)\big]. \tag{2.16}
$$

Also, the M_k's of Kesten (1993) (not to be confused with our M_n's) correspond to our $E(\hat{R}_h(v)|\mathscr{F}_k)$. By taking

$$
c = 2\log(2n+1), \tag{2.17}
$$

we get from (2.15),

$$
|\Delta_k|\le c, \tag{2.18}
$$

which is (1.27) in Kesten (1993). Furthermore, if we write (X, y^k, x) for $(X_{w^1},\dots,X_{w^{k-1}},y^k,x^{k+1},\dots,x^N)$ and $I_k(X,y^k,x)$ for the indicator function of the event that $w^k\in\pi(h,v)$ in this configuration, then

$$
\begin{aligned}
E(\Delta_k^2|\mathscr{F}_{k-1}) &= E\Big(\big[E\big(\hat{R}_h(v)|\mathscr{F}_k\big)-E\big(\hat{R}_h(v)|\mathscr{F}_{k-1}\big)\big]^2\Big|\mathscr{F}_{k-1}\Big)\\
&= \int F(dy^k)\bigg[\int\big(f(X_{w^1},\dots,X_{w^{k-1}},y^k,x^{k+1},\dots,x^N)\\
&\qquad -f(X_{w^1},\dots,X_{w^{k-1}},x^k,x^{k+1},\dots,x^N)\big)\prod_{i=k}^{n}F(dx^i)\bigg]^2\\
&\le (\log(2n+1))^2\int F(dy^k)\\
&\qquad \times\bigg[\int\big(I_k(X,y^k,x)+I_k^*(X,x)\big)\prod_{i=k}^{N}F(dx^i)\bigg]^2\\
&\le (\log(2n+1))^2\int\big[I_k(X,y^k,x)+I_k^*(X,x)\big]^2F(dy^k)\prod_{i=k}^{N}F(dx^i)\\
&\qquad\qquad \text{(by Schwarz' inequality)}\\
&\le 4(\log(2n+1))^2\int I_k^*(X,x)\prod_{i=k}^{N}F(dx^i)\\
&= 4(\log(2n+1))^2E\big(\mathbb{1}\big[w^k\in\pi(h,v)\big]\big|\mathscr{F}_{k-1}\big)\\
&= E(U_k|\mathscr{F}_{k-1}).
\end{aligned}
\tag{2.19}
$$

This verifies (1.28) in Kesten (1993). Now let

$$x_0 = 8hc^2 \geq e^2c^2 \tag{2.20}$$

[so that (1.29) in Kesten (1993) is satisfied] and note that if $x \geq x_0$, we have, by (2.17) and (2.20),

$$\begin{aligned}\sum_{k=1}^{N} U_k &= 4(\log(2n+1))^2 \sum_{k=1}^{N} \mathbb{1}[w^k \in \pi(h,v)] \\ &= 4(\log(2n+1))^2 |\pi(h,v)| \\ &= 4(\log(2n+1))^2 h \leq 8c^2 h = x_0 \leq x.\end{aligned} \tag{2.21}$$

Therefore, (1.30) in Kesten (1993) is trivially satisfied (for any choice of $C_1, C_2 > 0$). We now follow Theorem 3 there. From (2.9) it follows that

$$x := y\sqrt{h} \leq C_3 h^{3/2}(\log(2n+1))^3 \leq x_0^{3/2} \tag{2.22}$$

provided we take $C_3 \leq 8^{5/2}$. Inequality (1.32) in Kesten (1993) [applied to $\hat{R}_h(v) - E\hat{R}_h(v)$ and to $-\hat{R}_h(v) + E\hat{R}_h(v)$] now yields (2.10) for suitable constants $0 < C_4, C_5 < \infty$. □

3. Convergence of cylinder quantities. Our aim here is to prove the relations (1.8) and (1.9). This will be done by a subadditivity argument. First we prove some integrability properties of our variables, though.

LEMMA 3. *If* (1.1) *holds, then there exists a constant* $C_1 < \infty$ *such that*

$$E\left\{\left(\frac{1}{n}N_n\right)^2\right\} \leq C_1, \qquad n \geq 2. \tag{3.1}$$

PROOF. Let c_0 and $\bar{c}$ be as in Proposition A. Then

$$E\left\{\left((N_n - 2(1+\bar{c}c_0)n)^+\right)^2\right\} = 2n^2\int_0^\infty yP\{N_n \geq 2(1+\bar{c}c_0)n + yn\}dy.$$

Now

$$N_n \leq \max_{\substack{|\xi|=n \\ \mathbf{0}\in\xi}} \sum_{v\in\xi} (X_v \wedge \gamma(n)) + \sum_{v\in[-n,n]^d} [X_v - \gamma(n)]^+.$$

Therefore, by Proposition A, Schwarz' inequality and (1.1), we have for $n \geq 2$, $b = 1 + a/d$ and suitable $C_2, C_3 < \infty$,

$$
\begin{aligned}
E&\left\{\left((N_n - 2(1+\bar{c}c_0)n)^+\right)^2\right\} \\
&\leq 2n^2 \int_0^\infty yP\left(\sum_{v\in[-n,n]^d} [X_v - \gamma(n)]^+ \geq \left(1 + \bar{c}c_0 + \frac{y}{2}\right)n\right) dy \\
&\quad + 2n^2 \int_0^\infty y \exp\left\{-\left(\frac{\bar{c}}{8} + \frac{y}{16c_0}\right)(\log n)^b\right\} dy \\
&\leq 4E\left\{\sum_{v\in[-n,n]^d} [X_v - \gamma(n)]^+\right\}^2 \qquad (3.2) \\
&\quad + 2n^2 \left\{\frac{16c_0}{(\log n)^b}\right\}^2 \exp\left(-\frac{\bar{c}}{8}(\log n)^b\right) \\
&\leq 4(2n+1)^{2d}\left(E[X_0 - \gamma(n)]^+\right)^2 \\
&\quad + 4(2n+1)^d E\{X_0^2 \mathbb{1}[X_0 > \gamma(n)]\} + C_2 n^2.
\end{aligned}
$$

Now by (1.1) and (1.11),

$$
\begin{aligned}
E[X_0 - \gamma(n)]^+ &\leq [\gamma(n)]^{1-d}[\log \gamma(n)]^{-d-a} \int_{[\gamma(n),\infty)} x^d (\log x)^{d+a}\, dF(x) \\
&= o(n^{1-d}),
\end{aligned}
$$

and, similarly,

$$
E\{X_0^2 \mathbb{1}[X_0 > \gamma(n)]\} = o\left([\gamma(n)]^{2-d}[\log \gamma(n)]^{-d-a}\right) = o(n^{2-d}).
$$

Therefore, the right-hand side of (3.2) is at most $C_3 n^2$. □

The variables $R_n(\beta)$ and $V_n(\beta)$ as they stand are not superadditive, but the slightly modified versions of (2.7) and (2.8) are (more precisely, we show that these modifications are "subconvolutive").

LEMMA 4. *For all real x and integers $m, n > L$,*

$$
P\{Y_{n+m}(\beta) \geq x\} \geq P\{Y_n(\beta) + Y'_m(\beta) \geq x\}, \tag{3.3}
$$

$$
P\{Z_{n+m}(\beta) \geq x\} \geq P\{Z_n(\beta) + Z'_m(\beta) \geq x\}, \tag{3.4}
$$

where $Y'_m(\beta)$ and $Z'_m(\beta)$ are random variables with the same distribution as $Y_m(\beta)$ and $Z_m(\beta)$, respectively, and independent of the $\{Y_n(\beta)\}$ and $\{Z_n(\beta)\}$, respectively.

PROOF. We only prove the relation (3.3) for the Y's; the proof for the Z's is similar. For integers r and s let

$$\mathscr{F}(r,s) = \sigma\text{-field generated by } \{X_v: r \le v(1) \le s\}.$$

Fix $m, n > L$. Note that $R_{(n-L)}(\beta)$ is the maximum of $S(\pi)$ over the finitely many cylinder paths π from $\mathbf{0}$ to $H(\lfloor (n-L)\beta \rfloor)$ of length $n-L$. For such paths, $S(\pi)$ is $\mathscr{F}(0, \lfloor (n-L)\beta \rfloor)$-measurable, and so are the $R_{(n-L)}(\beta)$. Let $\pi_1 = (\mathbf{0}, \dots, w_{n-L-1})$ be a cylinder path for which $S(\pi_1) = R_{(n-L)}(\beta)$. If the distribution of X_0 is not continuous, there may be several possible choices for π_1, due to ties among the $S(\pi)$'s. In this case we order all paths in some arbitrary way and take for π_1 the first path in this ordering with $S(\pi_1) = R_{(n-L)}(\beta)$. Observe that the endpoint w_{n-L-1} of π_1 lies in $H(\lfloor (n-L)\beta \rfloor)$.

Next note that

$$\mathscr{F}' := \mathscr{F}(\lfloor (n-L)\beta \rfloor + 1, \lfloor (n-L)\beta \rfloor + 1 + \lfloor (m-L)\beta \rfloor)$$

is independent of $\mathscr{F}(0, \lfloor (n-L)\beta \rfloor)$, and therefore $S(\pi)$ is independent of $\mathscr{F}(0, \lfloor (n-L)\beta \rfloor)$ for any $\pi = (v_0, v_1, \dots, v_{m-L-1})$ for which

$$\begin{aligned} &\lfloor (n-L)\beta \rfloor + 1 \le v_i(1) \le \lfloor (n-L)\beta \rfloor + 1 + \lfloor (m-L)\beta \rfloor \\ &\qquad\qquad\qquad\qquad \text{for } 0 \le i \le m-L-1, \\ &v_{m-L-1}(1) = \lfloor (n-L)\beta \rfloor + 1 + \lfloor (m-L)\beta \rfloor. \end{aligned} \tag{3.5}$$

For fixed $v_0 \in H(\lfloor (n-L)\beta \rfloor + 1)$, the maximum $S(\pi)$ over such paths is therefore a random variable $Y'_m(v_0, \beta) = R'_{(m-L)}(v_0, \beta)$, independent of $Y_n(\beta)$ and with the same distribution as $Y_m(\beta)$. Let

$$\pi_2 = (w_{n-L-1} + e_1, z_1, \dots, z_{m-L-1})$$

be a cylinder path starting at $z_0 = w_{n-L-1} + e_1$ and satisfying (3.5) with v replaced by z and such that $S(\pi_2) = Y'_m(w_{n-L-1} + e_1, \beta)$. The concatenation

$$\pi_3 = (\mathbf{0}, \dots, w_{n-L-1}, w_{n-L-1} + e_1, z_1, \dots, z_{m-L-1})$$

of π_1 and π_2 is a cylinder path from $\mathbf{0}$ to $H(\lfloor (n-L)\beta \rfloor + 1 + \lfloor (m-L)\beta \rfloor)$ of length $n+m-2L$. Note that π_3 is self-avoiding by construction. The distribution of $S(\pi_3)$ is the same as that of $Y_n(\beta) + Y'_m(\beta)$, so that we can prove (3.3) by showing that

$$Y_{n+m}(\beta) \ge S(\pi_3). \tag{3.6}$$

By definition, $Y_{n+m}(\beta)$ is the maximum of $S(\pi)$ over cylinder paths of length $n+m-L$ from $\mathbf{0}$ to $H(\lfloor (n+m-L)\beta \rfloor)$. Although π_3 is not such a path, we shall show how to find such a path π_4 by adding L vertices at the end of π_3. This will prove that $Y_{n+m}(\beta) \ge S(\pi_4) \ge S(\pi_3)$ and hence that (3.6) holds. The construction of π_4 is very easy. As we have seen, the endpoint z_{m-L-1} of π_3 lies in $H(\lfloor (n-L)\beta \rfloor + 1 + \lfloor (m-L)\beta \rfloor)$. Now set

$$K = \lfloor (n+m-L)\beta \rfloor - \{\lfloor (n-L)\beta \rfloor + 1 + \lfloor (m-L)\beta \rfloor\}$$

and note that $1 \le L\beta - 2 \le K \le L\beta + 1 < L$ [see (2.6)]. We may therefore form π_4 by adding to π_3 K steps in the e_1 direction from z_{m-L-1}, and then

$L - K$ steps in the e_2 direction, so that

$$\pi_4 = (\mathbf{0}, w_1, \dots, w_{n-L-1}, w_{n-L-1} + e_1, \dots, z_{m-L-1}, z_{m-L-1} + e_1, \dots,$$
$$z_{m-L-1} + Ke_1, z_{m-L-1} + Ke_1 + e_2, \dots, z_{m-L-1} + Ke_1 + (L-K)e_2).$$

Because $K \geq 1$, π_4 is self-avoiding.

This proves (3.6) and (3.3). □

For typographical reasons we shall sometimes write $Y(n)$ or $Y(n, \beta)$ instead of Y_n or $Y_n(\beta)$ and we shall adopt a similar convention for Z.

LEMMA 5. *Assume that* (1.1) *holds. For each fixed* $0 < \beta < 1$ *there exist constants* $0 \leq M(\beta) \leq N(\beta) < \infty$ *such that for each* $\delta > 0$,

$$\sum_{n=1}^{\infty} n^{d+1} P\left\{\left|\frac{\hat{Y}(n,\beta)}{n} - M(\beta)\right| > \delta\right\} < \infty \tag{3.7}$$

and

$$\frac{Y_n(\beta)}{n} \to M(\beta) \quad w.p.1. \tag{3.8}$$

Similarly,

$$\sum_{n=1}^{\infty} n^{d+1} P\left\{\left|\frac{\hat{Z}(n,\beta)}{n} - N(\beta)\right| \geq \delta\right\} < \infty \tag{3.9}$$

and

$$\frac{Z_n(\beta)}{n} \to N(\beta) \quad w.p.1. \tag{3.10}$$

REMARK. The factor n^{d+1} can be replaced by any power of n in (3.7) and (3.9), but we only use the lemma in the form given here.

PROOF. Again, we restrict ourselves to (3.7) and (3.8). Fix $\beta \in (0, 1)$. We shall apply Lemma 1 and Proposition 2. By Lemma 1, for $n > L$,

$$\begin{aligned} 0 &\leq Y(n,\beta) - \hat{Y}(n,\beta) = R_{n-L}(\beta) - \hat{R}_{n-L}(\beta) \\ &\leq \max_{\substack{|\xi| = n-L \\ \mathbf{0} \in \xi}} \sum (S(\xi) - \hat{S}(\xi)) \\ &= o(n) \quad \text{w.p.1.} \end{aligned} \tag{3.11}$$

Thus, it suffices to prove (3.8) with Y replaced by $\hat{Y}$, and this will be immediate from (3.7). As for (3.7), this will follow from Proposition 2 (with $h = n$ and $y = (\delta/2)\sqrt{n}$), once we prove

$$M(\beta) := \lim \frac{1}{n} E\hat{Y}(n,\beta) \text{ exists and is finite.} \tag{3.12}$$

However, from (3.11), the bound $0 \le Y(n,\beta) - \hat{Y}(n,\beta) \le N_n$ and the integrability condition (3.1), we see that

$$\lim \frac{1}{n}\left[EY(n,\beta) - E\hat{Y}(n,\beta)\right] = 0 \tag{3.13}$$

and that

$$0 \le \frac{1}{n} EY(n,\beta) \le \frac{1}{n} EN_{n-L}$$

is bounded. Finally, from (3.3) we have

$$EY(n+m,\beta) \ge EY(n,\beta) + EY(m,\beta), \qquad n,m > L.$$

The standard proof for subadditive sequences [see Hille and Phillips (1957), Theorem 9.6.1] now implies

$$\lim \frac{1}{n} EY(n,\beta) \text{ exists.}$$

Now (3.12) follows from this and (3.13). □

We next prove a kind of continuity in β, in order to derive (1.9) from Lemma 5. By a small modification of a cylinder path π whose length is smaller than n, we will obtain a cylinder path π' of a prescribed width such that its length $|\pi'|$ is not much more than $|\pi|$, and whose weight satisfies $S(\pi') \ge S(\pi)$. For a path $\pi = (v_0, \dots, v_k)$, let $D(\pi) = v_k(1) - v_0(1)$.

LEMMA 6. *Assume that either*

$$0 < \beta \le \tfrac{1}{2}, \qquad \beta \le \beta' < 1, \tag{3.14}$$

or

$$\tfrac{1}{2} < \beta \le 1, \qquad 0 < \beta' \le \beta. \tag{3.15}$$

Let $\pi = (v_0, \dots, v_k)$ be a cylinder path with

$$|\pi| \le n, \qquad D(\pi) = \lfloor \beta|\pi| \rfloor, \qquad v_0 \in [-n,n]^d. \tag{3.16}$$

Then there exists a cylinder path $\pi' = (v_0, \dots, v_k, u_0, \dots, u_m)$, for suitable m, that contains π as a subpath and satisfies

$$\begin{aligned} &D(\pi') = \lfloor \beta'|\pi'| \rfloor, \qquad S(\pi') \ge S(\pi), \\ &\hat{S}(\pi') \ge \hat{S}(\pi), \qquad \textit{initial point of } \pi' = v_0 \in [-n,n]^d \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} 0 &\le |\pi'| - |\pi| \\ &\le \frac{3}{\beta'} + |\beta' - \beta| n \left\{ \mathbb{1}\left[\beta \le \frac{1}{2}\right] \frac{1}{1-\beta'} + \mathbb{1}\left[\beta > \frac{1}{2}\right] \frac{1}{\beta'} \right\}. \end{aligned} \tag{3.18}$$

PROOF. We follow more or less the construction of π_4 in Lemma 4. We form π' from π by adding j_1 steps in the e_1 direction and then j_2 steps in the

e_2-direction to the endpoint of π. This will give us a cylinder path π' with

$$(3.19)\qquad |\pi'| = |\pi| + j_1 + j_2, \qquad D(\pi') = D(\pi) + j_1.$$

In order to make sure that π' is self-avoiding, we want $j_1 \geq 1$, and in order to satisfy (3.17) we want

$$D(\pi') = D(\pi) + j_1 = \lfloor \beta|\pi| \rfloor + j_1 = \lfloor \beta'|\pi'| \rfloor = \lfloor \beta'|\pi| + \beta'(j_1 + j_2) \rfloor.$$

We satisfy these requirements by taking

$$(3.20)\qquad j_1 = \left\lceil \frac{1}{(1-\beta')}(\beta' - \beta)^+ |\pi| \right\rceil + 1$$

and then solve for j_2 from

$$(3.21)\qquad \lfloor \beta|\pi| \rfloor + j_1 = \lfloor \beta'|\pi| + \beta' j_1 + \beta' j_2 \rfloor.$$

Since

$$\lfloor \beta|\pi| \rfloor + j_1 \geq \lfloor \beta'|\pi| + \beta' j_1 \rfloor$$

for the j_1 of (3.20), (3.21) does indeed have a solution $j_2 \geq 0$.

It follows from (3.20) and (3.21) that

$$\begin{aligned}
|\pi'| - |\pi| = j_1 + j_2 &\leq \frac{\beta - \beta'}{\beta'}|\pi| + \frac{1}{\beta'} j_1 + \frac{1}{\beta'} \\
&\leq \left\{ \frac{\beta - \beta'}{\beta'} + \frac{1}{\beta'(1-\beta')}(\beta' - \beta)^+ \right\} |\pi| + \frac{3}{\beta'} \\
&\leq |\beta - \beta'| n \left\{ \mathbb{1}\left[\beta \leq \frac{1}{2}\right] \frac{1}{1-\beta'} + \mathbb{1}\left[\beta > \frac{1}{2}\right] \frac{1}{\beta'} \right\} + \frac{3}{\beta'},
\end{aligned}$$

so that (3.18) also holds. □

We also need the analog of Lemma 6 for lattice animals, instead of paths. If $\xi = \{v_0, \dots, v_k\}$ is a collection of vertices, we call $v_j \in \xi$ a *leftmost* (*rightmost*) point of ξ if $v_j(1) \leq v_i(1)$ [respectively, $v_j(1) \geq v_i(1)$] for $0 \leq i \leq k$. We now merely have to take v_0 and v_k as the left and rightmost points of a connected set ξ and again $D(\xi) = v_k(1) - v_0(1)$, to obtain the following analog of Lemma 6. The proof needs no change.

LEMMA 7. *Assume that* (3.14) *or* (3.15) *holds. Let* $\xi = \{v_0, \dots, v_k\}$ *be a connected set of vertices with* v_0 *and* v_k *as left and rightmost point, respectively. Assume that* (3.16) *holds with* $|\pi|$ *replaced by* $|\xi|$. *Then there exists a connected set* $\xi' = \{v_0, \dots, v_k, u_0, \dots, u_m\}$, *for suitable* m, *which contains* ξ *as a subset, with* v_0 *and* u_m *as left and rightmost points of* ξ', *and such that* (3.17) *and* (3.18) *hold with* π *and* π' *replaced by* ξ *and* ξ', *respectively.*

The next proposition proves the convergence of cylinder functions.

PROPOSITION 8. *If* (1.1) *holds, then w.p.*1,

$$(3.22)\qquad \frac{R_n}{n} \to M$$

and

$$\frac{V_n}{n} \to N, \tag{3.23}$$

with $M = \sup_{0<\beta<1} M(\beta)$ *and* $N = \sup_{0<\beta<1} N(\beta)$.

PROOF. Since, for each $\beta \in (0,1), R_n \geq R_n(\beta)$ *and* $R_n(\beta)/n \to M(\beta)$ w.p.1 (by Lemma 5), it follows that

$$\liminf_n \frac{R_n}{n} \geq \sup_{0<\beta<1} M(\beta) \quad \text{w.p.1.}$$

Similarly,

$$\liminf_n \frac{V_n}{n} \geq \sup_{0<\beta<1} N(\beta) \quad \text{w.p.1.}$$

Therefore, we only have to investigate $\limsup_n (1/n) R_n$ and $\limsup_n (1/n) V_n$ or, by virtue of Lemma 1, $\limsup_n (1/n) \hat{R}_n$ and $\limsup_n (1/n) \hat{V}_n$. Again we restrict ourselves to the first of these.

Let $\varepsilon > 0$ be given. Fix an integer $q \geq 8$ such that

$$\left(1 - \frac{4}{q}\right)(M + \varepsilon) \geq \left(M + \frac{3\varepsilon}{4}\right). \tag{3.24}$$

For later purposes it is useful not to restrict ourselves here to paths starting at the origin, so for the remainder of the proof we merely assume that $\pi = (v_0, \dots, v_{n-1})$ is a cylinder path with

$$|\pi| = n, \qquad v_0 \in [-n, n]^d, \tag{3.25}$$

and

$$\hat{S}(\pi) \geq (M + \varepsilon)|\pi|. \tag{3.26}$$

We can define β by

$$\beta = \frac{v_{n-1}(1) - v_0(1)}{|\pi|} = \frac{D(\pi)}{|\pi|}.$$

Then π satisfies (3.16) with this β. We take $\beta' = j/q$ for a $j \in \{1, \dots, q-1\}$ such that (3.14) or (3.15) holds and

$$|\beta' - \beta| \leq \frac{1}{q}.$$

The idea of the proof is to show that π' constructed as in Lemma 6 for this β' must satisfy

$$\hat{S}(\pi') \geq (M + \varepsilon/2)|\pi'| \geq (M(\beta') + \varepsilon/2)|\pi'|. \tag{3.27}$$

It will then be shown, by an application of (3.7), that (3.27) does not happen when $|\pi'|$ is large. This in turn implies that for large n, there are no cylinder paths π that satisfy (3.25) and (3.26). This essentially proves

$$\limsup \frac{R_n}{n} \le M + \varepsilon \quad \text{w.p.1.}$$

The details of the argument are fairly straightforward. For the sake of the argument let $\beta \le \frac{1}{2}$; the case $\beta > \frac{1}{2}$ will be left to the reader. Since $\beta \le \frac{1}{2}$ we may take $\beta' \ge \beta$, so that we will have

$$\frac{1}{q} \le \beta' \le \frac{1}{2} + \frac{1}{q}, \qquad 0 < \beta \le \beta' \le \beta + \frac{1}{q} \le \frac{3}{4}. \tag{3.28}$$

Now, if π' satisfies (3.17) and (3.18) and n is large enough that

$$\left(1 - \frac{3q}{n} - \frac{4}{q}\right)(M + \varepsilon) \ge \left(M + \frac{\varepsilon}{2}\right) \quad \text{and} \quad \frac{3q}{n} \le \frac{1}{2} \tag{3.29}$$

[which is possible by (3.24)], then

$$\begin{aligned}
\hat{S}(\pi') &\ge \hat{S}(\pi) \quad [\text{by } (3.17)] \\
&\ge (M + \varepsilon)|\pi| \quad [\text{by } (3.26)] \\
&\ge (M + \varepsilon)\left(|\pi'| - \frac{3}{\beta'} - \frac{\beta' - \beta}{1 - \beta'}n\right) \quad [\text{by } (3.18)] \\
&\ge (M + \varepsilon)\left(|\pi'| - 3q - \frac{4}{q}n\right) \quad [\text{by } (3.28)] \\
&\ge (M + \varepsilon)|\pi'|\left(1 - \frac{3q}{n} - \frac{4}{q}\right) \quad (\text{since } |\pi'| \ge |\pi| = n) \\
&\ge \left(M + \frac{\varepsilon}{2}\right)|\pi'| \quad [\text{by } (3.29)].
\end{aligned}$$

Thus (3.27) holds.

Recall that the path π' has the same initial point $v_0 \in [-n, n]^d$ as π. Moreover,

$$\begin{aligned}
n = |\pi| \le |\pi'| &\le |\pi| + 3q + \frac{4}{q}n \quad [\text{by } (3.18)] \\
&\le |\pi| + n \quad [\text{by } q \ge 8 \text{ and } (3.29)] \\
&= 2n.
\end{aligned}$$

Consequently, for n satisfying (3.29), the event

$$\mathscr{A}(n, \varepsilon) := \{\text{there exists a cylinder path } \pi \text{ satisfying } (3.25) \text{ and } (3.26)\} \tag{3.30}$$

is contained in

$$\bigcup_{v_0 \in [-n, n]^d} \bigcup_{j=1}^{q-1} \mathscr{A}'(n, \varepsilon; v_0, j, q),$$

where

$$\mathscr{A}'(n,\varepsilon;v_0,j,q) := \{\text{there exists a cylinder path } \pi' \text{ from the point } v_0 \text{ to the hyperplane } H(v_0(1)+\lfloor j/q|\pi'|\rfloor) \text{ with } n \le |\pi'| \le 2n \text{ and satisfying (3.27) with } \beta' = j/q\}.$$

Consequently,

$$P(\mathscr{A}(n,\varepsilon)) \le (2n+1)^d \sum_{j=1}^{q-1} \sum_{n\le m\le 2n} P\left(\hat{Y}_{m+L}\left(\frac{j}{q}\right) \ge \left(M\left(\frac{j}{q}\right)+\frac{\varepsilon}{2}\right)m\right).$$

It follows that for some constants $C_1, C_2 < \infty$,

$$\sum_{n=1}^{\infty} P(\mathscr{A}(n,\varepsilon)) \le C_1 + C_2 \sum_{j=1}^{q-1} \sum_{m=1}^{\infty} m^{d+1} P\left(\hat{Y}_{m+L}\left(\frac{j}{q}\right) \ge \left(M\left(\frac{j}{q}\right)+\frac{\varepsilon}{2}\right)m\right) < \infty,$$

where C_1 takes care of the small n's and the finiteness of the last double sum follows from (3.7). Thus $\mathscr{A}(n,\varepsilon)$ occurs w.p.1 for only finitely many n. By the definition (1.5) this means that w.p.1 $\hat{R}_n \le (M+\varepsilon)n$ eventually. In other words,

$$\limsup_n \frac{\hat{R}_n}{n} \le M+\varepsilon \quad \text{w.p.1.}$$

Since this holds for all $\varepsilon > 0$, (3.22) now follows by means of Lemma 1.

The proof of (3.23) is a straightforward adaptation of the preceding proof of (3.22). A slightly stronger statement is proved in the following proof of (1.3). □

We can now also complete the following proof.

PROOF OF (1.3). This really is contained in the proof of Proposition 8, since there we did not restrict to paths starting at the origin. Since $N_n \ge V_n$, we only have to consider $\limsup_n (1/n) N_n$ when proving the convergence w.p.1 in (1.3) [compare (3.23)]. Now observe that any lattice animal ξ that contains the origin and satisfies $|\xi| = n$ has a leftmost point somewhere in $[-n,n]^d$. We, therefore, have exactly as after (3.30), with N as in (1.9),

$$\left\{\hat{N}_k \ge (N+\varepsilon)\right\} \subseteq \bigcup_{v_0\in[-n,n]^d} \bigcup_{j=1}^{q-1} \bigcup_{m=n}^{2n} \mathscr{B}'(n,\varepsilon;v_0,j,m,q)$$

where

$$\mathscr{B}'(n,\varepsilon;v_0,j,m,q) := \{\text{there exists a connected set } \xi' \text{ with } |\xi'| = m \text{ which has } v_0 \text{ as its leftmost point and has its rightmost point in } H(v_0(1)+\lfloor (j/q)m\rfloor) \text{ and } \hat{S}(\xi') \ge (N(j/q)+\varepsilon/2)m\}.$$

Consequently, by (3.9),

$$\sum P\big(\hat{N}_n \geq (N+\varepsilon)n\big)$$
$$\leq C_1 + C_2 \sum_{j=1}^{q-1} \sum_{m=1}^{\infty} m^{d+1} P\left\{\hat{Z}_{m+L}\left(\frac{j}{q}\right) \geq \left(N\left(\frac{j}{q}\right) + \frac{\varepsilon}{2}\right) m\right\} < \infty.$$

Again by an appeal to Lemma 1, this implies the convergence w.p.1 in (1.3). The L^1 convergence follows from this, together with (3.1). Even though we decided to ignore the case when $d = 1$, we note that we cannot use (3.1) if X_0 does not have a second moment. However, if $d = 1$ and the X_i are i.i.d. with finite mean, then

$$\frac{1}{n}\sum_1^n E\{|X_i|\mathbb{1}[|X_i| \geq A]\} = E\{|X_0|I|[|X_0| \geq A]\}$$

can be made as small as desired by taking A large. We may therefore truncate our X_v's at a large A. This is enough to obtain the L^1 convergence when $d = 1$. □

REMARK. In some of the examples of Cox, Gandolfi, Griffin and Kesten (1993), Section 1, one of the coordinates plays a special role (e.g., the role of time). One may then want to restrict to paths along which this component is increasing. If one takes the special component to be the first one, this means that one only wants to consider the cylinder paths already introduced. In this situation Proposition 8 completes the proof of the basic convergence result; see, for instance, Glynn and Whitt (1991).

4. Nonoccurrence of large bad cylinder paths. Define $\hat{\mathscr{B}}(n, h, \varepsilon)$ as in (1.10), but with S replaced by $\hat{S}$. In this section we prove Lemma 9, which shows that cylinder paths π in $[-n, n]^d$ with $\hat{S}(\pi) \geq (M+\varepsilon)|\pi|$ cannot be too long.

LEMMA 9. *Assume* (1.1), *and let* $\varepsilon > 0$ *and* $\delta > 0$. *Then w.p.*1,

(4.1) $\hat{\mathscr{B}}(n, n^\delta, \varepsilon)$ *occurs for only finitely many* n*'s*.

PROOF. Fix $\varepsilon, \delta > 0$. If $\hat{\mathscr{B}}(n, n^\delta, \varepsilon)$ occurs, then $\hat{R}_h(v) \geq (M+\varepsilon)h$ [see (2.5) for notation] for some $n^\delta \leq h \leq n$ and $v \in [-n, n]^d$. Therefore,

$$\hat{\mathscr{B}}(n, n^\delta, \varepsilon) \subseteq \bigcup_{n^\delta \leq h \leq n} \bigcup_{v \in [-n,n]^d} \{\hat{R}_h(v) \geq (M+\varepsilon)h\}. \tag{4.2}$$

Next, observe that for all v,

$$E\big(\hat{R}_h(v)\big) \leq E(R_h)$$

and that, from Proposition 8, $h^{-1}R_h \to M$ w.p.1. In view of Lemma 3, this implies, uniformly in v,

$$\limsup_{h\to\infty} E\left\{\frac{\hat{R}_h(v)}{h}\right\} \le \lim_{h\to\infty} E\left\{\frac{R_h}{h}\right\} = M. \tag{4.3}$$

Consequently, for large n,

$$\hat{\mathscr{B}}(n, n^\delta, \varepsilon) \subseteq \bigcup_{n^\delta \le h \le n} \bigcup_{v\in[-n,n]^d} \left\{\hat{R}_h(v) - E\big(\hat{R}_h(v)\big) \ge \frac{\varepsilon}{2}h\right\}. \tag{4.4}$$

We now use Proposition 2 in which we take $y = (\varepsilon/2)\sqrt{h}$. For n large enough and all $n^\delta \le h \le n$,

$$y = \frac{\varepsilon}{2}\sqrt{h} \le C_3 h(\log(2n+1))^3.$$

We thus obtain that, for n large enough,

$$\begin{aligned} P\{\hat{\mathscr{B}}(n, n^\delta, \varepsilon)\} &\le \sum_{n^\delta \le h \le n} \sum_{v\in[-n,n]^d} C_4 \exp\left(-C_5\frac{\varepsilon}{2}\frac{\sqrt{h}}{\log(2n+1)}\right) \\ &\le (2n+1)^d n C_4 \exp\left(-C_5 \frac{\varepsilon}{2}\frac{n^{\delta/2}}{\log(2n+1)}\right), \end{aligned}$$

so that

$$\sum_{n=1}^{\infty} P\{\hat{\mathscr{B}}(n, n^\delta, \varepsilon)\} < \infty.$$

Thus, (4.1) follows from the Borel–Cantelli lemma. □

REMARK. It is tempting to apply the same argument to estimate

$$P\{|\hat{M}_n - Mn| \ge \varepsilon n\}.$$

There is certainly no difficulty to obtain a good estimate for

$$P\left\{\left|\hat{M}_n - E\{\hat{M}_n\}\right| \ge \varepsilon n\right\}$$

in the same way as in Proposition 2. The only stumbling block to a quick proof of (1.2) from this is that we do not have a direct way to prove that

$$\frac{1}{n}E\{\hat{M}_n\} \to M.$$

This forces us to follow the rather circuitous route of the next section.

5. Decomposition into cylinder paths. In this section we complete the proof of (1.2) by decomposing a path π more or less into cylinder paths. These cylinder paths will be pieces of the path between a leftmost and a rightmost point either of the path itself or of a subpath obtained in an earlier stage of the decomposition.

LEMMA 10. *Let $\varepsilon > 0$ and let $\pi = (v_0, v_1, \dots, v_{h-1})$ be a path of length $h \le n$, contained in $[-n, n]^d$ and with $\hat{S}(\pi) \ge (M + \varepsilon)|\pi|$, where M is as in Proposition 8. Suppose that $\hat{\mathscr{B}}(n, n^{1/(2d)}, \varepsilon/2)$ does not occur. Then π can be decomposed into the following pieces:*

(5.1) *a path $\sigma = (v_0, v_1, \dots, v_l)$ with v_l a leftmost or a rightmost point of σ and $|v_l(1) - v_i(1)| < n^{1/(2d)}$, for $0 \le i \le l$;*

(5.2) *another path $\tau = (v_m, \dots, v_{h-1})$ with v_m a leftmost or a rightmost point of τ and $|v_m(1) - v_i(1)| < n^{1/(2d)}$ for $m \le i \le h-1$;*

(5.3) *a number of cylinder paths $\nu_1, \nu_2, \dots, \nu_r$ with width in the first coordinate direction at least $n^{1/(2d)}$, each of which satisfies*

$$\hat{S}(\nu_j) < \left(M + \frac{\varepsilon}{2}\right)|\nu_j|;$$

(5.4) *at most one cylinder path ν_{r+1} with width in the first coordinate direction smaller than $n^{1/(2d)}$.*

Moreover, if any two of these paths have a vertex in common, then this vertex is an endpoint of both paths. No vertex belongs to more than two of these paths.

REMARK. The paths σ and τ in (5.1) and (5.2) are in general *not* cylinder paths.

PROOF. As a first step in our decomposition of π we take $v_{l(1)}$ and $v_{r(1)}$ as a leftmost and a rightmost point of π, respectively. For the sake of the argument, let $l(1) \le r(1)$. Then π is the concatenation of the three paths

$$\rho_1 := (v_0, \dots, v_{l(1)}),$$
$$\rho_2 := (v_{l(1)}, \dots, v_{r(1)}),$$
$$\rho_3 := (v_{r(1)}, \dots, v_{h-1}).$$

Here, and in the following, it is convenient to use a broader (and, perhaps, improper) use of the word concatenation, under which we allow that the paths to be concatenated have the last vertex of one path equal to the first vertex of the other path. By construction, ρ_2 is a cylinder path and it cannot satisfy both

$$|v_{r(1)}(1) - v_{l(1)}(1)| \ge n^{1/(2d)} \quad \text{and} \quad \hat{S}(\rho_2) \ge \left(M + \frac{\varepsilon}{2}\right)|\rho_2| \tag{5.5}$$

or else $\hat{\mathscr{B}}(n, n^{1/(2d)}, \varepsilon/2)$ would occur. In fact, (5.5) would imply $|\rho_2| \ge |v_{r(1)}(1) - v_{l(1)}(1)| \ge n^{1/(2d)}$. Note also that

$$v_{l(1)}(1) \le v_i(1) \le v_{r(1)}(1) \quad \text{for all } 0 \le i \le h-1. \tag{5.6}$$

Therefore, the paths ρ_1 and ρ_3 have a width less than or equal to $|v_{r(1)}(1) - v_{l(1)}(1)|$ in the first coordinate direction. Moreover, ρ_1 has the endpoint $v_{l(1)}$ as a leftmost point, and ρ_3 has the initial point $v_{r(1)}$ as a rightmost point.

The intention is to repeat this kind of decomposition into cylinder paths plus one path with initial point v_0 and another with final point v_{h-1}, until the latter two paths have a width less than $n^{1/(2d)}$ in the first coordinate direction. Note that we never decompose a cylinder path any further, but only the two paths that start at v_0 or end at v_{h-1}, respectively, and these two are decomposed further only as long as their width in the first coordinate direction is at least $n^{1/(2d)}$. For instance, if ρ_1 has width less than $n^{1/(2d)}$ in the first coordinate axis, then we do not decompose it further, and similarly for ρ_3. For the sake of the argument assume thus that the width of both ρ_1 and ρ_3 in the first coordinate direction is at least $n^{1/(2d)}$. *We then decompose both ρ_1 and ρ_3 as follows. We take* $v_{r(1,2)}$ as a rightmost point of ρ_1, and $v_{l(3,2)}$ as a leftmost point of ρ_3. We get four new paths at this second stage; see Figure 1. The path ρ_1 breaks up into

$$\rho_{1,2} := (v_0, \dots, v_{r(1,2)}) \quad \text{and} \quad \rho_{2,2} := (v_{r(1,2)}, \dots, v_{l(1)}),$$

and ρ_3 breaks up into

$$\rho_{3,2} := (v_{r(1)}, \dots, v_{l(3,2)}) \quad \text{and} \quad \rho_{4,2} := (v_{l(3,2)}, \dots, v_{h-1}).$$

Here, $\rho_{2,2}$ and $\rho_{3,2}$ are again cylinder paths. Again, the analog of (5.5) cannot occur for either $\rho_{2,2}$ or $\rho_{3,2}$, or else $\hat{\mathscr{B}}(n, n^{1/(2d)}, \varepsilon/2)$ would occur.

We can now continue decomposing those paths among $\rho_{1,2}$ and $\rho_{4,2}$, which have a width in the first coordinate direction at least $n^{1/(2d)}$. This process stops when we have achieved a decomposition of π into one path satisfying (5.1), one satisfying (5.2) and some other paths satisfying either the condition in (5.3) or the one in (5.4). To see that at the end of the process there is at most one path satisfying the condition in (5.4), note that at each stage, the path that starts at v_0, $(v_0, v_1, \dots, v_u)$ say, has its endpoint as a leftmost or a

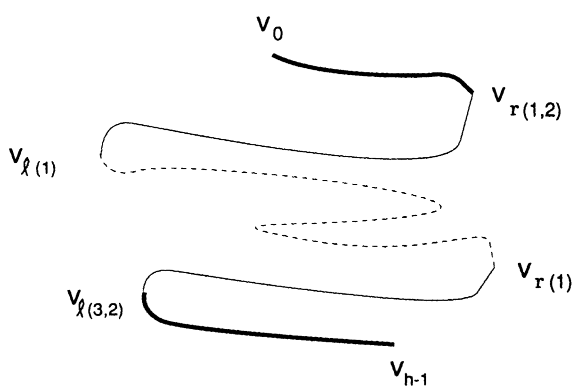

FIG. 1. *The second stage of the decomposition of π. The dashed piece is ρ_2. The boldly drawn pieces near v_0 and v_{h-1} are $\rho_{(1,2)}$ and $\rho_{(4,2)}$, respectively.*

rightmost point. If we decompose it into a piece $(v_0, \dots, v_t)$ and a cylinder path $(v_t, \dots, v_u)$, then the width of $(v_0, \dots, v_t)$ in the first coordinate direction is no more than that of $(v_t, \dots, v_u)$ [compare with (5.6)]. Since we only decompose $(v_0, v_1, \dots, v_u)$ further if its width in the first coordinate direction is at least $n^{1/(2d)}$, the resulting cylinder path $(v_t, \dots, v_u)$ also has width at least $n^{1/(2d)}$. This can only fail at the first step, namely ρ_1 may have width $< n^{1/(2d)}$. The same argument applies to the piece that ends at v_{h-1}. Therefore the only path satisfying (5.4) can be ζ_1.

The last property of the intersections of the subpaths is clear from the way the paths were obtained; the only vertices that can belong to two subpaths are the successive leftmost and rightmost points encountered at the various stages of the decomposition of π. This proves the lemma. □

It is now convenient to introduce the following notation:

$$\begin{aligned}\hat{L}_n &= \hat{L}_n(\varepsilon)\\ &:= \max\{\hat{S}(\pi)\colon \pi \text{ is a cylinder path in } [-n, n]^d \text{ of width in the first coordinate direction less than } n^{1/(2d)},\ |\pi| \le n, \text{ but } \hat{S}(\pi) \ge (M + \varepsilon/2)\,|\pi|\};\end{aligned} \tag{5.7}$$

$$\hat{\mathscr{H}}(n, \varepsilon) := \{\hat{L}_n \ge n^{1/2}\},$$

$$\gamma := 2^{-d}(8 + 8\bar{c}c_0)^{-1},$$

$$\hat{\mathscr{E}}(n, \varepsilon) := \{\text{there exists a path } \pi \text{ in } [-n, n]^d \text{ whose width in the first coordinate direction is less than } n^{1/(2d)}, \text{ but } \hat{S}(\pi) \ge (M + \varepsilon/2)|\pi| \text{ and } \gamma\varepsilon n^{1-1/(8d)} \le |\pi| \le n\}, \tag{5.8}$$

$$\hat{\mathscr{D}}(n, \varepsilon) := \{\text{there exists a path } \pi \text{ in } [-n, n]^d \text{ such that } |\pi| < \gamma\varepsilon n^{1-1/(8d)}, \text{ but } \hat{S}(\pi) \ge (\varepsilon/8) n^{1-1/(8d)}\},$$

with $\bar{c}$ as in Proposition A and $\varepsilon > 0$ arbitrary.

LEMMA 11. *Let $\varepsilon > 0$. There exists an $n_1(\varepsilon) < \infty$ such that if $n \ge n_1$, the following holds. If there exists a path π in $[-n, n]^d$ with $\hat{S}(\pi) \ge (M + \varepsilon)|\pi|$ and with length $n^{1-1/(8d)} \le |\pi| \le n$, and if $\hat{\mathscr{B}}(n, n^{1/(2d)}, \varepsilon/2)$ does not occur, then*

$$\hat{\mathscr{H}}(n, \varepsilon) \cup \hat{\mathscr{E}}(n, \varepsilon) \cup \hat{\mathscr{D}}(n, \varepsilon) \tag{5.9}$$

must occur.

PROOF. Let π be a path with the stated properties and assume $\hat{\mathscr{B}}(n, n^{1/(2d)}, \varepsilon/2)$ does not occur. Then decompose π according to Lemma 9. From (5.3) we have r paths ν_j with width larger than $n^{1/(2d)}$, and a fortiori length larger than $n^{1/(2d)}$. Since each vertex belongs to at most two of the paths ν_j in (5.3) (in fact only an endpoint of a ν_j can coincide with an endpoint of exactly one other ν_i), we find

$$r n^{1/(2d)} \le 2n \quad \text{or} \quad r \le 2n^{1-1/(2d)}. \tag{5.10}$$

If v_l is the endpoint of σ, as in (5.1), and v_m is the initial point of τ, as in (5.2), then

$$|\sigma| = l + 1, \qquad |\tau| = |\pi| - m, \tag{5.11}$$

and also, from (5.3) and (5.4),

$$\begin{aligned}\sum_{j=1}^{r+1} |\nu_j| &\le (m - l + 1) + r + 1 \\ &\le (m - l + 1) + 2(n^{1-1/(2d)} + 1),\end{aligned} \tag{5.12}$$

once more because only the endpoints of the ν_j can be counted twice in the left-hand side. Combined with the fact that

$$\sum_{j=1}^{r+1} \hat{S}(\nu_j) \le \left(M + \frac{\varepsilon}{2}\right) \sum_{j=1}^{r+1} |\nu_j| + \hat{L}_n, \tag{5.13}$$

these relations show

$$\begin{aligned}\hat{S}(\pi) &\le \hat{S}(\sigma) + \hat{S}(\tau) + \sum_{j=1}^{r+1} \hat{S}(\nu_j) \\ &\le \hat{S}(\sigma) + \hat{S}(\tau) + \left(M + \frac{\varepsilon}{2}\right)(m - l + 1) \\ &\quad + 2\left(M + \frac{\varepsilon}{2}\right)(n^{1-1/(2d)} + 1) + \hat{L}_n \\ &= \hat{S}(\sigma) + \hat{S}(\tau) + \left(M + \frac{\varepsilon}{2}\right)(|\pi| - |\sigma| - |\tau|) \\ &\quad + 2\left(M + \frac{\varepsilon}{2}\right)(n^{1-1/(2d)} + 2) + \hat{L}_n.\end{aligned} \tag{5.14}$$

Now assume that $\hat{S}(\pi) \ge (M + \varepsilon)|\pi| \ge (M + \varepsilon)n^{1-1/(8d)}$, but that $\hat{\mathscr{H}}(n, \varepsilon)$ does not occur (i.e., $\hat{L}_n < n^{1/2}$). Then, for sufficiently large n this, together with (5.14), implies

$$\begin{aligned}\hat{S}(\sigma) + \hat{S}(\tau) &\ge \frac{\varepsilon}{2}|\pi| + \left(M + \frac{\varepsilon}{2}\right)(|\sigma| + |\tau|) \\ &\quad - 2\left(M + \frac{\varepsilon}{2}\right)(n^{1-1/(2d)} + 2) - n^{1/2} \\ &\ge \frac{\varepsilon}{2}|\pi| + \left(M + \frac{\varepsilon}{2}\right)(|\sigma| + |\tau|) - 2n^{1-1/(4d)}.\end{aligned}$$

Consequently, one of the events

$$\begin{aligned}\hat{S}(\sigma) &\ge \left(M + \frac{\varepsilon}{2}\right)|\sigma| + \left(\frac{\varepsilon}{4}|\pi| - n^{1-1/(4d)}\right) \\ &\ge \left(M + \frac{\varepsilon}{2}\right)|\sigma| + \frac{\varepsilon}{8}n^{1-1/(8d)}\end{aligned}$$

or

$$\hat{S}(\tau) \geq \left(M + \frac{\varepsilon}{2}\right)|\tau| + \frac{\varepsilon}{8} n^{1-1/(8d)}$$

occurs. For the sake of the argument assume that the first one occurs. Then either

$$|\sigma| \geq \gamma \varepsilon n^{1-1/(8d)}$$

occurs, in which case $\hat{\mathscr{E}}(n, \varepsilon)$ occurs, or

$$|\sigma| < \gamma \varepsilon n^{1-1/(8d)}$$

occurs, in which case $\hat{\mathscr{D}}(n, \varepsilon)$ occurs. □

We already know, from Lemma 9, that w.p.1 $\mathscr{B}(n, n^{1/(2d)}, \varepsilon/2)$ occurs only finitely often. Thus one of the hypotheses of the last two lemmas is, w.p.1, valid for all large n. Our next lemma shows that also $\hat{\mathscr{H}}(n, \varepsilon)$ and $\hat{\mathscr{D}}(n, \varepsilon)$ occur w.p.1 only finitely often.

LEMMA 12. *For each* $\varepsilon > 0$,

$$P\{\hat{\mathscr{H}}(n, \varepsilon) \cup \hat{\mathscr{D}}(n, \varepsilon) \text{ occurs for infinitely many } n\} = 0.$$

PROOF. First consider $\hat{\mathscr{D}}(n, \varepsilon)$. By decomposing with respect to the location of the starting point of the path π and with respect to its length, we have

$$P(\hat{\mathscr{D}}(n, \varepsilon)) \leq (2n+1)^d \sum_{l \leq \gamma \varepsilon n^{1-1/(8d)}} P\left\{ \max_{\pi \in \Pi_0(l)} \sum_{v \in \pi} (X_v \wedge 2 \log n) \geq \frac{\varepsilon}{8} n^{1-1/(8d)} \right\}. \tag{5.15}$$

Since

$$\sum_{v \in \pi} (X_v \wedge 2 \log n) \leq 2l \log n \tag{5.16}$$

for any $\pi \in \Pi_0(l)$, the sum over l can be restricted to $l \geq C_6 \varepsilon n^{1-1/(8d)}$ $(\log n)^{-1}$. Therefore, by Proposition A,

$$\begin{aligned} P\{\hat{\mathscr{D}}(n, \varepsilon)\} &\leq (2n+1)^d \sum_{l \leq \gamma \varepsilon n^{1-1/(8d)}} \exp\left\{ -\frac{1}{8c_0}\left(\frac{\varepsilon}{8l} n^{1-1/(8d)} - 1\right)(\log l)^b \right\} \\ &\leq \exp\{-C_7 (\log n)^b\}, \end{aligned}$$

where, as before, $b = 1 + a/d$. Thus,

$$\sum_n P(\hat{\mathscr{D}}(n, \varepsilon)) < \infty.$$

Next, assume $\hat{\mathscr{H}}(n, \varepsilon)$ occurs. Then, there exists a cylinder path ν in $[-n, n]^d$, of width in the first coordinate direction less than $n^{1/(2d)}$, but with

$$\hat{S}(\nu) \geq \left(\left(M + \frac{\varepsilon}{2}\right)|\nu|\right) \vee n^{1/2}.$$

If $|\nu| \geq n^{1/4}$, then $\hat{\mathscr{B}}(n, n^{1/4}, \varepsilon/2)$ would occur, so that (by virtue of Lemma 9) only the possibility that $|\nu| < n^{1/4}$ remains. However, this situation cannot arise because, as in (5.16), for any such $\nu \subset [-n, n]^d$,

$$\hat{S}(\nu) = \sum_{v \in \nu} \hat{X}_v \leq |\nu| 2 \log n \leq n^{1/4} 2 \log n < n^{1/2}.$$

Lemmas 9–12 show that, apart from a set of measure zero,

$$\begin{aligned}\left\{\limsup \frac{\hat{M}_n}{n} \geq M + \varepsilon\right\} &\subseteq \{\text{for infinitely many } n\text{'s there exists a path } \pi \text{ in } [-n, n]^d, \\ &\qquad \text{with } |\pi| = n \text{ and } \hat{S}(\pi) \geq (M + \varepsilon)|\pi|\} \\ &\subseteq \{\hat{\mathscr{E}}(n, \varepsilon) \text{ occurs for infinitely many } n\text{'s}\}.\end{aligned}$$

Therefore, in order to show that

$$\limsup_{n \to \infty} \frac{\hat{M}_n}{n} \leq M, \tag{5.17}$$

it suffices to show that for each $\varepsilon > 0$,

$$P\{\hat{\mathscr{E}}(n, \varepsilon) \text{ occurs infinitely often}\} = 0. \tag{5.18}$$

This will be proved in Lemma 13. This will finish the proof of convergence w.p.1 in (1.2). In fact, if (5.17) holds, by Lemma 1, also

$$\limsup_{n \to \infty} \frac{M_n}{n} \leq M \quad \text{w.p.1.}$$

Also, $M_n \geq R_n$ and (1.9) imply that

$$\liminf_{n \to \infty} \frac{M_n}{n} \geq M \quad \text{w.p.1.}$$

It only remains to verify (5.18).

LEMMA 13. *If* (1.1) *holds, then* (5.18) *holds for every* $\varepsilon > 0$.

PROOF. Define, for each $1 \leq j \leq d$,

$\hat{\mathscr{E}}(n, \varepsilon, j) = \{$there exists a path π in $[-n, n]^d$ whose width in each of the first j coordinate directions is less than $n^{1/(2d)}$, but $\hat{S}(\pi) \geq (M + 2^{-j}\varepsilon)|\pi|$ and $\gamma^j \varepsilon^j n^{1-1/(8d)} \leq |\pi| \leq n\}$.

Then, $\hat{\mathscr{E}}(n,\varepsilon) = \hat{\mathscr{E}}(n,\varepsilon,1)$. We claim for every $\varepsilon > 0$, apart from a set of probability zero,

$$\{\hat{\mathscr{E}}(n,\varepsilon,j) \text{ occurs infinitely often}\} \subseteq \{\hat{\mathscr{E}}(n,\varepsilon,j+1) \text{ occurs infinitely often}\}. \tag{5.19}$$

This will prove (5.18) because $\hat{\mathscr{E}}(n,\varepsilon,d)$ is empty for large n. Indeed, there cannot be a self-avoiding path of length at least $\gamma^d\varepsilon^d n^{1-1/(8d)}$ but of width less than $n^{1/(2d)}$ in each coordinate direction, since a hypercube of edge size less than $n^{1/(2d)}$ contains at most $n^{1/2}$ vertices. Therefore, (5.19) implies that

$$P\{\hat{\mathscr{E}}(n,\varepsilon,j) \text{ occurs infinitely often}\} = 0$$

for all $j = 1,\dots,d$, and (5.18) is just this statement for $j = 1$.

The proof of (5.19) is basically the same decomposition of paths as given in Lemma 10, but with the role of the first coordinate direction being taken over by the $(j+1)$th coordinate direction. We now start with a path π that has width less than $n^{1/(2d)}$ in each of the first j coordinate directions. Thus π is contained in a set

$$[a_1, a_1 + n^{1/(2d)}] \times \cdots \times [a_j, a_j + n^{1/(2d)}] \times [-n,n]^{d-j} \tag{5.20}$$

for some integers $a_i \in [-n,n]$, $1 \le i \le j$. We now look for vertices v' and v'' on π with minimal and maximal $(j+1)$th coordinates. The piece between v' and v'' is a cylinder path in the $(j+1)$th direction, and the other two pieces may be decomposed further. However, all pieces obtained in the decomposition lie in the hypercube (5.20). The decomposition is stopped because either the analog of $\hat{\mathscr{B}}(n, n^{1/(2d)}, 2^{-j-1}\varepsilon)$, with the role of the first coordinate taken over by the $(j+1)$th coordinate, occurs, or because we end up with a collection of paths as in (5.1)–(5.4) of Lemma 10, but now with σ and τ having width less than $n^{1/(2d)}$ in all the first $(j+1)$ coordinate directions. From here to the proof of (5.19) is practically identical to the proofs of Lemmas 11 and 12. $\square$

Lemma 13 completes the proof of the convergence w.p.1 in (1.2). The L^1 convergence follows from this and Lemma 3, as in (1.3).

6. Equivalence with ρ-percolation. In Cox, Gandolfi, Griffin and Kesten (1993), the model of ρ-percolation introduced by Menshikov and Zuyev (1992) was discussed. In this model, vertices of $\mathbb{Z}^d$ are independently occupied with probability p, and vacant with probability $1-p$. For $0 \le \rho \le 1$, *ρ-percolation* occurs if with strictly positive probability there exists an infinite self-avoiding path $\pi = (v = \mathbf{0}, v_1, v_2, \dots)$ such that

$$\liminf_{n\to\infty} \frac{1}{n}[\#\text{ of occupied vertices among } v_0, v_1, \dots, v_{n-1}] \ge \rho. \tag{6.1}$$

Cox, Gandolfi, Griffin and Kesten (1993) announced that the occurrence of ρ-percolation is equivalent to the fact that $M \geq \rho$, with M as defined in (1.2). Here we give a proof of this statement. To unify the notation, define random variables $\{X_v,\ v \in \mathbb{Z}^d\}$ such that $X_v = 1$ if v is occupied and $X_v = 0$ if v is vacant. The number of occupied vertices among $v_0, v_1, \dots, v_{n-1}$ becomes simply $\sum_{i=0}^{n-1} X_{v_i}$. We write $M^{(p)}$ for $\lim(1/n)M_n$ for these random variables.

THEOREM 2. *If $\{X_v\colon v \in \mathbb{Z}^d\}$ are i.i.d. random variables such that $P(X_v = 1) = p = 1 - P(X_v = 0)$ for some $p \in [0,1]$, then the following three statements are equivalent*:

(i) *ρ-percolation occurs*;
(ii) *w.p.1 there exists an infinite path $\pi = (v_0, v_1, \dots)$ such that*

$$Z(\pi) := \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} X_{v_i} \geq \rho \tag{6.2}$$

[*the existence of the limit here is part of the statement in* (6.2)],
(iii) *The following inequality holds w.p.*1:

$$M^{(p)} = \lim_{n\to\infty} \frac{1}{n} M_n \geq \rho, \tag{6.3}$$

where M_n is as defined in the abstract.

PROOF. By the definitions of ρ-percolation and M_n, (i) implies (iii). Obviously, (ii) implies (i), so that we just need to verify that (iii) implies (ii).

Suppose that $M^{(p)} \geq \rho$. For any finite path $\pi = (v_0, v_1, \dots)$ we have, by the definition of M_n,

$$\sum_{i=0}^{n-1} X_{v_i} \leq M_n$$

so that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} X_{v_i} \leq \lim_{n\to\infty} \frac{1}{n} M_n = M^{(p)}. \tag{6.4}$$

Therefore, all we have to prove is that, w.p.1, there exists an infinite path $\pi = (v_0, v_1, \dots)$ such that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} X_{v_i} \geq M^{(p)}. \tag{6.5}$$

and (6.2) will be satisfied with $Z(\pi) = M^{(p)} \geq \rho$. The following lemma gives the main step for finding π.

LEMMA 14. *For all $\varepsilon > 0$, there exist a $\beta(\varepsilon) \in (0,1)$ and $\nu(\varepsilon), n(\varepsilon) < \infty$ such that for all w there exists with probability at least $1 - \varepsilon$ an infinite path*

$\tilde{\pi} = (w_0 = w, w_1, w_2, \dots)$ *with the properties*

$$w_{k\nu(\varepsilon)}(1) = w_0(1) + k(\lfloor \beta(\varepsilon)\nu(\varepsilon) \rfloor + 1), \quad k = 0, 1, \dots, \tag{6.6}$$

$$\begin{gathered} w_0(1) + k(\lfloor \beta(\varepsilon)\nu(\varepsilon) \rfloor + 1) \\ \le w_i(1) < w_0(1) + (k+1)(\lfloor \beta(\varepsilon)\nu(\varepsilon) \rfloor + 1) \\ \text{for } k\nu(\varepsilon) \le i < (k+1)\nu(\varepsilon),\ k \ge 0, \end{gathered} \tag{6.7}$$

$$\sum_0^{h-1} X_{w_i} \ge h\left(M^{(p)} - \frac{3\varepsilon}{4}\right) \quad \text{for } h \ge n(\varepsilon). \tag{6.8}$$

PROOF. Since $M^{(p)} = \sup_{0<\beta<1} M(\beta)$ [see (1.9)], we can choose $\beta = \beta(\varepsilon) \in (0,1)$ such that

$$M(\beta) \ge M^{(p)} - \frac{\varepsilon}{4},$$

and then, by virtue of (3.8), or equivalently (1.8), we can find $\nu = \nu(\varepsilon)$ such that

$$ER_\nu(\beta) \ge \nu\left(M^{(p)} - \frac{\varepsilon}{2}\right). \tag{6.9}$$

To construct $\tilde{\pi}$ we shall take $w = \mathbf{0}$ without loss of generality. We successively choose cylinder paths $\tilde{\pi}^{(i)}$ of length ν in the following way: $\tilde{\pi}^{(0)} = (\mathbf{0}, w_1, \dots, w_{\nu-1})$ is an optimal cylinder path with endpoint in $H(\lfloor \beta\nu \rfloor)$, that is,

$$0 \le w_i(1) \le \lfloor \beta\nu \rfloor, \qquad 0 \le i \le \nu - 1 \quad \text{and} \quad w_{\nu-1}(1) = \lfloor \beta\nu \rfloor,$$

and $S(\tilde{\pi}^{(0)})$ is maximal among all such paths, that is,

$$S(\tilde{\pi}^{(0)}) = R_\nu(\beta).$$

After that we take $w_\nu = w_{\nu-1} + e_1$, where e_1 is the first coordinate vector. Assume that the vertices $w_0, \dots, w_{l\nu}$ have been chosen in such a way that (6.6) holds for $k \le l$ and (6.7) holds for $k < l$. Then choose $\tilde{\pi}^{(l)} = (w_{l\nu}, \dots, w_{l\nu+\nu-1})$ as an optimal cylinder path satisfying

$$l(\lfloor \beta\nu \rfloor + 1) = w_{l\nu}(1) \le w_i(1) \le w_{l\nu}(1) + \lfloor \beta\nu \rfloor, \qquad l\nu \le i \le (l+1)\nu - 1,$$
$$w_{(l+1)\nu-1}(1) = (l+1)(\lfloor \beta\nu \rfloor + 1) - 1$$

and such that $S(\tilde{\pi}^{(l)})$ is maximal among all such paths. After that we take $w_{(l+1)\nu} = w_{(l+1)\nu-1} + e_1$. Then (6.6) also holds for $k = l+1$ and (6.7) also holds when k equals l. Let $\tilde{\pi}$ be the infinite path obtained by concatenating the paths $\tilde{\pi}(l)$ for all l. This path $\tilde{\pi}$ satisfies (6.6) and (6.7).

We have to show that we can choose $n(\varepsilon)$ such that (6.8) holds for all $h \ge n(\varepsilon)$ with a probability at least $1 - \varepsilon$. Clearly this will follow from

$$\liminf_{h\to\infty} \frac{1}{h} \sum_0^{h-1} X_{w_i} \ge M^{(p)} - \frac{\varepsilon}{2} \quad \text{w.p.1} \tag{6.10}$$

In turn, it suffices to show that the random variables

$$\sum_{k\nu}^{(k+1)\nu-1} X_{w_i} \tag{6.11}$$

are i.i.d., each with the distribution of $R_\nu(\beta)$ and hence with expectation at least $\nu(M^{(p)} - \varepsilon/2)$ [see (6.9)]. To prove this we observe that the choice of $\tilde{\pi}^{(0)}, \dots, \tilde{\pi}^{(k-1)}$, or of the vertices $\mathbf{0} = w_0, \dots, w_{k\nu-1})$, depends only on the values X_w with $w(1) < k(\lfloor \beta\nu \rfloor + 1)$ [cf. (6.7)]. After that $w_{k\nu}$ is chosen as $w_{k\nu-1} + e_1$, without examining $X_{w_{k\nu}}$. On the other hand, $\tilde{\pi}^{(k)} = (w_{k\nu}, \dots, w_{(k+1)\nu-1})$ depends only on the location of $w_{k\nu}$ and on X_w with $k(\lfloor \beta\nu \rfloor + 1) \le w(1) < (k+1)(\lfloor \beta\nu \rfloor + 1)$. This implies that (6.11) is indeed independent of $\{X_{w_i}: 0 \le i < k\nu\}$. The sum in (6.11) has the same distribution as $R_\nu(\beta)$ by construction of $\tilde{\pi}^{(k)}$. This was shown explicitly for $k = 0$ and the argument does not depend on k. $\square$

We return to the proof of (6.5) for a suitable π. We choose $\varepsilon_l = 1/(l+1)^2$ and take [with $\nu(\cdot)$ and $n(\cdot)$ as in Lemma 14]

$$\beta_l = \beta(\varepsilon_l), \qquad \nu_l = \nu(\varepsilon_l), \qquad m_l \text{ a multiple of } \nu_l,$$

with m_l increasing so fast that $m_l \ge n(\varepsilon_l)$ and

$$m_l\left(M^{(p)} - \frac{3\varepsilon_l}{4}\right) \ge \left\{\sum_0^l m_i + n(\varepsilon_{l+1})\right\}\left(M^{(p)} - \varepsilon_l\right). \tag{6.12}$$

The construction of π is now very similar to that of $\tilde{\pi}$ in the last lemma. The path $\pi^{(0)}$ will be chosen as the vertices $(v_0 = \mathbf{0}, v_1, \dots, v_{m_0})$ according to the construction in the proof of Lemma 14 for $w = \mathbf{0}$, $\varepsilon = \varepsilon_0$. Since m_0 is a multiple of ν_0, choice of these vertices depends only on the X_w with

$$w(1) < \frac{m_0}{\nu_0}(\lfloor \beta_0 \nu_0 \rfloor + 1) = v_{m_0}(1)$$

and

$$v_{m_0} = v_{m_0-1} + e_1.$$

We now continue from v_{m_0} with the first m_1 vertices of a path constructed as in Lemma 14 with v_{m_0} and ε_1 for w and ε, respectively. (Note that we used an infinite path in Lemma 14 merely to simplify the formulation of this lemma. Here we only construct the first m_1 vertices of such a path and never look at the later vertices. Note also that in this construction v_{m_0} appears both as the last point of $\pi^{(0)}$ and as the first point of $\pi^{(1)}$. This duplication is for convenience of description only; π contains v_{m_0} only once.) After l steps we will have picked vertices $v_0 = \mathbf{0}, v_1, \dots, v_{m_0+m_1+\cdots+m_{l-1}}$ in such a way that they depend only on the X_w with

$$\begin{aligned} w(1) &< \sum_0^{l-1} \frac{m_i}{\nu_i}(\lfloor \beta_i \nu_i \rfloor + 1) \\ &= v_{m_0+\cdots+m_{l-1}}(1). \end{aligned}$$

We next add a path $\pi^{(l)}$ of m_l steps by the method of Lemma 14 with $w = v_{m_0+\cdots+m_{l-1}}$, $\varepsilon = \varepsilon_l$.

To complete the proof we show that the resulting infinite path π satisfies (6.5). We define $\mu_l = \sum_0^{l-1} m_i$ and the events

$$\mathscr{E}_l = \left\{ \sum_{\mu_l \le i < h} X_{v_i} \ge (h - \mu_l)\left(M^{(p)} - \frac{3}{4}\varepsilon_l\right) \quad \text{for } \mu_l + n(\varepsilon_l) \le h \le \mu_{l+1} \right\}.$$

By our choice of $\pi^{(l)}$ and by virtue of Lemma 14,

$$P\{\mathscr{E}_l \text{ fails}\} \le P\{\pi^{(l)} \text{ does not have the property (6.8) for } \varepsilon = \varepsilon_l\} \le \varepsilon_l.$$

Since $\sum \varepsilon_l < \infty$, the Borel–Cantelli lemma shows that w.p.1 $\mathscr{E}_l$ occurs for all but finitely many l. This quickly implies (6.5), for if $\mathscr{E}_l$ occurs for $l \ge l_0$, then for $l \ge l_0$, $h \ge \mu_{l+1}$,

$$\begin{aligned}\sum_{i=0}^{h-1} X_{v_i} &\ge \sum_{i=\mu_l}^{\mu_{l+1}-1} X_{v_i} \ge (\mu_{l+1} - \mu_l)\left(M^{(p)} - \frac{3\varepsilon_l}{4}\right) \\ &= m_l\left(M^{(p)} - \frac{3\varepsilon_l}{4}\right) \ge \mu_{l+1}\left(M^{(p)} - \varepsilon_l\right) \quad \text{[see (6.12)]}.\end{aligned}$$

In particular, for $\mu_{l+1} \le h < \mu_{l+1} + n(\varepsilon_{l+1})$,

$$\sum_{i=0}^{h-1} X_{v_i} \ge \frac{\mu_{l+1}}{\mu_{l+1} + n(\varepsilon_{l+1})} h\left(M^{(p)} - \varepsilon_l\right) \ge \frac{\left(M^{(p)} - \varepsilon_l\right)}{\left(M^{(p)} - \frac{3}{4}\varepsilon_l\right)} h\left(M^{(p)} - \varepsilon_l\right)$$

$$\text{[again by (6.12)]}.$$

Also for $\mu_{l+1} + n(\varepsilon_{l+1}) \le h < \mu_{l+2}$,

$$\begin{aligned}\sum_{i=0}^{h-1} X_{v_i} &= \sum_{0}^{\mu_{l+1}-1} X_{v_i} + \sum_{\mu_{l+1}}^{h-1} X_{v_i} \\ &\ge \mu_{l+1}\left(M^{(p)} - \varepsilon_l\right) + (h - \mu_{l+1})\left(M^{(p)} - \tfrac{3}{4}\varepsilon_{l+1}\right) \\ &\ge h\left(M^{(p)} - \varepsilon_l\right)\end{aligned}$$

(since $\mathscr{E}_{l+1}$ also occurs). Now is (6.5) obvious. □

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