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THE FBM ITO'S FORMULA THROUGH ANALYTIC CONTINUATION

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Abstract The Fractional Brownian Motion can be extended to complex values of the parameter α for $\operatorname{Re}\alpha > \frac{1}{2}$. This is a useful tool. Indeed, the obtained process depends holomorphically on the parameter, so that many formulas, as Itô formula, can be extended by analytic continuation. For large values of $\operatorname{Re}\alpha$, the stochastic calculus reduces to a deterministic one, so that formulas are very easy to prove. Hence they hold by analytic continuation for $\operatorname{Re}\alpha \le 1$, containing the classical case $\alpha = 1$.

Keywords Wiener space, Sobolev space, Stochastic integral, Fractional Brownian Motion, Itô's formula

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1 Introduction

Many authors ([1-4,6-10,18-20,23,24,27]) have studied different kinds of FBM (fractional Brownian motion). An important problem was to find a nice extension of the Itô-Skorohod formula. In the regular case (more regular than the classical one), Dai-Heyde [5], Decreusefond-Ustünel [6] gave a formula based on the divergence operator. The more difficult singular case was also studied by Alòs-León-Mazet-Nualart [2-4] who gave a formula in a general context.

Note that many of the cited authors prefer to deal with an other kind of FBM, associated to the so-called Hurst parameter H which is real and corresponds to our α through the relation $H = \operatorname{Re}\alpha - \frac{1}{2}$.

Recall that the LFBM (Liouville Fractional Brownian Motion) is defined by

$$W_t^\alpha = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} dW_s \quad .$$

There are three main ideas in the present paper. The first is to deal with the Liouville spaces, $\mathcal{J}^{\alpha,p}$ which are the images of $L^p([0,T])$ under the Liouville kernel defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds \quad .$$

This idea seems to go back to our paper [14] of 1996, in relation with Hölder continuous functions, and Young-Stieltjes integration. Another advantage of the Liouville space, as pointed in [14], is to give the natural isomorphism

$$\mathcal{J}^{\alpha,p}(L^p(\mu)) = L^p(\mu, \mathcal{J}^{\alpha,p})$$

where μ is an arbitrary measure (for example the Wiener measure). This gives a nice interpretation of Kolmogorov's lemma, and this gives also some natural Banach spaces of solutions of trace problems.

For a β-Hölder continuous function φ, we redefine the FBM-Wiener integral as the natural extension of the linear map

$$X_T^\alpha(G) = \int_0^T \varphi(t) dW_t^\alpha(G) = \int_0^T \varphi(t) dI^{\alpha-1}G(t)$$

for a $G \in \mathcal{J}^{1,2}$ which is the Cameron-Martin space of the Wiener measure, the integral being taken in a little more precise sense than that of Young.

The second idea, after the deep study of [17,22,25], is to use the Itô-Skorohod integral, and to define

$$X_T^\alpha(\omega) = \int_0^T u_t(\omega) \odot dW_t^\alpha(\omega) \quad .$$

Here u is a β-Hölder process with values in a Gaussian Sobolev space, and X_T^α is the Gaussian divergence of a suitable FBM-Wiener integral.

Observe that if u is not adapted, X^α is not in general. Hence we get a true stochastic calculus and an Itô-Skorohod formula for anticipative processes.

The third idea is to use complex values of the parameter α, for $\operatorname{Re}\alpha > \frac{1}{2}$. The interesting property is that all the preceding objects are holomorphic functions of α. So that our first goal is to show that there exists a differential formula in the ordinary pathwise calculus for $\operatorname{Re}\alpha > \frac{3}{2}$. We then extend its validity by analytic continuation on the domain of definition of every term. To this end, we only have to prove that each involved term has a meaning. It appears that doing so, we easily get for $\alpha = 1$ the so-called Itô-Skorohod stochastic formula, and for $\operatorname{Re}\alpha > \frac{3}{4}$ the fractional Itô-Skorohod formula we were looking for.

Notice that for $\operatorname{Re}\alpha < 1$, the definition of the remaining term(s) in the Itô formula, needs singular integrals which exist in the sense of Hadamard (*Parties finies de Hadamard*). Finally for the Itô formula, a natural domain for $(\alpha, \beta) \subset \mathbb{C} \times \mathbb{R}$ is defined by the conditions

$$\operatorname{Re}\alpha > \frac{3}{4}, \quad 0 < \beta < 1, \quad \operatorname{Re}\alpha + \beta > 1 \quad .$$

Note that in [2], the Itô formula for the LFBM is only stated under the stronger condition $\alpha + \beta/2 > 1$. Actually for the reason as above (true stochastic calculus), it is not reasonable to consider other values than $\beta < \operatorname{Re}\alpha - \frac{1}{2}$. In conclusion, a natural stochastic calculus can only be obtained for $\operatorname{Re}\alpha > \frac{3}{4}$.

Observe that the point $(\frac{3}{4}, 1)$ is the most left limit point of the natural domain.

As a matter of fact, in all the paper, the only stochastic analysis elements we use, are the Wiener integral and the Sobolev Gaussian space.

For adapted processes, it would be interesting to know that if the domain could be extended by considering simultaneously the method in use in [1-4] (cutting the Liouville kernel to obtain semi-martingales), and analytic extension of integrals.

Of course, it would be possible to extend this formulas to n-dimensional FBM. There would be no new difficulties, except in writing formulas.

In conclusion, we can say that we have an ordinary pathwise differential calculus for $\operatorname{Re}\alpha > \frac{3}{2}$, a "Young stochastic" calculus for $\operatorname{Re}\alpha > 1$, and a "Young-Hadamard stochastic" calculus for $\operatorname{Re}\alpha > \frac{3}{4}$.

2 Recall on the Liouville space

Throughout the paper, α is a complex parameter such that $\operatorname{Re}\alpha > \frac{1}{2}$, β is a real number (the order of Hölder continuity) between 0 and 1, and p is a real number (the Hölder exponent of integrability) strictly between 1 and $+\infty$ when no other precision.

We use the same notations as in [14]. For $\operatorname{Re}\alpha > 0$, the Liouville integral is defined by convolution

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds$$

with the locally integrable function $k_\alpha(t) = t_+^{\alpha-1}/\Gamma(\alpha)$. Thus we obtain a holomorphic semigroup of continuous operators of $L^p_{\mathrm{loc}}([0,+\infty[, dt, \mathbb{C})$ for $p \geq 1$, and then of $L^p([0,T])$ as the values of $I^\alpha f$ on $[0,T]$ do only depend on the values of f on $[0,T]$.

The Liouville space $\mathcal{J}^{\alpha,p}$ is the image of the complex space $L^p([0,T])$ under I^α. For $\operatorname{Re}\alpha > 1/p$, we have $k_\alpha \in L^q_{\mathrm{loc}}$ so that $\mathcal{J}^{\alpha,p}$ is contained in the space $\mathcal{C}^0$ of complex continuous functions on

$[0, T]$ vanishing at 0. Notice that $\mathcal{J}^{\alpha,p}$ only depends on the real part $\operatorname{Re}\alpha$. Indeed, for real γ, the Fourier-Schwartz transform

$$\hat{k}_{\,\mathrm{i}\gamma}(\xi) = (2\pi|\xi|)^{-\mathrm{i}\gamma} \exp[\pi\gamma \operatorname{Sign}(\xi)/2]$$

is a FL^p multiplier according to the Marcinkiewicz theorem [21], so that $I^{\mathrm{i}\gamma}$ is a continuous operator of $L^p([0, T])$. It then easily follows by the semi-group property that $\mathcal{J}^{\alpha,p} = \mathcal{J}^{\alpha+\mathrm{i}\gamma,p} = \mathcal{J}^{\operatorname{Re}\alpha,p}$.

Note that for $\operatorname{Re}\alpha > \frac{1}{2}$, the I^α's are Hilbert-Schmidt.

The natural norm of $\mathcal{J}^{\alpha,p}$ is given by

$$\|I^\alpha f\|_{\alpha,p} = N_p(f)$$

where N_p is the norm in $L^p([0, T])$. Obviously $\mathcal{J}^{\alpha,p}$ is a separable Banach space which is reflexive (of L^p type).

For $\beta \in]0, 1[$, let $\mathcal{C}^\beta$ the space of β-Hölder continuous functions on $[0, T]$ *in the restricted sense*, that is those functions such that

$$\Phi(s, t) = \frac{\varphi(t) - \varphi(s)}{|t - s|^\beta}$$

is a continuous function. This is a separable Banach space under the norm

$$\|\varphi\|_\beta = \|\varphi\|_\infty + \|\Phi\|_\infty \quad .$$

As it was proved in our paper [14], for exponents satisfying the inequalities $1 > \beta > \gamma > \gamma - 1/p > \beta' > 0$, the following inclusions hold

$$\mathcal{C}^\beta \subset \mathbb{R} + \mathcal{J}^{\gamma,\infty} \subset \mathbb{R} + \mathcal{J}^{\gamma,p} \subset \mathcal{C}^{\beta'} \quad . \tag{1}$$

Now, let B be a complex separable Banach space. Most of the above properties also hold for B-valued functions. For the property concerning the identity $\mathcal{J}^{\alpha,p}(B) = \mathcal{J}^{\operatorname{Re}\alpha,p}(B)$, we need an extra property. We say that a Banach space is a B_p-space if it is isomorphic with a closed subspace of an L^p space. Hence the required equality holds true for a B_p-space.

Note that every separable Hilbert space is a B_p-space (even if $p \neq 2$), and that a B_2 space is a Hilbert space.

In all of the paper, every involved functional Banach space is separable, and the expression "absolutely convergent integral" of a Banach space valued function means that the function is integrable in the Bochner sense.

3 Recall on the Wiener space

We denote Ω the standard Wiener space with the Wiener measure μ, that is the space of $\mathbb{R}$-valued continuous trajectories ω or ϖ defined on $[0, +\infty[$ and vanishing at 0. The standard Brownian motion is denoted W_t. The μ-expectation is denoted $\mathbb{E}$. The first Wiener chaos, that is the space of μ-measurable linear functions on Ω (*cf.* . [11], th.22 and [12], th.11) is denoted by

H^1. As for every Gaussian space, the gradient or differential ∇ can be defined. In the particular case of the Wiener space, for every elementary Wiener functional $F(\omega) = \varphi(W_{t_1}, \ldots, W_{t_n})$

$$\nabla F(\omega, \varpi) = \sum_i \partial_i \varphi(W_{t_1}(\omega), \ldots, W_{t_n}(\omega)) W_{t_i}(\varpi) \quad .$$

Notice that ∇F is linear in the second slot.

The Gaussian Sobolev space $\mathcal{D}^{1,p} = \mathcal{D}^{1,p}(\Omega, \mu)$ is the completion of elementary functions under the Sobolev norm defined by

$$\|F\|_{1,p}^p = \mathbb{E}(|F|^p) + \iint_{\Omega \times \Omega} |\nabla F(\omega, \varpi)|^p \, d\mu(\omega) d\mu(\varpi) \quad .$$

The divergence operator is defined by transposition. If G is a Wiener-Sobolev functional on $\Omega \times \Omega$, $\operatorname{div} G$ is defined on Ω by

$$\mathbb{E}(F \operatorname{div} G) = \iint G(\omega, \varpi) \nabla F(\omega, \varpi) \, d\mu(\omega) d\mu(\varpi) \quad .$$

Thanks to the theorem of divergence continuity, this definition makes sense for $G \in \mathcal{D}^{1,p}(\Omega \times \Omega)$ and $p > 1$.

In fact, the only interesting values of the divergence are achieved on the functional G which are linear in the second slot ϖ. For the particular functions which are of the form $G = \Phi(\omega) X(\varpi)$ where X is linear (*i.e.* belongs to the first Wiener chaos), one has

$$\operatorname{div} G(\omega) = \Phi(\omega) X(\omega) - \widetilde{\mathbb{E}}(\nabla \Phi(\omega, \cdot) X(\cdot))$$

where $\widetilde{\mathbb{E}}$ is the partial expectation w.r. to ϖ.

4 The holomorphic FBM

Let α a complex number such that $\operatorname{Re} \alpha > \frac{1}{2}$. We define the complex FBM by the Wiener integral

$$W_t^\alpha = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha - 1} dW_s$$

that can be symbolically written

$$W^\alpha = I^\alpha \dot{W} = k_\alpha * \dot{W}$$

where $\dot{W}$ is for the white noise on $[0, +\infty[$, that is the "derivative" of the standard Brownian Motion.

This can be justified in the following way: the Cameron-Martin space of the Wiener space is $\mathcal{J}^{1,2} = I^1(L^2([0, +\infty[))$, so that $W_t^\alpha(\omega)$ is exactly the μ-measurable linear extension of the bounded linear form (*cf.* [11], th.38)

$$G \to I^{\alpha - 1} G(t) = I^\alpha \dot{G}(t) = k_\alpha * \dot{G}(t)$$

for $G \in \mathcal{J}^{1,2}$. Now we have

$$\mathbb{E}|W_t^\alpha|^2 = \frac{t^{2\operatorname{Re}\alpha - 1}}{(2\operatorname{Re}\alpha - 1)|\Gamma(\alpha)|^2} \quad .$$

At this point, one could prouve that the FBM is a Gaussian process with values in the space of holomorphic functions on $\operatorname{Re}\alpha > \frac{1}{2}$, with γ-Hölder continuous trajectories for $\gamma < \operatorname{Re}\alpha - \frac{1}{2}$. In fact we shall prove more general results in the next section.

Only observe for the moment that for $\operatorname{Re}\alpha > \frac{3}{2}$, W_t^α has $\mathcal{C}^1$-trajectories, which is obvious since the Brownian motion has continuous trajectories.

5 The main lemma

Recall that in [14], we defined the Young-Stieltjes integral

$$\int_0^T \varphi d\psi$$

for $\varphi \in \mathcal{C}^\beta$ and $\psi \in \mathcal{C}^\gamma$ for $\beta + \gamma > 1$, and the result was $\mathcal{C}^\gamma$ w.r. to T. Now, if $g \in L^2$, we want to define

$$\int_0^T \varphi dI^\alpha g \quad .$$

Unfortunately, as $I^\alpha g$ is only $\mathcal{C}^{\alpha' - \frac{1}{2}}$, for $\alpha' < \operatorname{Re}\alpha$, it seems that we are obliged to assume $\beta + \operatorname{Re}\alpha > \frac{3}{2}$ for the existence of the Young integral. Nevertheless we have a more precise result given by the next lemma, which involves an analytic extension, that is in fact a "*Partie finie de Hadamard*".

Before enouncing this main lemma, it is convenient to introduce some domains of constant use in the sequel.

$$D_0 = \{\operatorname{Re}\alpha > \frac{1}{2}\}, \qquad D_1 = \{(\alpha, \beta) \in \mathbb{C} \times \mathbb{R} /\ \operatorname{Re}\alpha > \frac{1}{2},\ \beta > 0\ \operatorname{Re}\alpha + \beta > 1\}$$

$$D_1(\beta) = \{\alpha \in \mathbb{C} /\ (\alpha, \beta) \in D_1\} \quad .$$

1 Lemma : *Let $\varphi \in \mathcal{C}^\beta(B)$, and $g \in L^2_{loc}$ where B is a Banach space. Consider the integral*

$$x_t^\alpha = \int_0^t \varphi(s) dI^\alpha g(s) \quad .$$

It converges absolutely for $\operatorname{Re}\alpha > \frac{3}{2}$ and is of class $\mathcal{C}^1$ in t. Moreover it admits a unique holomorphic extension in the domain $D_1(\beta)$. This extension is absolutely continuous for $\operatorname{Re}\alpha \geq 1$. For $\operatorname{Re}\alpha \leq \frac{3}{2}$, $0 < \gamma < \operatorname{Re}\alpha - \frac{1}{2}$ and $0 \leq s \leq t \leq T$, we have

$$|x_t^\alpha - x_s^\alpha| \leq K_T(\alpha, \beta, \gamma) \|\varphi\|_\beta N_2(g 1_{[0,T]}) |t - s|^\gamma \tag{2}$$

where $K_T(\alpha, \beta, \gamma)$ is locally bounded (and even continuous) on the admissible domain

$$T > 0, \quad (\alpha, \beta) \in D_1, \qquad \operatorname{Re}\alpha \leq \frac{3}{2}, \qquad 0 < \gamma < \operatorname{Re}\alpha - \frac{1}{2} \ .$$

Proof: The assertion concerning the case $\mathrm{Re}\,\alpha > \frac{3}{2}$ is obvious. Now one has

$$x_t^\alpha = \frac{1}{\Gamma(\alpha-1)} \int_0^t \varphi(s) ds \int_0^s g(r)(s-r)^{\alpha-2} dr \quad .$$

Write $x_t^\alpha = y_t^\alpha + z_t^\alpha$ with

$$y_t^\alpha = \frac{1}{\Gamma(\alpha-1)} \int_0^t \varphi(r) g(r) dr \int_r^t (s-r)^{\alpha-2} ds$$

$$z_t^\alpha = \frac{1}{\Gamma(\alpha-1)} \int_0^t ds \int_0^s g(r)\Phi(r,s)(s-r)^{\alpha+\beta-2} dr$$

where we introduced the continuous function (*cf.* section 2)

$$\Phi(r,s) = \frac{\varphi(s)-\varphi(r)}{|s-r|^\beta} \quad .$$

First observe that y_t^α is exactly $I^\alpha(\varphi g)(t)$, hence it has a holomorphic extension until $\mathrm{Re}\,\alpha > \frac{1}{2}$, which belongs to $\mathcal{J}^{\alpha,2}$. Then for $\mathrm{Re}\,\alpha > 1$, it is absolutely continuous.

Now, the double integral defining z_t^α converges absolutely and is holomorphic until $\mathrm{Re}\,\alpha > 1-\beta$. It remains to prove inequality (2). Put $a = \mathrm{Re}\,\alpha$. First assume that $a \leq 1$. We get

$$\left|\frac{dz_t^\alpha}{dt}\right| \leq \frac{\|\Phi\|_\infty}{|\Gamma(\alpha-1)|} \int_0^t |g(r)|(s-r)^{a+\beta-2} dr = \frac{\|\Phi\|_\infty \Gamma(a+\beta-1)}{|\Gamma(\alpha-1)|} I^{a+\beta-1}|g|(t)$$

which belongs to $L^2_{\mathrm{loc}}(dt, B)$. Hence $z_t^\alpha \in \mathcal{J}^{1,2}(B) \subset \mathcal{J}^{\alpha,2}(B)$ for $a \leq 1$. Moreover z_t^α is absolutely continuous.

From these different inclusions, the following inequality follows

$$\|x^\alpha\|_{\mathcal{J}^{\alpha,2}(B)} \leq K_T(\alpha,\beta)\|\varphi\|_\beta N_2(g1_{[0,T]})$$

where $K_T(\alpha,\beta)$ is locally bounded on D_1. Inequality (2) follows from the inclusions (1).

Now assume that $1 \leq a \leq \frac{3}{2}$. As y^α belongs to $\mathcal{J}^{\alpha,2}(B)$, we get by the same inclusions

$$|y_t^\alpha - y_s^\alpha| \leq K_T^1(\alpha,\beta,\gamma)\|\varphi\|_\beta N_2(g1_{[0,T]})|t-s|^\gamma$$

with a locally bounded $K_T^1(\alpha,\beta,\gamma)$. On the other hand, we have for $0 \leq \tau \leq t \leq T$

$$|z_t^\alpha - z_\tau^\alpha| \leq \frac{\|\varphi\|_\beta}{|\Gamma(a-1)|} \int_\tau^t ds \int_0^s |g(r)|(s-r)^{a-2} dr$$

$$|z_t^\alpha - z_\tau^\alpha| \leq \|\varphi\|_\beta \int_\tau^t (I^{a-1}|g|)(r) dr = \|\varphi\|_\beta [I^a|g|(t) - I^a|g|(\tau)] .$$

Hence, replacing τ with s, we get the required

$$|z_t^\alpha - z_s^\alpha| \leq K_T^2(\alpha,\beta,\gamma)\|\varphi\|_\beta N_2(g1_{[0,T]})|t-s|^\gamma$$

with another $K_T^2(\alpha,\beta,\gamma)$. The proof is complete.

2 Remarks : a) Observe that z_t^α vanishes for $\alpha = 1$, so that x_t^α reduces to the obvious formula $x_t^\alpha = \int_0^t \varphi(s) g(s) ds$.

b) For $\operatorname{Re}\alpha \leq \frac{3}{2}$, it would be more correct to write

$$x_t^\alpha = (\mathrm{Y}) \int_0^t \varphi(t) dI^\alpha g \qquad \text{for } \operatorname{Re}\alpha + \beta > \tfrac{3}{2}, \quad \text{and}$$

$$x_t^\alpha = (\mathrm{Pf}) \int_0^t \varphi(t) dI^\alpha g \qquad \text{for } \operatorname{Re}\alpha + \beta > 1$$

but in general, we shall omit these tedious notations. The context shall recall the meaning of the singular integrals.

6 The FBM Wiener integral

The following definition is equivalent to the one given in [14], page 12.

3 Definition : *For $(\alpha, \beta) \in D_1$ and $\varphi \in \mathcal{C}^\beta$, we denote by*

$$X_t^\alpha(\omega) = \int_0^t \varphi(s) dW_s^\alpha(\omega)$$

the unique element of the first Wiener chaos which represents the bounded linear functional (cf. [11] and [12])

$$X_t^\alpha(G) = \int_0^t \varphi(s) I^{\alpha-1} \dot{G}(s) ds = \int_0^t \varphi(s) dI^\alpha \dot{G}(s)$$

for $G \in \mathcal{J}^{1,2}$ (that is the Cameron-Martin space of (Ω, μ)).

In the case $\varphi = 1$, we recover W_t^α (take $\beta > \frac{1}{2}$). In the case $\alpha = 1$, we recover the ordinary Wiener integral.

Now, it follows from the formulas of the main lemma with the same notations, that we have $X_t^\alpha = Y_t^\alpha + Z_t^\alpha$ with

$$Y_t^\alpha = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) dW_s$$

$$Z_t^\alpha = \frac{1}{\Gamma(\alpha-1)} \int_0^t dW_r \int_r^t \Phi(r,s)(s-r)^{\varepsilon-1} ds$$

so that with $a = \operatorname{Re}\alpha$ and $a + \beta - 1 = \varepsilon > 0$

$$N_2(X_t^\alpha) \leq \frac{\|\varphi\|_\infty t^{a-\frac{1}{2}}}{|\Gamma(\alpha)|\sqrt{2a-1}} + \frac{\|\Phi\|_\infty t^{\varepsilon+\frac{1}{2}}}{|\Gamma(\alpha-1)|\sqrt{\varepsilon(2\varepsilon+1)}} \quad .$$

From this inequality we infer that X_t^α makes sense even if φ is H-valued for a separable Hilbert space H.

4 Theorem : *Assume that $\varphi \in \mathcal{C}^\beta$ is H-valued. For $(\alpha, \beta) \in D_1$ and for $0 < \gamma < \operatorname{Inf}(1, \operatorname{Re}\alpha - \frac{1}{2})$, then X_t^α is a γ-Hölder Gaussian process with values in the space of holomorphic functions $\mathcal{H}(D_1(\beta), H)$. Moreover, a.e. trajectory is γ-Hölder with values in $\mathcal{H}(D_1(\beta), H)$.*

Proof: According to inequality (2) of lemma 1, we have for the restriction of X_t^α to the Cameron-Martin space $\mathcal{J}^{1,2}$ of μ

$$|X_t^\alpha(G) - X_s^\alpha(G)| \leq K_T(\alpha,\gamma,\beta)\|\varphi\|_\beta N_2(\dot{G})|t-s|^\gamma \quad .$$

As the first Wiener chaos H^1 is naturally isometrically isomorphic to the dual space of the Cameron-Martin space (*cf.* [11,12]) we get the norm in $H^1 = (\mathcal{J}^{1,2})^*$

$$\|X_t^\alpha - X_s^\alpha\|_{H^1} \leq K_T(\alpha,\gamma,\beta)\|\varphi\|_\beta |t-s|^\gamma$$

that is

$$N_2(X_t^\alpha - X_s^\alpha) \leq K_T(\alpha,\gamma,\beta)\|\varphi\|_\beta |t-s|^\gamma \ . \tag{3_2}$$

As X_t^α is Gaussian, we get for every $p \geq 2$

$$N_p(X_t^\alpha - X_s^\alpha) \leq \sqrt{p-1}\ K_T(\alpha,\gamma,\beta)\|\varphi\|_\beta |t-s|^\gamma \ . \tag{3_p}$$

Integrating over a compact set $L \subset D_1(\beta)$, w.r. to the Lebesgue measure σ on $\mathbb{C}$, we get

$$\left[\int_L \mathbb{E}|X_t^\alpha - X_s^\alpha|^p d\sigma(\alpha)\right]^{1/p} \leq \sqrt{p-1}\ \|\varphi\|_\beta |t-s|^\gamma \left[\int_L K_T(\alpha,\beta,\gamma)^p d\sigma(\alpha)\right]^{1/p} < \infty \quad .$$

The right member is finite. As the topology of $\mathcal{H}(D_1(\beta), H)$ is induced by $L^p_{\text{loc}}(D_1(\beta),\sigma,H)$, the Kolmogorov lemma gives all the results.

5 Remarks : a) The coefficient $\sqrt{p-1}$ follows from an easy extension of the Nelson inequalities (*cf.* [16], remarque 9) to Gaussian vectors.

b) This applies to the case $X_t^\alpha = W_t^\alpha$ (take $\varphi = 1$ and $\beta > \frac{1}{2}$ so that $\alpha \in D_0$). This is an improvement of a result of [5], where it is proved an analogous result for W_t^α, but only for $\mathcal{C}^\infty$-functions of real α.

c) As in our article [14], we could extend some of these considerations to the fractional Brownian sheet, and get the separately Hölder continuity for the sheet with values in H-valued holomorphic functions.

6 Corollary : *For* $\operatorname{Re}\alpha > n + \frac{1}{2}$, X_t^α *has* $\mathcal{C}^n$*-trajectories.*

Proof: This is true for $W_t^\alpha = I^1 W_t^{\alpha-1}$. This extends to X_t^α by the definition of X_t^α.

7 Theorem : *Assume that* $(\alpha,\beta) \in D_1$, $\beta > 1/p$, $\operatorname{Re}\alpha + \beta > 1 + 1/p$ *with* $p \geq 2$, *and that* φ *belongs to* $\mathcal{C}^\beta(B)$ *where* B *is a* B_p*-space. Then the conclusions of the previous theorem still hold.*

Proof: It suffices to deal with the case $B_p = L^p(\xi)$ for a bounded and separable measure ξ (*i.e.* $L^1(\xi)$ is separable). As we have $\mathcal{C}^\beta(L^p(\xi)) \subset L^p(\xi, \mathcal{C}^{\beta'-1/p})$ for $1/p < \beta' < \beta$, we see that ξ-a.e. every path $s \to \varphi(s,x)$ is $(\beta' - 1/p)$-Hölder continuous. Hence put

$$X_t^\alpha(\omega,x) = \int_0^t \varphi(s,x)\, dW_s^\alpha(\omega)$$

$$\mathbb{E}|X_t^\alpha(\omega,x) - X_s^\alpha(\omega,x)|^p \leq K_T^p(\alpha,\beta,\beta',\gamma)\|\varphi(\cdot,x)\|^p_{\beta'-1/p}|t-s|^{\gamma p}$$

$$\int \mathbb{E}|X_t^\alpha(\omega,x)-X_s^\alpha(\omega,x)|^p\,d\xi(x) \le K_T^p(\alpha,\beta,\beta',\gamma)\,|||\varphi|||_{\beta'-1/p}^p\,|t-s|^{\gamma p}$$

where $|||\varphi|||_{\beta'-1/p}$ stands for the norm of $L^p(\xi,\mathcal{C}^{\beta'-1/p})$. It follows

$$N_p(X_t^\alpha - X_s^\alpha) \le K_T^1(\alpha,\beta,\gamma)\,\|\varphi\|_\beta\,|t-s|^\gamma$$

with another constant $K_T^1(\alpha,\beta,\gamma)$ and where $\|\varphi\|_\beta$ stands for the norm in $\mathcal{C}^\beta(L^p(\xi))$.

Also remark that as $\mathcal{C}^\beta(L^p(\xi)) \subset \mathcal{C}^\beta(L^2(\xi))$, the definitions of X_t^α given by theorems 4 and 7 agree.

7 The fractional Itô-Skorohod integral

For $(\alpha,\beta)\in D_1$, consider the FBM-Wiener integral w.r. to ϖ

$$\widetilde{X}_t^\alpha(\omega,\varpi) = \int_0^t u_s(\omega)\,dW_s^\alpha(\varpi)$$

where $u \in \mathcal{C}^\beta(\mathcal{D}^{1,2}(\mu))$. According to theorem 4, this is a $\mathcal{D}^{1,2}$-valued FBM-Wiener integral. Then we can put

8 Definition : *Let $(\alpha,\beta)\in D_1$. The FBM-Itô-Skorohod integral of u is defined by*

$$X_t^\alpha = \int_0^t u_s \odot dW_s = \operatorname{div} \widetilde{X}_t^\alpha \quad .$$

As the FBM-Wiener integral $\widetilde{X}_t^\alpha$ belongs $\mathcal{C}^\gamma(\mathcal{D}^{1,2})$ for every $\gamma < \operatorname{Re}\alpha - \frac{1}{2}$, not only the divergence is well defined but also the result is a process which belongs to $\mathcal{C}^\gamma(L^2(\mu))$.

9 Theorem : *Let $(\alpha,\beta)\in D_1$, and that $u \in \bigcap_p \mathcal{C}^\beta(\mathcal{D}^{1,p})$. Then X_t^α belongs to $\bigcap_p L^p(\mu,\mathcal{C}^\gamma)$ for every γ such that $0<\gamma<\operatorname{Re}\alpha-\frac{1}{2}$. Moreover, for a fixed γ, X belongs to $\bigcap_p L^p(\mu,\mathcal{C}^\gamma(\mathcal{H}))$ where $\mathcal{H}$ is the space of holomorphic functions on $\operatorname{Re}\alpha > \frac{1}{2}$.*

Proof: Applying theorem 7 for $p>1/\beta$ and the continuity of the divergence yields

$$N_p(X_t^\alpha - X_s^\alpha) \le c_p\|\widetilde{X}_t^\alpha - \widetilde{X}_s^\alpha\|_{\mathcal{D}^{1,p}} \le c_p K_T^1(\alpha,\beta,\gamma)\|u\|_{\mathcal{C}^\beta(D^{1,p})}|t-s|^\gamma$$

for $\gamma < \operatorname{Re}\alpha - 1/2$.

10 Corollary : *Almost every trajectory of X_t^α is Hölder continuous with values in holomorphic functions on $D_1(\beta)$.*

8 The little Itô formula

11 Lemma : *Let F be a polynomial. If X belongs to the first Wiener chaos H^1 with $\mathcal{C}^1$-trajectories, then*

$$F(X_t) = F(X_0) + \operatorname{div}\int_0^t F'(X_s(\omega))\,dX_s(\varpi) + \frac{1}{2}\int_0^t F''(X_s)\,d\mathbb{E}(X_s^2)\ . \tag{4}$$

Proof: Compute the divergence which is worth

$$\int_0^t F'(X_s(\omega))\dot{X}_s(\omega)ds - \int_0^t \widetilde{\mathbb{E}}[\nabla(F'(X_s))(\omega,\cdot)\dot{X}_s(\cdot)]ds$$

that is

$$F(X_t) - F(X_0) - \int_0^t F''(X_s(\omega))\widetilde{\mathbb{E}}X_s(\cdot)\dot{X}_s(\cdot)ds \quad .$$

12 Theorem : *Let* $\operatorname{Re}\alpha > \frac{3}{4}$, *and let* F *be a polynomial, one has*

$$F(W_t^\alpha) = F(0) + \int_0^t F'(W_s^\alpha) \odot dW_s^\alpha + \frac{1}{2}\int_0^t F''(W_s^\alpha)\frac{s^{2\alpha-2}}{\Gamma(\alpha)^2}ds \ . \tag{5}$$

Proof: Note that for $\alpha > \frac{3}{2}$ formula (5) is nothing but formula (4). The second step consists to remark that formula (5) makes sense for every complex α in the domain $\{\operatorname{Re}\alpha > \frac{3}{4}\}$ in view of theorem 9, since $F(W_s^\alpha)$ and $F'(W_s^\alpha)$ are β-Hölder for every $0 < \beta < \operatorname{Re}\alpha - \frac{1}{2}$. Indeed one has $(\alpha, \beta) \in D_1$ for such a β. Hence the equality holds true by analytic continuation for $\operatorname{Re}\alpha > \frac{3}{4}$.

13 Remarks : a) This proves *a posteriori* that the Itô-Skorohod integral

$$\int_0^t G(W_s^\alpha) \odot dW_s^\alpha$$

has an analytic extension all over $D_0 = \{\operatorname{Re}\alpha > \frac{1}{2}\}$ for every polynomial G. This remark is not so trivial if we deal with the n-dimensional Brownian motion. Some analogous properties will come below.

b) If F is not a polynomial, formula (5) extends by routine arguments, for real $\alpha > \frac{3}{4}$, to a suitable subspace of $\mathcal{C}^2$-functions F.

9 The FBM Itô-Skorohod differential

14 Proposition : *Let* $(\alpha, \beta) \in D_1$, *and let* u *a process belonging to* $\mathcal{C}^\beta([0,T], \mathcal{D}^{1,2})$. *Put*

$$X_t^\alpha = \int_0^t u_s \odot dW_s^\alpha \quad .$$

If X^α *vanishes (for every* t*), then* $u = 0$.

Proof: According to the definition, we have

$$\mathbb{E}\left[\nabla G(\omega, \varpi)\int_0^T u_t(\omega)dW_t^\alpha(\varpi)\right] = 0$$

for every $G \in \mathcal{D}^{1,2}$. Take $G(\omega) = \exp(f(\omega))$ where

$$f(\omega) = \int_0^T \psi(t)dW_t$$

belongs to H^1. We get

$$\int_0^T \mathbb{E}(\mathrm{e}^f\, u_t) I^{\alpha-1}\psi(t)\,dt = \mathbb{E}\left[\mathrm{e}^{f(\omega)} \int_0^T u_t(\omega) f(\varpi)\, dW_t^\alpha(\varpi)\right] = 0$$

for $\psi \in \mathcal{C}^1([0,T])$. Varying T we get

$$\mathbb{E}(\mathrm{e}^f\, u_t)\mathbb{E}(I^{\alpha-1}\psi(t)) = 0$$

for every $t \leq T$. For $\psi' > 0$ we then have

$$\mathbb{E}(\mathrm{e}^f\, u_t) = 0$$

for every $t \leq T$. As ψ runs through a total set in $L^2([0,T])$, f runs through a total set in H^1, and e^f runs through a total set in $L^2(\Omega,\mu)$. Hence we have $u_t = 0$.

15 Theorem : *Let u_t and v_t belonging to $\mathcal{C}^\beta(\mathcal{D}^{1,p})$ for $p \geq 4$. The FBM-Itô-Skorohod integral of v_t w.r. to X_t^α is defined by*

$$Y_t^\alpha = \int_0^T v_t(\omega) \odot dX_t^\alpha(\omega) = \int_0^T v_t(\omega) u_t(\omega) \odot dW_t^\alpha(\omega) \quad .$$

Then we have

$$Y_t^\alpha = \int_0^T v_t(\omega) \odot dX_t^\alpha(\omega) = \mathrm{div} \int_0^T v_t(\omega)\, d\widetilde{X}_t^\alpha(\omega,\varpi) \tag{6}$$

where

$$\widetilde{X}_t^\alpha(\omega,\varpi) = \int_0^t u_s(\omega)\, dW_s^\alpha(\varpi) \quad .$$

Finally we have the following computational rule

$$dY_t^\alpha = v_t \odot dX_t = (u_t v_t) \odot dW_t^\alpha \ . \tag{6'}$$

Proof: First observe that the last term in the right member of formula (6) is non-ambiguous thanks to proposition 14, and this is a FBM-Wiener integral. Secondly the regularity conditions are satisfied for $(\alpha,\beta) \in D_1$, and all the quantities are holomorphic w.r. to α.

Hence it suffices to prove formulas (6) and (6′) for real large enough values of α.

In this case formula (6) reads

$$Y_t = \mathrm{div} \int_0^T v_t(\omega) u_t(\omega) \dot{W}_t^\alpha(\varpi)\, dt = \int_0^T (u_t v_t)(\omega) \odot dW_t^\alpha(\omega)$$

and this is also formula (6′).

10 The main FBM Itô-Skorohod formula

Recall that the domains D_1 and $D_1(\beta)$ were defined by

$$D_1 = \{(\alpha,\beta) \in \mathbb{C} \times \mathbb{R} /\ \ \mathrm{Re}\,\alpha > \frac{1}{2},\ \ 0 < \beta < 1,\ \ \mathrm{Re}\,\alpha + \beta > 1\}$$

$$D_1(\beta) = \{\alpha \;/\; (\alpha,\beta) \in D_1\}\;.$$

We deal with a process

$$X_t^\alpha = \int_0^t u_s \odot dW_s^\alpha$$

satisfying the condition

$$u \in \bigcap_p \mathcal{C}^\beta(\mathcal{D}^{2,p})$$

and a polynomial F. We introduce the following domains

$$D_2 = \{(\alpha,\beta) \in \mathbb{C}\times\mathbb{R}/ \;\; \mathrm{Re}\,\alpha > \frac{3}{4}, \;\; 0<\beta<1, \quad \mathrm{Re}\,\alpha+\beta>1\}$$
$$D_2(\beta) = \{\alpha\Big/ \;(\alpha,\beta)\in D_2\}\;.$$

It is well known that in the Itô formula are involved many terms. So, before claiming the formula, we need to analyze the existence of the two following terms. The first one is

$$\int_0^T F'(X_t^\alpha)\odot dX_t^\alpha\;.$$

According to theorem 9, X_t^α belongs to $\bigcap_p \mathcal{C}^\gamma(L^p(\mu))$ for every $0<\gamma<\mathrm{Re}\,\alpha-\frac{1}{2}$, so that, for the existence of this term we need to assume that $(\alpha,\alpha-\frac{1}{2})\in D_1$, that is $\mathrm{Re}\,\alpha > \frac{3}{4}$.
The second one is

$$\int_0^T F''(X_t^\alpha)d\widetilde{\mathbb{E}}[\widetilde{X}_t^{\alpha 2}]\;. \tag{7}$$

More generally we have
16 Proposition : *Let $v_t \in \bigcap_p \mathcal{C}^\beta(L^p(\mu))$. Then*

$$\int_0^T v_t d\widetilde{\mathbb{E}}[\widetilde{X}_t^{\alpha 2}]$$

is holomorphically extendable for $\alpha\in D_1(\beta)$.
Proof: Denote Δ_0^T the simplex defined by the condition

$$(r,s,t)\in\Delta_0^T \quad \text{if} \quad 0\le r\le s\le t\le T \;\;.$$

For $\mathrm{Re}\,\alpha>1$, we have

$$\int_0^T v_t d\widetilde{\mathbb{E}}[\widetilde{X}_t^{\alpha 2}] = \frac{1}{\Gamma(\alpha-1)^2}\iiint_{\Delta_0^T} u_t v_t u_s (t-r)^{\alpha-2}(s-r)^{\alpha-2}drdsdt$$

where the triple integral absolutely converges. We then have to apply the following lemma
17 Lemma : *Let $\varphi,\psi\in\mathcal{C}^\beta(B)$ where B is a Banach space. Let b a bilinear map with values in another Banach space B_1, that we denote $b(\varphi,\psi)=\varphi\psi$. Then*

$$J^\alpha = \frac{1}{\Gamma(\alpha-1)^2}\iiint_{\Delta_0^T}\varphi(s)\psi(t)(t-r)^{\alpha-2}(s-r)^{\alpha-2}dr\,ds\,dt$$

converges absolutely for $\mathrm{Re}\,\alpha > 1$ *and admits a holomorphic extension for* $\alpha \in D_1(\beta)$.

Proof: Write $\quad \varphi(s)\psi(t) = (\varphi(s)-\varphi(r))(\psi(t)-\psi(r)) + \varphi(r)\psi(t) + \psi(r)(\varphi(s)-\varphi(r))$
so that $J^\alpha = J_1^\alpha + J_2^\alpha + J_3^\alpha$. The first term J_1^α absolutely converges for $\alpha+\beta > 1$. One has

$$\begin{aligned}
J_2^\alpha &= \frac{1}{\Gamma(\alpha-1)^2} \iiint_\Delta \varphi(r)\psi(t)(t-r)^{\alpha-2}(s-r)^{\alpha-2}\,drdsdt \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\alpha-1)} \iint \varphi(r)\psi(t)(t-r)^{2\alpha-3}\,drdt \\
&= \frac{\Gamma(2\alpha-1)}{2\Gamma(\alpha)^2} \int_0^T (I^{2\alpha-2}\varphi)(t)\psi(t)\,dt \\
&= \frac{\Gamma(2\alpha-1)}{2\Gamma(\alpha)^2} \left[\frac{\varphi(0)}{2\Gamma(2\alpha-1)} \int_0^T \psi(t)t^{2\alpha-2}\,dt + \int_0^T (I^{2\alpha+\beta'-2}f)(t)\psi(t)\,dt \right]
\end{aligned}$$

where $\beta' < \beta$, and f is a B-valued continuous function such that $\varphi - \varphi(0) = I^{\beta'}f$.

The first integral in the right hand side absolutely converges for $\alpha > \frac{1}{2}$. According to lemma 1, the second one holomorphically extends for $\beta + 2\alpha + \beta' - 1 > 1$, hence $\alpha+\beta > 1$ since β' can be arbitrarily close to β.

$$\begin{aligned}
J_3^\alpha &= \frac{1}{\Gamma(\alpha-1)^2} \iiint [\varphi(s)-\varphi(r)]\psi(r)(t-r)^{\alpha-2}(s-r)^{\alpha-2}\,drdsdt \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\alpha-1)} \iint [\varphi(s)-\varphi(r)]\psi(r)(s-r)^{\alpha-2}\left[(T-r)^{\alpha-1} - (s-r)^{\alpha-1}\right] drds
\end{aligned}$$

that is $J_3^\alpha = J_{31}^\alpha - J_{32}^\alpha$. Now J_{31}^α converges absolutely. It remains J_{32}^α.

$$J_{32}^\alpha = \frac{1}{\Gamma(\alpha)\Gamma(\alpha-1)} \iint [\varphi(s)-\varphi(r)]\psi(r)(s-r)^{2\alpha-3}\,drds = J_{321}^\alpha - J_{322}^\alpha \quad .$$

As for J_2^α, one finds that J_{321}^α holomorphically extends for $\alpha+\beta > 1$. Finally we have

$$J_{322}^\alpha = \frac{\Gamma(2\alpha-1)}{2\Gamma(\alpha)^2} \quad I^{2\alpha-1}(\varphi\psi)(T)$$

so that we are done.

18 Remarks : For $\alpha = 1$ every integral vanishes except J_{322}^α, and we recover

$$J^1 = J_{322}^1 = \frac{1}{2}\int_0^T \varphi(t)\psi(t)\,dt \quad .$$

Now we can claim the Itô formula.

19 Theorem : *Let* $(\alpha,\beta) \in D_2$. *Let* $u \in \bigcap_p \mathcal{C}^\beta(\mathcal{D}^{2,p})$, *and let* F *be a polynomial. Consider*

$$X_t^\alpha = \int_0^t u_s \odot dW_s^\alpha \quad .$$

Then we have the FBM Itô-Skorohod formula

$$
\begin{aligned}
F(X_T^\alpha(\omega)) = F(0) + \int_0^T F'(X_t^\alpha(\omega)) \odot dX_t^\alpha(\omega) + \frac{1}{2}\int_0^T F''(X_t^\alpha(\omega)) d\widetilde{\mathbb{E}}[\widetilde{X}_t^{\alpha 2}] \\
+ \int_0^T F''(X_t^\alpha) u_t dt \int_0^t \frac{(t-r)^{\alpha-2}}{\Gamma(\alpha-1)} dr \int_0^t \nabla_r u_s \odot dW_s^\alpha .
\end{aligned}
\tag{8}
$$

This formula can also be written

$$
\begin{aligned}
F(X_T^\alpha(\omega)) = F(0) + \int_0^T F'(X_t^\alpha(\omega)) \odot dX_t^\alpha(\omega) \\
+ \int_0^T F''(X_t^\alpha) u_t dt \int_0^t \frac{(t-r)^{\alpha-2}}{\Gamma(\alpha-1)} \nabla_r X_t^\alpha dr \ .
\end{aligned}
\tag{8'}
$$

Proof: By the preceding considerations, we know that in formulas (8) and (8′), every term but maybe the last makes sense and is holomorphic with respect to α.

First we prove formulas (8) and (8′) for $\mathrm{Re}\,\alpha$ large enough (for example $\mathrm{Re}\,\alpha > 5$). In this case every computation can be made pathwise (as for the little Itô formula). We then get

$$
\begin{aligned}
\int_0^T F'(X_t^\alpha(\omega)) u_t(\omega) \odot dW_t^\alpha(\omega) = \mathrm{div} \int_0^T F'(X_t^\alpha(\omega)) u_t(\omega) dW_t^\alpha(\varpi) \\
= \int_0^T F'(X_t^\alpha(\omega)) u_t(\omega) \dot{W}_t^\alpha(\omega) dt - \widetilde{\mathbb{E}} \int_0^T F'(X_t^\alpha(\omega)) \nabla u_t(\omega,\varpi) \dot{W}_t^\alpha(\varpi) dt \\
- \widetilde{\mathbb{E}} \int_0^T F''(X_t^\alpha(\omega)) u_t(\omega) \nabla X_t^\alpha(\omega,\varpi) \dot{W}_t^\alpha(\varpi) dt \ .
\end{aligned}
$$

On the other hand, we have

$$
\dot{X}_t^\alpha(\omega) dt = u_t(\omega) \dot{W}_t^\alpha(\omega) dt - \widetilde{\mathbb{E}}\left(\nabla u_t(\omega,\varpi) \dot{W}_t^\alpha(\varpi)\right) dt
$$

so that the sum of the two first terms of the right hand side is worth

$$
F(X_T^\alpha) - F(0) \quad .
$$

We then obtain

$$
F(X_T^\alpha) = F(0) + \int_0^T F'(X_t^\alpha) u_t \odot dW_t^\alpha + \widetilde{\mathbb{E}} \int_0^T F''(X_t^\alpha(\omega)) u_t(\omega) \nabla X_t^\alpha(\omega,\varpi) \dot{W}_t^\alpha(\varpi) dt \quad .
$$

It remains to compute

$$
J(\omega) = \widetilde{\mathbb{E}} \int_0^T F''(X_t^\alpha(\omega)) u_t(\omega) \nabla X_t^\alpha(\omega,\varpi) \dot{W}_t^\alpha(\varpi) dt \quad .
$$

We have

$$
\nabla X_t^\alpha(\omega,\varpi) = \int_0^t \nabla u_s(\omega,\varpi) \odot dW_s^\alpha(\omega) + \int_0^t u_s(\omega) dW_s^\alpha(\varpi)
$$

so that $J(\omega)$ splits into two terms $J_1(\omega)$ and $J_2(\omega)$

$$J_1(\omega) = \int_0^T F''(X_t^\alpha(\omega))u_t(\omega)dt \ \ \widetilde{\mathbb{E}}\left[\dot{W}_t^\alpha(\cdot)\int_0^t \nabla u_s(\omega,\cdot) \odot dW_s^\alpha(\omega)\right]$$
$$J_2(\omega) = \widetilde{\mathbb{E}}\int_0^T F''(X_t^\alpha)u_t\dot{W}_t^\alpha dt \int_0^t u_s\dot{W}_s^\alpha ds = \frac{1}{2}\int_0^T F''(X_t^\alpha)d\widetilde{\mathbb{E}}\left[\widetilde{X}_t^{\alpha 2}\right] \ .$$

Now we compute $J_1(\omega)$. First we have

$$\widetilde{\mathbb{E}}\left[\dot{W}_t^\alpha(\cdot)\int_0^t \nabla u_s(\omega,\cdot)\odot dW_s^\alpha(\omega)\right] = \int_0^t \widetilde{\mathbb{E}}\left[\nabla u_s(\omega,\cdot)\dot{W}_t^\alpha(\cdot)\right]\odot dW_s^\alpha(\omega) \ .$$

Indeed, for every test functional $G(\omega) \in \mathcal{D}^{1,2}$, we have

$$\begin{aligned}
&\mathbb{E}\left[G(\omega)\dot{W}_t^\alpha(\varpi)\int_0^t \nabla u_s(\omega,\varpi)\odot dW_s^\alpha(\omega)\right] \\
=\ &\mathbb{E}\left[\nabla G(\omega,\widehat{\omega})\dot{W}_t^\alpha(\varpi)\int_0^t \nabla u_s(\omega,\varpi)dW_s^\alpha(\widehat{\omega})\right] \\
=\ &\mathbb{E}\left[\nabla G(\omega,\widehat{\omega})\int_0^t \widetilde{\mathbb{E}}\left[\nabla u_s(\omega,\varpi)\dot{W}_t^\alpha(\varpi)\right]dW_s^\alpha(\widehat{\omega})\right] \\
=\ &\mathbb{E}\left[G(\omega)\int_0^t \widetilde{\mathbb{E}}\left[\nabla u_s(\omega,\varpi)\dot{W}_t^\alpha(\varpi)\right]\odot dW_s^\alpha(\omega)\right] \ .
\end{aligned}$$

Using the Wiener representation of ∇u_s w.r. to ϖ that is

$$\nabla u_s(\omega,\varpi) = \int_0^T \nabla_r u_s(\omega)dW_r(\varpi)$$

we get

$$\widetilde{\mathbb{E}}\left[\nabla u_s(\omega,\varpi)\dot{W}_t^\alpha(\varpi)\right] = \int_0^t \nabla_r u_s(\omega)\frac{(t-r)^{\alpha-2}}{\Gamma(\alpha-1)}dr \ .$$

Hence we find

$$J_1(\omega) = \int_0^T F''(X_t)u_t dt\int_0^t \frac{(t-r)^{\alpha-2}}{\Gamma(\alpha-1)}dr\int_0^t \nabla_r u_s\odot dW_s^\alpha \ .$$

Now we prove formula (8′). We return to

$$J(\omega) = \widetilde{\mathbb{E}}\int_0^T F''(X_t^\alpha(\omega))u_t(\omega)\nabla X_t^\alpha(\omega,\varpi)\dot{W}_t^\alpha(\varpi)dt \ .$$

As above we get

$$\widetilde{\mathbb{E}}[\nabla X_t^\alpha(\omega,\cdot)\dot{W}_t^\alpha(\cdot)] = \int_0^t \nabla_r X_t^\alpha(\omega)\frac{(t-r)^{\alpha-2}}{\Gamma(\alpha-1)}dr \ .$$

This yields

$$J(\omega) = \int_0^T F''(X_t^\alpha(\omega))u_t(\omega)dt\int_0^t \nabla_r X_t^\alpha(\omega)\frac{(t-r)^{\alpha-2}}{\Gamma(\alpha-1)}dr$$

and formula (8′) is proved.

So, formulas are proved for $\mathrm{Re}\,\alpha$ large enough.

Every term but the last admits an analytic continuation for $\alpha \in D_2(\beta)$, as we have seen above. Hence the last term (in the formulas (8) and (8′)) has also an analytic continuation. This establishes the formulas, and the following corollary.

20 Corollary : *Let F be a polynomial. Then the integral*

$$\int_0^T F(X_t^\alpha) u_t dt \int_0^t \frac{(t-r)^{\alpha-2}}{\Gamma(\alpha-1)} dr \int_0^t \nabla_r u_s \odot dW_s^\alpha$$

admits an analytic continuation for $\alpha \in D_2(\beta)$.

Proof: it suffices to notice that every polynomial is the second derivative of another polynomial, and to apply formula (8).

11 Recovering the case $\alpha = 1$

Note that the last integral in formula (8) is singular, even in the case $\alpha = 1$. Nevertheless, the previous corollary proves that the symbolic writing

$$Y = \int_0^T v_t dt \int_0^t \nabla_t u_s \odot dW_s^\alpha$$

makes sense for $v_t = F(X_t) u_t$ if F is a polynomial.

In this section we prove that such a formula can be justified under an additional trace hypothesis.

First we prove a lemma

21 Lemma : *Let B be a Banach space and $q > 1$. Consider a function $\Phi(r,t)$ which belongs to the space $L^q([0,T], dr, \mathcal{C}^\beta(B))$. Then the following integral*

$$J^\alpha = \frac{1}{\Gamma(\alpha-1)} \int_0^T dt \int_0^t (t-r)^{\alpha-2} \Phi(r,t) dr$$

makes sense and is holomorphic w.r. to α for $\mathrm{Re}\,\alpha + \beta > 1/q$. Moreover, its value for $\alpha = 1$ is

$$J^1 = \int_0^T \Phi(t,t) dt$$

where $\Phi(t,t)$ belongs to $L^q([0,T], dr)$.

Proof: First observe that the trace $\Phi(r,r)$ exists and belongs to $L^q(B)$. Put

$$\Phi(r,t) - \Phi(r,r) = \Psi(r,t)|t-r|^\beta, \qquad A(r) = \operatorname*{Sup}_t |\Psi(r,t)|_B \quad .$$

By the hypothesis, $A(r)$ belongs to $L^q(dr)$. Put $J^\alpha = J_1^\alpha + J_2^\alpha$ with

$$J_1^\alpha = \frac{1}{\Gamma(\alpha-1)} \int_0^T dt \int_0^t \Psi(r,t)(t-r)^{\alpha+\beta-2} dr$$

$$J_2^\alpha = \frac{1}{\Gamma(\alpha-1)} \int_0^T dt \int_0^t \Phi(r,r)(t-r)^{\alpha-2} dr \quad .$$

The first J_1^α is absolutely convergent by the majoration ($a = \operatorname{Re}\alpha$)

$$\int_0^T A(r)dr \int_r^T (t-r)^{a+\beta-2} dt = \int_0^T A(r) \frac{(T-r)^{a+\beta-1}}{a+\beta-1} dr < +\infty$$

for $a+\beta > 1/q$. For the other J_2^α, we have

$$J_2^\alpha = \frac{1}{\Gamma(\alpha-1)} \int_0^T \Phi(r,r)dr \int_r^T (t-r)^{\alpha-2} dt = \frac{1}{\Gamma(\alpha)} \int_0^T \Phi(r,r)(T-r)^{\alpha-1} dr$$

for $\operatorname{Re}\alpha > 1$. The right hand side extends analytically for $\operatorname{Re}\alpha > 1/q$.

It remains to compute J^1. For $\alpha = 1$, J_1^1 vanishes, so that J^1 reduces to $\int_0^T \Phi(r,r)dr$.

22 Theorem : *Let $(\alpha,\beta) \in D_2$. Assume that $u, v \in \bigcap_p \mathcal{C}^\beta(\mathcal{D}^{2,p})$. Put*

$$Y^\alpha(\omega) = \int_0^T v_t dt \int_0^t \frac{(t-r)^{\alpha-2}}{\Gamma(\alpha-1)} dr \int_0^t \nabla_r u_s \odot dW_s^\alpha$$

which is the last term of formula (8). Assume the following additional trace hypothesis, that is $u - u_0 \in \mathcal{J}^{\beta+\frac{1}{2},2}(\mathcal{D}^{2,2})$. Then the integral is absolutely convergent for $\operatorname{Re}\alpha > 1$. Moreover Y^α has an analytic continuation for $\alpha \in D_1(\beta)$. Its value for $\alpha = 1$ is worth

$$Y(\omega) = \operatorname*{Lim}_{\alpha \searrow 1} Y^\alpha(\omega) = \int_0^T v_t dt \int_0^t \nabla_t u_s \odot dW_s^\alpha \quad .$$

Proof: We must analyze the integral

$$\int_0^T v_t dt \int_0^t \frac{(t-r)^{\alpha-2}}{\Gamma(\alpha-1)} dr \int_0^t \nabla_r u_s \odot dW_s^\alpha \ . \tag{9}$$

To this end, put for $\operatorname{Re}\alpha > \frac{3}{2}$

$$Z_{r,t}^\alpha = \int_0^t \nabla_r u_s \odot dW_s^\alpha$$

where $\nabla_r u_s$ is the Wiener representation

$$\nabla u_s(\omega,\varpi) = \int_0^T \nabla_r u_s(\omega) dW_r(\varpi) \quad .$$

We have $\nabla u - \nabla u_0 \in \mathcal{J}^{\beta+\frac{1}{2},2}(\mathcal{D}^{1,2})$, so that by the above Wiener representation $\nabla_r u_t - \nabla_r u_0$ belongs to $L^2(dr, \mathcal{J}^{\beta+\frac{1}{2},2}(\mathcal{D}^{1,2}))$. Thanks to theorem 4, we see that

$$\widetilde{Z}_{r,t}^\alpha(\omega,\varpi) = \int_0^t \nabla_r u_s(\omega) dW_s^\alpha(\varpi)$$

belongs to $L^2(dr, \mathcal{C}^\gamma(\mathcal{D}^{1,2}))$ for every $\gamma < \operatorname{Re}\alpha - \frac{1}{2}$. Hence $Z_{r,t}^\alpha = \operatorname{div} \widetilde{Z}_{r,t}^\alpha$ belongs to $L^2(dr, \mathcal{C}^\gamma(L^2(\mu)))$. Now put $\Phi(r,t) = v_t Z_{r,t}$ which belongs to $L^2(dr, \mathcal{C}^\gamma(L^q(\mu)))$ for every $q < 2$. Applying the previous lemma gives the result.

23 Corollary (The classical Itô-Skorohod formula, *cf.* [22])**:** *With the additional trace hypothesis, we have*

$$F(X_T(\omega)) = F(0) + \int_0^T F'(X_t(\omega)) \odot dX_t(\omega) + \frac{1}{2}\int_0^T F''(X_t(\omega))u_t^2\,dt \\ + \int_0^T F''(X_t)u_t\,dt \int_0^t \nabla_t u_s \odot dW_s\,. \tag{10}$$

24 Proposition : *If u is an adapted process, and for $\alpha = 1$, the last term vanishes, so that we recover the classical Itô formula.*

Proof: It suffices to remark that $\nabla_t u_s$ vanishes a.e. on the set $\{(s,t)/t > s\}$.

25 Remarks : a) It should be observed that the last term in formula $(8')$ is the sum of the two last terms in formula (8), so that there is no need to look after it.

b) By routine arguments, we can replace F by a $\mathcal{C}^2$ function which is bounded with its two first derivatives, for $(\alpha,\beta) \in D_2$.

12 A more complete formula

26 Theorem : *Let $u, v \in \bigcap_p \mathcal{C}^\beta(\mathcal{D}^{2,p})$, and let $F(x)$ be a polynomial. Suppose that the trace additional property is satisfied for u or v. Consider*

$$X_t^\alpha = \int_0^t u_s \odot dW_s^\alpha, \qquad Y_t = X_t^\alpha + \int_0^t v_s\,ds$$

for α such that $(\alpha,\beta) \in D_2$. Then we have the FBM-Itô-Skorohod formula

$$\begin{aligned} F(Y_T^\alpha(\omega)) = F(0) &+ \int_0^T F'(Y_t^\alpha(\omega)) \odot dX_t^\alpha(\omega) + \int_0^T F'(Y_t^\alpha(\omega))v_t\,dt \\ &+ \frac{1}{2}\int_0^T F''(Y_t(\omega))d\widetilde{\mathbb{E}}[\widetilde{X}_t^{\alpha 2}] \\ &+ \int_0^T F''(Y_t^\alpha)u_t\,dt \int_0^t \frac{(t-r)^{\alpha-2}}{\Gamma(\alpha-1)}\,dr \int_0^t \nabla_r u_s \odot dW_s^\alpha \\ &+ \int_0^T F''(Y_t^\alpha)u_t\,dt \int_0^t \frac{(t-r)^{\alpha-2}}{\Gamma(\alpha-1)}\,dr \int_0^t \nabla_r v_s\,ds. \end{aligned} \tag{11}$$

Proof: Similar to the one of formula (8), without new difficulties.

27 Remark : According to [4], [8] and [20] the FBM B_t^H with Hurst parameter H is worth

$$B_t^H = W_t^{H+\frac{1}{2}} + Z_t$$

where Z_t is a pathwise absolutely continuous process, so that formula (11) proves an Itô formula for B_t^H.

13 The FBM Stratonovich integral

28 Theorem : *Let $X_t^\alpha = \int_0^t u_s \odot dW_s^\alpha$, and $Y_t \in \bigcap_p \mathcal{C}^\beta(\mathcal{D}^{1,p})$. For $\operatorname{Re}\alpha > \frac{3}{2}$, we have*

$$\int_0^T Y_t \frac{dX_t^\alpha}{dt} dt = \int_0^T Y_t \odot dX_t^\alpha + \int_0^T u_t dt \int_0^t \nabla_r Y_t \frac{(t-r)^{\alpha-2}}{\Gamma(\alpha-1)} dr \ . \tag{12}$$

Proof: As in the proof of theorem 19 we have

$$\begin{aligned} u_t \dot{W}_t^\alpha dt &= u_t \odot dW_t^\alpha + \widetilde{\mathbb{E}}\left(\nabla u_t(\omega,\cdot)\dot{W}_t^\alpha(\cdot)\right) dt \\ Y_t u_t \dot{W}_t^\alpha dt &= (Y_t u_t) \odot dW_t^\alpha + \widetilde{\mathbb{E}}\left(\nabla(Y_t(\omega)u_t(\omega,\cdot))\dot{W}_t^\alpha(\cdot)\right) dt \\ Y_t \frac{dX_t^\alpha}{dt} dt &= (Y_t u_t) \odot dW_t^\alpha + \widetilde{\mathbb{E}}\left(\nabla(Y_t(\omega)u_t(\omega,\cdot))\dot{W}_t^\alpha(\cdot)\right) dt \\ &\quad - \widetilde{\mathbb{E}}\left(Y_t \nabla u_t(\omega,\cdot)\dot{W}_t^\alpha(\cdot)\right) dt \ . \end{aligned}$$

So that we get

$$Y_t \frac{dX_t^\alpha}{dt} dt = Y_t u_t \odot dW_t^\alpha + u_t(\omega)\widetilde{\mathbb{E}}\left(\nabla Y_t(\omega,\cdot)\dot{W}_t^\alpha(\varpi)\right) dt \ .$$

Replacing $\widetilde{\mathbb{E}}\left(\nabla Y_t(\omega,\cdot)\dot{W}_t^\alpha(\cdot)\right) dt$ by its value from the proof of theorem 19 completes the proof.

Now we are in a position to put

29 Theorem and definition: *Let $u, v \in \bigcap_p \mathcal{C}^\beta(\mathcal{D}^{2,p})$. Let G be a polynomial. Put*

$$Y_t^\alpha = G(Z_t^\alpha), \quad \text{where} \quad Z_t^\alpha = \int_0^t v_s \odot dW_t^\alpha \ .$$

Suppose in addition that $v \in \mathcal{J}^{\beta+\frac{1}{2},2}(\mathcal{D}^{2,2})$. Then the ordinary integral $\int_0^T Y_t^\alpha dX_t^\alpha$ admits an analytic continuation for $\alpha \in D_2(\beta)$ which is by definition the Stratonovich integral

$$\int_0^T Y_t^\alpha \circ dX_t^\alpha \ .$$

Proof: As in the proof of theorem 19, we have for $\operatorname{Re}\alpha > \frac{3}{2}$

$$\begin{aligned} \int_0^t \nabla_r G(Z_t^\alpha) \frac{(t-r)^{\alpha-2}}{\Gamma(\alpha-1)} dr &= G'(Z_t^\alpha) \int_0^t v_s ds \int_0^s \frac{((t-r)(s-r))^{\alpha-2}}{\Gamma(\alpha-1)^2} dr \\ &\quad + \ G'(Z_t^\alpha) \int_0^t \frac{(t-r)^{\alpha-2}}{\Gamma(\alpha-1)} dr \int_0^t \nabla_r v_s \odot dW_s^\alpha . \end{aligned}$$

Hence we get

$$\begin{aligned} \int_0^T Y_t \circ dX_t^\alpha = &\int_0^T Y_t \odot dX_t^\alpha \\ &+ \int_0^T u_t G'(Z_t^\alpha) dt \int_0^t v_s ds \int_0^s \frac{((t-r)(s-r))^{\alpha-2}}{\Gamma(\alpha-1)^2} dr \\ &+ \int_0^T u_t G'(Z_t^\alpha) \int_0^t \frac{(t-r)^{\alpha-2}}{\Gamma(\alpha-1)} dr \int_0^t \nabla_r v_s \odot dW_s^\alpha. \end{aligned}$$

In the right hand member, the Skorohod integral extends for $\alpha \in D_2(\beta)$, the following term extends for $\alpha \in D_2(\beta)$ thanks to lemma 17. The last term extends by the extra trace property for v, thanks to theorem 22. The proof is complete.

30 Theorem : *Under the hypotheses of theorem 28, for every polynomial F we have the FBM Itô Stratonovich formula*

$$F(X_T^\alpha) = F(0) + \int_0^T F'(X_t^\alpha) \circ dX_t^\alpha \ .$$

Proof: Put $Y_t^\alpha = F'(X_t^\alpha)$, so that Y_t^α satisfies the hypotheses of the last theorem, and applies the analytic continuation from the case $\mathrm{Re}\,\alpha > \frac{3}{2}$.

Notes Added After Proof: After the acceptance of this paper, we have learned of the following relevant and interesting preprint:

M. Gradinaru, F. Russo, P. Vallois (2001) Generalized covariations, local time and stratonovitch Itô's formula for fractional Brownian motion with Hurst index $H \geq 1/4$. *Preprint of the Institut Elie Cartan*, 2001/no38, Université de Nancy.

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