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Nonequilibrium moderate deviations from hydrodynamics of the simple symmetric exclusion process*

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Abstract

In this paper, we give the moderate deviation principle from the hydrodynamic limit of the simple symmetric exclusion process on the 1-dimensional torus starting from a nonequilibrium state, which extends the result given in Gao and Quastel (2003) about the case where the process starts from an equilibrium state. The exponential tightness of the scaled density field of the process and a replacement lemma play key roles in the proof of the main result. We utilize Grownwall's inequality and the upper bound of the large deviation principle given in Kipnis, Olla and Varadhan (1989) to prove the above exponential tightness and the replacement lemma respectively in the absence of the invariance of the initial distribution.

Keywords: simple symmetric exclusion process; nonequilibrium; moderate deviation; exponential tightness; replacement lemma.
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1 Introduction

In this paper, we will give moderate deviation principles from hydrodynamic limits of the simple symmetric exclusion process on the one dimensional torus starting from a nonequilibrium state, which extend the result given in [1] about the equilibrium case. We first recall the definition of the simple symmetric exclusion process. For each integer $N \geq 1$, let $\mathbb{T}^N = \{0, 1, 2, \dots, N-1\}$. The simple symmetric exclusion process $\{\hat{\eta}_t\}_{t\geq 0}$ on $\mathbb{T}^N$ is a continuous-time Markov process with state space $\mathbb{X}^N = \{0,1\}^{\mathbb{T}^N}$ and generator $\hat{\mathcal{L}}_N$ given by

$$\hat{\mathcal{L}}_N f(\eta) = \sum_{x\in\mathbb{T}^N} \left[f(\eta^{x,x+1}) - f(\eta)\right] \tag{1.1}$$

for any $\eta \in \mathbb{X}^N$ and f from $\mathbb{X}^N$ to $\mathbb{R}$, where $\eta^{x,x+1} \in \mathbb{X}^N$ is defined as

$$\eta^{x,x+1}(y) = \begin{cases} \eta(y) & \text{if } y \neq x, x+1, \\ \eta(x+1) & \text{if } y = x, \\ \eta(x) & \text{if } y = x+1. \end{cases}$$

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Note that, throughout this paper operations on $\mathbb{T}^N$ are under the $(\mathrm{mod}\ N)$-meaning. For example, $(N-1)+1=0$.

According to Equation (1.1), the simple symmetric exclusion process describes a traffic flow on $\mathbb{T}^N$. At each $x \in \mathbb{T}^N$, there is at most one particle. All particles perform random walks on $\mathbb{T}^N$. In detail, a particle at x jumps to the neighbor $y = x \pm 1$ at rate 1 when y is not occupied by other particles.

Here we recall invariant measures of the simple symmetric exclusion process. For $0 < \alpha < 1$, let ν_α be the product measure on $\mathbb{T}^N$ under which $\{\eta(x)\}_{0\leq x\leq N-1}$ are independent and $\nu_\alpha(\eta(x)=1)=\alpha$ for all $0\leq x\leq N-1$, then by Equation (1.1), it is easy to check that ν_α is a reversible distribution and hence an invariant distribution of $\{\hat{\eta}_t\}_{t\geq 1}$. The measure ν_α is called the global invariant measure of the process. For a given integer $1\leq K\leq N-1$, let

$$\nu_K(\cdot) = \nu_\alpha\left(\cdot\Big|\sum_{x\in\mathbb{T}^N}\eta(x)=K\right).$$

Since the total number of particles is conserved for the simple symmetric exclusion process, we have

$$1_{\{\sum_{x\in\mathbb{T}^N}\hat{\eta}_t(x)=K\}} = 1_{\{\sum_{x\in\mathbb{T}^N}\hat{\eta}_0(x)=K\}}$$

for ant $t\geq 0$, where 1_A is the indicator function of the event A. As a result, for any f from $\mathbb{X}^N$ to $\mathbb{R}$,

$$\begin{aligned}\mathbb{E}_{\nu_\alpha}\left(f(\hat{\eta}_t)1_{\{\sum_{x\in\mathbb{T}^N}\hat{\eta}_0(x)=K\}}\right) &= \mathbb{E}_{\nu_\alpha}\left(f(\hat{\eta}_t)1_{\{\sum_{x\in\mathbb{T}^N}\hat{\eta}_t(x)=K\}}\right)\\ &= \mathbb{E}_{\nu_\alpha}\left(f(\hat{\eta}_0)1_{\{\sum_{x\in\mathbb{T}^N}\hat{\eta}_0(x)=K\}}\right)\end{aligned}$$

and hence

$$\begin{aligned}\mathbb{E}_{\nu_K}f(\hat{\eta}_t) &= \frac{\mathbb{E}_{\nu_\alpha}\left(f(\hat{\eta}_t)1_{\{\sum_{x\in\mathbb{T}^N}\hat{\eta}_0(x)=K\}}\right)}{\nu_\alpha(\sum_{x\in\mathbb{T}^N}\eta(x)=K)}\\ &= \frac{\mathbb{E}_{\nu_\alpha}\left(f(\hat{\eta}_0)1_{\{\sum_{x\in\mathbb{T}^N}\hat{\eta}_0(x)=K\}}\right)}{\nu_\alpha(\sum_{x\in\mathbb{T}^N}\eta(x)=K)} = \mathbb{E}_{\nu_K}f(\hat{\eta}_0).\end{aligned}$$

In conclusion, ν_K is an invariant measure of the process which is called the local invariant measure. Note that ν_K is independent of the choice of α since the process is irreducible and has finite states conditioned on the total number of particles being K.

For other basic properties of the simple symmetric exclusion process, readers could see Chapter 8 of [5] and Part III of [6] for a detailed survey.

Now we recall hydrodynamic limits of the simple symmetric exclusion process. From now on, we write $\hat{\eta}_t^N$ instead of $\hat{\eta}_t$ to emphasize the N-dependence. We denote $\hat{\eta}^N_{tN^2}$ by η_t^N. Then, the generator $\mathcal{L}_N$ of $\{\eta_t^N\}_{t\geq 0}$ is given by $\mathcal{L}_N = N^2\hat{\mathcal{L}}_N$. Let $\mathbb{T}=[0,1)$ be the one dimensional torus. We denote by μ_t^N the empirical density field of η_t^N, i.e.,

$$\mu_t^N(du) = \frac{1}{N}\sum_{x\in\mathbb{T}^N}\eta_t^N(x)\delta_{\frac{x}{N}}(du),$$

where δ_a is the Dirac measure concentrated at a. The following hydrodynamic limit theorem is given in Chapter 4 of [3].

Proposition 1.1 (Kipnis and Landim, [3]). *If* $\{\eta_0^N(x)\}_{0\leq x\leq N-1}$ *are independent and*

$$P(\eta_0^N(x)=1) = \phi\left(\frac{x}{N}\right)$$

for some $\phi \in C(\mathbb{T})$ *and all* $N \geq 1, x \in \mathbb{T}^N$*, then* $\mu_t^N(f)$ *converges in probability to* $\int_{\mathbb{T}} \rho_t(u)f(u)du$ *as* $N \to +\infty$ *for any* $t \geq 0$ *and* $f \in C(\mathbb{T})$*, where* $\{\rho_t\}_{t\geq 0}$ *is the unique weak solution to the heat equation*

$$\begin{cases} \partial_t \rho(t,u) = \partial_{uu}^2 \rho(t,u) & \textit{for any } t \geq 0, \\ \rho_0 = \phi. \end{cases} \tag{1.2}$$

The proof of Proposition 1.1 utilizes Dynkin's martingale formula to show that any weak limit of a subsequence of $\{\mu_t^N\}_{N\geq 1}$ is absolutely continuous with respect to Lebesgue measure and the corresponding Radon-Nikodym derivative is a weak solution to Equation (1.2).

It is natural to further investigate central limit theorems, large and moderate deviations from hydrodynamic limits given in Proposition 1.1. The central limit theorem from the hydrodynamic limit is also called the fluctuation. It is shown in Chapter 11 of [3] that the fluctuation of the simple symmetric exclusion process is driven by a generalized Ornstein-Uhlenbeck process introduced in [2]. The large deviation principle from the hydrodynamic limit of the simple symmetric exclusion process is given in [4], the proof of which utilizes an exponential martingale strategy. A moderate deviation principle is given in [1] in the case where the initial distribution of the process is ν_α, the proof of which extends the strategy given in [4] and relies heavily on the fact that ν_α is an invariant measure of the process.

In this paper, we will extend the result given in [1] to cases where initial distributions of our processes are not invariant. The proof of our main result still utilizes the exponential martingale strategy as that in [1] but two technical details are improved. We give new approaches to check a replacement lemma and the exponential tightness of the scaled density field of the process, where the invariance of the initial distribution is not needed. For mathematical details, see Section 3.

2 Main result

In this section, we give our main result. For later use, we first introduce some notations and definitions. Let $\mathbb{T} = [0,1)$ be the one dimensional torus defined as in Section 1. Throughout this paper, operators on $\mathbb{T}$ are under $(\mathrm{mod}\, 1)$-meaning. For example, $0.2 - 0.3 = 0.9$. We denote the dual of $C^\infty(\mathbb{T})$ endowed with the weak topology by $\mathcal{S}$, i.e., $\nu_n \to \nu$ in $\mathcal{S}$ if and only if

$$\lim_{n\to+\infty} \nu_n(f) = \nu(f)$$

for any $f \in C^\infty(\mathbb{T})$. Let $\{a_N\}_{N\geq 1}$ be a positive sequence such that

$$\lim_{N\to+\infty} \frac{a_N}{N} = \lim_{N\to+\infty} \frac{\sqrt{N}}{a_N} = 0.$$

Throughout this paper, we adopt the following assumption.

Assumption (A): $\{\eta_0^N(x)\}_{x\in\mathbb{T}^N}$ are independent and

$$P\left(\eta_0^N(x) = 1\right) = \phi\left(\frac{x}{N}\right)$$

for all $x \in \mathbb{T}^N$, where $\phi \in C(\mathbb{T})$ such that $0 < \phi(u) < 1$ for all $u \in \mathbb{T}$.

Let $\mathbb{E}$ be the expectation operator. For any $t \geq 0$ and $N \geq 1$, we define the scaled density field θ_t^N as

$$\theta_t^N(du) = \frac{1}{a_N} \sum_{x\in\mathbb{T}^N} \left(\eta_t^N(x) - \mathbb{E}\eta_t^N(x)\right) \delta_{\frac{x}{N}}(du).$$

We can consider θ_t^N as a random element in $\mathcal{S}$ such that

$$\theta_t^N(f) = \frac{1}{a_N}\sum_{x\in\mathbb{T}^N}\left(\eta_t^N(x) - \mathbb{E}\eta_t^N(x)\right) f\left(\frac{x}{N}\right)$$

for any $f \in C^\infty(\mathbb{T})$. For a given $T > 0$, we denote $\{\theta_t^N\}_{0\leq t\leq T}$ by θ^N. Then θ^N is a random element in $\mathcal{D}([0,T],\mathcal{S})$, which is the set of càdlàg functions from $[0,T]$ to $\mathcal{S}$ endowed with the Skorokhod topology.

We first give the moderate deviation rate function of the dynamic of the process. For any $W \in \mathcal{D}([0,T],\mathcal{S})$, we define

$$\begin{aligned} I_{dyn}(W) = \sup_{F\in C^{1,\infty}([0,T]\times\mathbb{T})} \Bigg\{ & W_T(F_T) - W_0(F_0) - \int_0^T W_s((\partial_s + \partial_{uu}^2)F_s)ds \\ & - \int_0^T\int_{\mathbb{T}} \rho_s(u)(1-\rho_s(u))(\partial_u F_s(u))^2 ds du \Bigg\}, \end{aligned} \tag{2.1}$$

where $\{\rho_t\}_{t\geq 0}$ is the unique weak solution to Equation (1.2). Note that, in Equation (2.1),

$$\left((\partial_s + \partial_{uu}^2)F_s\right)(u) = \partial_s F(s,u) + \partial_{uu}^2 F(s,u)$$

for any $u \in \mathbb{T}$ and hence $(\partial_s + \partial_{uu}^2)F_s \in C^\infty(\mathbb{T})$ for any given $0 \leq s \leq T$. Then, since $W \in \mathcal{D}([0,T],\mathcal{S})$, $\{W_s((\partial_s + \partial_{uu}^2)F_s)\}_{0\leq s\leq T}$ is a real-valued càdlàg function on $[0,T]$. Secondly, we give the moderate deviation rate function of the intial state of the process. For any $\nu \in \mathcal{S}$, we define

$$I_{ini}(\nu) = \sup_{f\in C^\infty(\mathbb{T})}\left\{\nu(f) - \frac{1}{2}\int_{\mathbb{T}}\phi(u)\left(1-\phi(u)\right)f^2(u)du\right\}. \tag{2.2}$$

Now we can give our main result.

Theorem 2.1. *Under Assumption (A),*

$$\limsup_{N\to+\infty}\frac{N}{a_N^2}\log P\left(\theta^N \in C\right) \leq -\inf_{W\in C}\left(I_{ini}(W_0) + I_{dyn}(W)\right) \tag{2.3}$$

for any closed set $C \subseteq \mathcal{D}([0,T],\mathcal{S})$ and

$$\liminf_{N\to+\infty}\frac{N}{a_N^2}\log P\left(\theta^N \in O\right) \geq -\inf_{W\in O}\left(I_{ini}(W_0) + I_{dyn}(W)\right) \tag{2.4}$$

for any open set $O \subseteq \mathcal{D}([0,T],\mathcal{S})$.

If $\phi \equiv \alpha$ for some $\alpha \in (0,1)$, then $\rho_s(u)(1-\rho_s(u)) \equiv \alpha(1-\alpha)$ and hence Theorem 2.1 reduces to Theorem 1.1 of [1].

The proof of Theorem 2.1 is divided into two sections. In Section 3, we give a replacement lemma and show that $\{\theta^N\}_{N\geq 1}$ are exponentially tight. To prove the above replacement lemma, we utilize the upper bound of the large deviation principle given in [4]. To check the exponential tightness of $\{\theta^N\}_{N\geq 1}$, we utilize Grownwall's inequality. According to Grownwall's inequality, the exponential tightness of $\{\theta^N\}_{N\geq 1}$ is a consequence of the exponential tightness of another class of stochastic processes $\{Z_t^N :\ 0 \leq t \leq T\}_{N\geq 1}$, which make a class of exponential martingales $\{\mathcal{M}_t^N :\ 0 \leq t \leq T\}_{N\geq 1}$ can be written as $\mathcal{M}_t^N = \exp\left(\frac{a_N^2}{N}\left(Z_t^N + O(1)\right)\right)$ for all $N \geq 1$. As a result, the exponential tightness of $\{Z_t^N :\ 0 \leq t \leq T\}_{N\geq 1}$ follows from Doob's inequality and consequently the exponential tightness of $\{\theta^N\}_{N\geq 1}$ is derived. In Section 4, we complete the proof of Theorem 2.1. The replacement lemma and the exponential tightness of $\{\theta^N\}_{N\geq 1}$ given in Section 3 are analogues of Lemmas 2.1 and 3.2 of [1] respectively, which makes the strategy introduced in [1] apply to the case discussed in this paper.

3 Exponential tightness and replacement lemma

In this section, we prove following two lemmas.

Lemma 3.1. *Under Assumption (A), $\{\theta^N\}_{N\geq 1}$ are exponentially tight.*

Lemma 3.2. *Under Assumption (A), for any $\epsilon > 0$ and $G \in C^{1,0}([0,T]\times\mathbb{T})$,*

$$\limsup_{N\to+\infty}\frac{N}{a_N^2}\log P\left(\left|\int_0^T\frac{1}{N}\sum_{x\in\mathbb{T}^N}\eta_s^N(x)\eta_s^N(x+1)G_s\left(\frac{x}{N}\right)ds\right.\right.$$
$$\left.\left.-\int_0^T\int_{\mathbb{T}}\rho_s^2(u)G_s(u)duds\right|>\epsilon\right)=-\infty. \tag{3.1}$$

Lemmas 3.1 and 3.2 are analogues of Lemmas 2.1 and 3.2 of [1] respectively. In [1], the simple symmetric exclusion process η_t^N follows an invariant distribution ν_α at any moment t. With this property, Lemma 2.1 of [1] is proved by utilizing Jensen's inequality and Cauchy-Schwarz inequality. Lemma 3.2 of [1] is proved by utilizing Garsia-Rademich-Rumsey inequality. For mathematical details, see [1]. Above approaches do not apply to cases discussed in this paper since the initial distribution of the simple symmetric exclusion process in this paper is not invariant. Hence we prove above two lemmas in different ways than those in [1]. By utilizing Grownwall's inequality, we reduce the check of the exponential tightness of θ^N to that of the logarithm of an exponential martingale and then Lemma 3.1 holds according to Doob's inequality. For mathematical details, see Subsection 3.1. By utilizing the upper bound of the large deviation principle given in [4], Lemma 3.2 follows from the fact that the minimum of the large deviation rate function given in [4] on a closed set without $\{\rho_t\}_{0\leq t\leq T}$ is strictly positive. For mathematical details, see Subsection 3.2.

3.1 Proof of Lemma 3.1

In this subsection, we prove Lemma 3.1. We first introduce some notations and definitions for later use. For integer $n\geq 0$, we denote $\cos(2n\pi u)$ by $e_n(u)$. For integer $n\geq 1$, we denote $\sin(2n\pi u)$ by $e_{-n}(u)$. For $n\in\mathbb{Z}$, $N\geq 1$, $c>0$ and $t\geq 0$, we define

$$Y_t^{N,n,c}=\exp\left(\frac{a_N^2}{N}\theta_t^N(ce_n)\right).$$

Furthermore, we define

$$\mathcal{M}_t^{N,n,c}=\frac{Y_t^{N,n,c}}{Y_0^{N,n,c}}\exp\left(-\int_0^t\frac{(\partial_s+\mathcal{L}_N)Y_s^{N,n,c}}{Y_s^{N,n,c}}ds\right).$$

Note that $\partial_t Y_t^{N,n,c}$ in the above definition does not refer to $\lim_{\Delta t\to 0}\frac{Y_{t+\Delta t}^{N,n,c}-Y_t^{N,n,c}}{\Delta t}$ since $\{Y_t^{N,n,c}\}_{t\geq 0}$ is not continuous. According to our notations, $Y_t^{N,n,c}=V_c(t,\eta_t^N)$, where $V_c(t,\eta)=\exp\left(\frac{ca_N}{N}\sum_{x\in\mathbb{T}^N}\left(\eta(x)-\mathbb{E}\eta_t^N(x)\right)e_n(x/N)\right)$. Then, for given η, $V_c(\cdot,\eta)\in C^1([0,T])$ and

$$\partial_t V_c(t,\eta)=V_c(t,\eta)\left(\frac{ca_N}{N}\sum_{x\in\mathbb{T}^N}\left(-\frac{d}{dt}\mathbb{E}\eta_t^N(x)\right)e_n(x/N)\right).$$

So, in the above definition of $\mathcal{M}_t^{N,n,c}$, we denote $\partial_t V_c(t,\eta)\Big|_{\eta=\eta_t^N}$ by $\partial_t Y_t^{N,n,c}$.

By Feynman-Kac formula, $\{\mathcal{M}_t^{N,n,c}\}_{t\geq 0}$ is a martingale with mean 1.

Now we give the proof of Lemma 3.1

Proof of Lemma 3.1. Since $\mathrm{span}\{e_n: \ -\infty < n < +\infty\}$ are dense in $C^\infty(\mathbb{T})$, according to the criterion given in [7], to complete this proof we only need to show that

$$\limsup_{M\to+\infty}\limsup_{N\to+\infty}\frac{N}{a_N^2}\log P\left(\sup_{0\le t\le T}|\theta_t^N(e_n)|>M\right)=-\infty \tag{3.2}$$

and

$$\limsup_{\delta\to 0}\limsup_{N\to+\infty}\frac{N}{a_N^2}\log\sup_{\sigma\in\mathcal{T}}P\left(\sup_{0\le t\le\delta}\left|\theta_{t+\sigma}^N(e_n)-\theta_\sigma^N(e_n)\right|>\epsilon\right)=-\infty \tag{3.3}$$

for any $n\in\mathbb{Z}$ and $\epsilon>0$, where $\mathcal{T}$ is the set of stopping times of $\{\eta_t^N\}_{t\ge 0}$ bounded by T.

We first check Equation (3.2). According to Chapman-Kolmogorov equation, we have

$$\begin{aligned}
\frac{\partial_s Y_s^{N,n,1}}{Y_s^{N,n,1}} &= \frac{\partial_s V_1(s,\eta)\big|_{\eta=\eta_s^N}}{Y_s^{N,n,1}}\\
&= -\frac{a_N}{N}\sum_{x\in\mathbb{T}^N}N^2\left(\mathbb{E}\eta_s^N(x+1)+\mathbb{E}\eta_s^N(x-1)-2\mathbb{E}\eta_s^N(x)\right)e_n(x/N)\\
&= -\frac{a_N}{N}\sum_{x\in\mathbb{T}^N}\mathbb{E}\eta_s^N(x)(\Delta^N e_n(x/N)),
\end{aligned}$$

where $\Delta^N e_n(u)=N^2\left(e_n(u+\frac{1}{N})+e_n(u-\frac{1}{N})-2e_n(u)\right)$. Using the definition of $\mathcal{L}_N$ and the fact that

$$V_1(s,\eta^{x,x+1})-V_1(s,\eta)=V_1(s,\eta)\left(e^{\frac{a_N}{N}\zeta^N}-1\right),$$

where

$$\begin{aligned}
\zeta^N &= \left[\left(\eta(x+1)-\mathbb{E}\eta_t^N(x)\right)e_n(x/N)+\left(\eta(x)-\mathbb{E}\eta_t^N(x+1)\right)e_n((x+1)/N)\right]\\
&\quad-\left[\left(\eta(x)-\mathbb{E}\eta_t^N(x)\right)e_n(x/N)+\left(\eta(x+1)-\mathbb{E}\eta_t^N(x+1)\right)e_n((x+1)/N)\right]\\
&=\left(\eta(x+1)-\eta(x)\right)\left(e_n(x/N)-e_n((x+1)/N)\right),
\end{aligned}$$

we have

$$\begin{aligned}
\frac{\mathcal{L}_N Y_s^{N,n,1}}{Y_s^{N,n,1}} &= \frac{N^2\sum_{x\in\mathbb{T}^N}\left(V_1\left(s,(\eta_s^N)^{x,x+1}\right)-V_1(s,\eta_s^N)\right)}{V_1(s,\eta_s^N)}\\
&= N^2\sum_{x\in\mathbb{T}^N}\left(\exp\left(\frac{a_N}{N}(\eta_s^N(x+1)-\eta_s^N(x))(e_n(x/N)-e_n((x+1)/N))\right)-1\right).
\end{aligned}$$

Using that $e^x-1=x+\frac{x^2}{2}+O(x^3)$, we have

$$\frac{\mathcal{L}_N Y_s^{N,n,1}}{Y_s^{N,n,1}}=\mathrm{I}+\frac{a_N^2}{N}\left(\mathrm{II}+o(1)\right),$$

where

$$\begin{aligned}
\mathrm{I} &= N^2\sum_{x\in\mathbb{T}^N}\frac{a_N}{N}\left(\eta_s^N(x+1)-\eta_s^N(x)\right)\left(e_n(x/N)-e_n((x+1)/N)\right)\\
&= \frac{a_N}{N}\sum_{x\in\mathbb{T}^N}\eta_s^N(x)(\Delta^N e_n(x/N))
\end{aligned}$$

and

$$\mathrm{II}=\frac{N^2}{2N}\sum_{x\in\mathbb{T}^N}(\eta_s^N(x+1)-\eta_s^N(x))^2\left(e_n\left(\frac{x}{N}\right)-e_n\left(\frac{x+1}{N}\right)\right)^2.$$

Then, according to the expression of $\frac{\partial_s Y_s^{N,n,1}}{Y_s^{N,n,1}}$ given above,

$$\frac{\partial_s Y_s^{N,n,1}}{Y_s^{N,n,1}} + \mathrm{I} = \frac{a_N}{N} \sum_{x\in\mathbb{T}^N} \left(\eta_s^N(x) - \mathbb{E}\eta_s^N(x)\right)\left(\Delta^N e_n(x/N)\right) = \frac{a_N^2}{N}\theta_s^N\left(\Delta^N e_n\right).$$

Consequently, we have

$$\mathcal{M}_t^{N,n,1} = \exp\left(\frac{a_N^2}{N}\left(\theta_t^N(e_n) - \theta_0^N(e_n) - \int_0^t \theta_s^N(\Delta^N e_n)ds - \varepsilon_{1,t}^{N,n} + o(1)\right)\right),$$

where

$$\varepsilon_{1,t}^{N,n} = \frac{1}{2}\int_0^t \frac{N^2}{N}\sum_{x\in\mathbb{T}^N}(\eta_s^N(x+1) - \eta_s^N(x))^2\left(e_n\left(\frac{x}{N}\right) - e_n\left(\frac{x+1}{N}\right)\right)^2 ds$$

and hence $\sup_{t\leq T}|\varepsilon_{1,t}^{N,n}| \leq 2n^2\pi^2 T$ according to Lagrange's mean value theorem. Let $Z_t^{N,n} = \theta_t^N(e_n) - \theta_0^N(e_n) - \int_0^t \theta_s^N(\partial_{uu}^2 e_n)ds$, then

$$\mathcal{M}_t^{N,n,1} = \exp\left(\frac{a_N^2}{N}\left(Z_t^{N,n} - \varepsilon_{1,t}^{N,n} + o(1)\right)\right), \tag{3.4}$$

since $|\Delta^N e_n(u) - \partial_{uu}^2 e_n(u)| = O(N^{-1})$. According to the fact that $\partial_{uu}^2 e_n = -(2n\pi)^2 e_n$ and Grownwall's inequality,

$$|\theta_t^N(e_n)| \leq \left(|\theta_0^N(e_n)| + \sup_{0\leq t\leq T} Z_t^{N,n}\right)e^{(2n\pi)^2 T}$$

for any $0 \leq t \leq T$. According to Assumption (A), it is easy to check that

$$\limsup_{M\to+\infty}\limsup_{N\to+\infty}\frac{N}{a_N^2}\log P\left(|\theta_0^N(e_n)| > M\right) = -\infty.$$

Hence, to prove Equation (3.2) we only need to show that

$$\limsup_{M\to+\infty}\limsup_{N\to+\infty}\frac{N}{a_N^2}\log P\left(\sup_{0\leq t\leq T} Z_t^{N,n} > M\right) = -\infty. \tag{3.5}$$

By Equation (3.4) and Doob's inequality, for sufficiently large N,

$$\begin{aligned} P\left(\sup_{0\leq t\leq T} Z_t^{N,n} > M\right) &\leq P\left(\sup_{0\leq t\leq T}\mathcal{M}_t^{N,n,1} \geq \exp\left(\frac{a_N^2}{N}(M - 2n^2\pi^2 T - 1)\right)\right) \\ &\leq \exp\left(-\frac{a_N^2}{N}(M - 2n^2\pi^2 T - 1)\right) \end{aligned}$$

and hence Equation (3.5) holds. Consequently, Equation (3.2) holds.

Now we check Equation (3.3). According to the definition of $Z_t^{N,n}$,

$$\left|\theta_{t+\sigma}^N(e_n) - \theta_\sigma^N(e_n)\right| \leq \sup_{0\leq s\leq\delta}\left(Z_{\sigma+s}^{N,n}(e_n) - Z_\sigma^{N,n}(e_n)\right) + (2n\pi)^2\delta\sup_{0\leq s\leq T+\delta}|\theta_s^N(e_n)|$$

for $0 \leq t \leq \delta$. Hence, by Equation (3.2), to prove Equation (3.3) we only need to show that

$$\limsup_{\delta\to 0}\limsup_{N\to+\infty}\frac{N}{a_N^2}\log\sup_{\sigma\in\mathcal{T}} P\left(\sup_{0\leq s\leq\delta}\left(Z_{\sigma+s}^{N,n}(e_n) - Z_\sigma^{N,n}(e_n)\right) > \epsilon\right) = -\infty. \tag{3.6}$$

According to an analysis similar with that leading to Equation (3.4), we have

$$\frac{\mathcal{M}_{t+\sigma}^{N,n,c}}{\mathcal{M}_\sigma^{N,n,c}} = \exp\left(\frac{a_N^2}{N}\left(c(Z_{\sigma+t}^{N,n}(e_n) - Z_\sigma^{N,n}(e_n)) - c^2\varepsilon_{2,t}^{N,n} + o(1)\right)\right) \tag{3.7}$$

for any $c>0$, where $\varepsilon_{2,t}^{N,n}=\frac{1}{2}\int_\sigma^{\sigma+t}\frac{N^2}{N}\sum_{x\in\mathbb{T}^N}(\eta_s^N(x+1)-\eta_s^N(x))^2(e_n(\frac{x}{N})-e_n(\frac{x+1}{N}))^2ds$ and hence $|\varepsilon_{2,t}^{N,n}|\leq 2n^2\pi^2\delta$ for $t\leq\delta$. Therefore, by Doob's inequality,

$$\begin{aligned}
&P\left(\sup_{0\leq s\leq\delta}\Big(Z_{\sigma+s}^{N,n}(e_n)-Z_\sigma^{N,n}(e_n)\Big)>\epsilon\right)\\
&\leq P\left(\sup_{0\leq s\leq\delta}\frac{\mathcal{M}_{\sigma+s}^{N,n,c}}{\mathcal{M}_\sigma^{N,n,c}}\geq\exp\left(\frac{a_N^2}{N}\left(\frac{1}{2}c\epsilon-2n^2\pi^2c^2\delta\right)\right)\right)\leq\exp\left(-\frac{a_N^2}{N}\left(\frac{1}{2}c\epsilon-2n^2\pi^2c^2\delta\right)\right)
\end{aligned}$$

for sufficiently large N and hence

$$\limsup_{\delta\to0}\limsup_{N\to+\infty}\frac{N}{a_N^2}\log\sup_{\sigma\in\mathcal{T}}P\left(\sup_{0\leq s\leq\delta}\Big(Z_{\sigma+s}^{N,n}(e_n)-Z_\sigma^{N,n}(e_n)\Big)>\epsilon\right)\leq-\frac{1}{2}c\epsilon.$$

Since c is arbitrary, let $c\to+\infty$ and then Equation (3.6) holds. Consequently, Equation (3.3) holds and the proof is complete. □

3.2 Proof of Lemma 3.2

In this subsection, we prove Lemma 3.2. We first recall the upper bound of the large deviation principle given in [4]. We denote by $\mathbb{M}$ the set of measures ν on $\mathbb{T}$ such that $\nu(\mathbb{T})\leq 1$. Let $\mathbb{M}$ be endowed with the weak topology. For any $\nu\in\mathbb{M}$, let $J_{ini}(\nu)$ be defined as

$$J_{ini}(\nu)=\sup_{f_1,f_2\in C(\mathbb{T})}\left\{\nu(f_1-f_2)+\int_{\mathbb{T}}f_2(u)du-\int_{\mathbb{T}}\log\left(\phi(u)e^{f_1(u)}+(1-\phi(u))e^{f_2(u)}\right)du\right\},$$

where we denote $\int_{\mathbb{T}}f(u)\nu(du)$ by $\nu(f)$ for any $\nu\in\mathbb{M},f\in C(\mathbb{T})$. Note that the above definition of J_{ini} is equivalent with that given in Equation (4.3) of [4] in the sense that we identify a measurable function $h:\mathbb{T}\to[0,1]$ with $\nu\in\mathbb{M}$ such that $\nu(A)=\int_A h(u)du$ for any measurable $A\subseteq\mathbb{T}$. Let $\mathcal{D}([0,T],\mathbb{M})$ be the set of càdlàg functions from $[0,T]$ to $\mathbb{M}$ endowed with the Skorokhod topology. For any $W\in\mathcal{D}([0,T],\mathbb{M})$, let $J_{dyn}(W)$ be defined as

$$\begin{aligned}
J_{dyn}(W)=\sup_{F\in C^{1,2}([0,T]\times\mathbb{T})}\Bigg\{&W_T(F_T)-W_0(F_0)-\int_0^T W_s((\partial_s+\partial_{uu}^2)F_s)ds\\
&-\int_0^T\int_{\mathbb{T}}\frac{dW_s}{du}(u)(1-\frac{dW_s}{du}(u))(\partial_uF_s(u))^2dsdu\Bigg\}
\end{aligned}$$

if W_s is absolutely continuous with respect to Lebesgue measure for all $0\leq s\leq T$ and $J_{dyn}(W)=+\infty$ otherwise. The following proposition is given in [4].

Proposition 3.3 (Kipnis, Olla and Varadhan, [4]). *Let μ_t^N be defined as in Section 1 and $\mu^N=\{\mu_t^N\}_{0\leq t\leq T}$, then*

$$\limsup_{N\to+\infty}\frac{1}{N}\log P\left(\mu^N\in C\right)\leq-\inf_{W\in C}(J_{dyn}(W)+J_{ini}(W_0))$$

for any closed set $C\subseteq\mathcal{D}([0,T],\mathbb{M})$.

Now we introduce some notations and definitions for later use. For any $t\geq0$, let $\mu_t(du)=\rho_t(u)du$, where $\{\rho_t\}_{t\geq0}$ is the unique weak solution to Equation (1.2) defined as in Section 1. We further define

$$\mu=\{\mu_t\}_{0\leq t\leq T}.$$

For any $f \in C(\mathbb{T})$ and $u \in \mathbb{T}$, let $\tau_u f$ be the element in $C(\mathbb{T})$ such that $\tau_u f(v) = f(v-u)$ for any $v \in \mathbb{T}$. For any $\epsilon > 0$ and $f \in C(\mathbb{T})$, we define

$$\mathcal{C}_{\epsilon,f} = \left\{ W \in \mathcal{D}([0,T],\mathbb{M}) : \sup_{\substack{u\in\mathbb{T},\\ 0\leq s\leq T}} |W_s(\tau_u f) - \mu_s(\tau_u f)| \geq \epsilon \right\}.$$

The following two lemmas are crucial for the proof of Lemma 3.2.

Lemma 3.4. *For any closed set $C \subseteq \mathcal{D}([0,T],\mathbb{M})$, if $C \not\ni \mu$, then*

$$\limsup_{N\to+\infty} \frac{N}{a_N^2} \log P(\mu^N \in C) = -\infty.$$

Lemma 3.5. *For any $\epsilon > 0$ and $f \in C(\mathbb{T})$, $\mathcal{C}_{\epsilon,f}$ is a closed subset of $\mathcal{D}([0,T],\mathbb{M})$.*

We prove Lemmas 3.4 and 3.5 at the end of this subsection. Now we utilize Lemmas 3.4 and 3.5 to prove Lemma 3.2.

Proof of Lemma 3.2. The weak solution $\{\rho_t\}_{t\geq 0}$ to Equation (1.2) has a $C([0,+\infty)\times\mathbb{T})$-valued version

$$\rho_t(u) = \mathbb{E}\phi(u+\sqrt{2}B_t),$$

where $\{B_t\}_{t\geq 0}$ is the standard Brownian motion starting at 0. Using that G and $\{\rho_t\}_{0\leq t\leq T}$ are uniformly continuous, for any $\epsilon > 0$, there exists $\delta_1 = \delta_1(\epsilon)$ such that

$$\left| \frac{1}{2\delta N} \sum_{-\delta N\leq j\leq \delta N} G_s\left(\frac{x+j}{N}\right) - G_s\left(\frac{x}{N}\right) \right| \leq \epsilon$$

and

$$\left| \frac{1}{2\delta} \int_{-\delta}^{\delta} \rho_s(u+v)dv - \rho_s(u) \right| \leq \epsilon$$

for any $\delta \leq \delta_1, x \in \mathbb{T}^N, 0 \leq s \leq T, u \in \mathbb{T}$ and sufficiently large N. Then, according to the fact that

$$\begin{aligned}
&\frac{1}{N}\sum_{x\in\mathbb{T}^N} \eta_s^N(x)\eta_s^N(x+1)\left(\frac{1}{2\delta N}\sum_{-\delta N\leq j\leq\delta N} G_s\left(\frac{x+j}{N}\right)\right)\\
&= \frac{1}{N}\sum_{x\in\mathbb{T}^N}\left(\frac{1}{2\delta N}\sum_{-\delta N\leq j\leq\delta N}\eta_s^N(x+j)\eta_s^N(x+j+1)\right) G_s\left(\frac{x}{N}\right),
\end{aligned}$$

to prove Lemma 3.2 we only need to show that

$$\begin{aligned}
\limsup_{\delta\to 0}\limsup_{N\to+\infty}\frac{N}{a_N^2}\log P\Bigg(\Bigg|&\int_0^T \frac{1}{N}\sum_{x\in\mathbb{T}^N} V_s^N(\eta^N,x,\delta)G_s\left(\frac{x}{N}\right)ds \\
&-\int_0^T\int_{\mathbb{T}}\left(\frac{1}{2\delta}\int_{-\delta}^{\delta}\rho_s(u+v)dv\right)^2 G_s(u)duds\Bigg| > \epsilon\Bigg) = -\infty \qquad (3.8)
\end{aligned}$$

for any $\epsilon > 0$, where

$$V_s^N(\eta^N,x,\delta) = \frac{1}{2\delta N}\sum_{-\delta N\leq j\leq\delta N}\eta_s^N(x+j)\eta_s^N(x+j+1).$$

According to Theorem 2.1 of [4],

$$\limsup_{\delta\to 0}\limsup_{N\to+\infty}\frac{1}{N}\log P\Bigg(\Bigg|\int_0^T \frac{1}{N}\sum_{x\in\mathbb{T}^N} V_s^N(\eta^N,x,\delta)G_s\left(\frac{x}{N}\right)ds - \int_0^T \frac{1}{N}\sum_{x\in\mathbb{T}^N}\left(U_s^N(\eta^N,x,\delta)\right)^2 G_s\left(\frac{x}{N}\right)ds\Bigg| > \epsilon\Bigg) = -\infty,$$

where

$$U_s^N(\eta^N,x,\delta) = \frac{1}{2\delta N}\sum_{-\delta N\le j\le \delta N}\eta_s^N(x+j) = \mu_s^N(\frac{1}{2\delta}\tau_{\frac{x}{N}}1_{[-\delta,\delta]}).$$

Hence, to prove Equation (3.8) we only need to show that

$$\limsup_{N\to+\infty}\frac{N}{a_N^2}\log P\Bigg(\Bigg|\int_0^T \frac{1}{N}\sum_{x\in\mathbb{T}^N}\left(U_s^N(\eta^N,x,\delta)\right)^2 G_s\left(\frac{x}{N}\right)ds - \int_0^T\int_{\mathbb{T}}\left(\frac{1}{2\delta}\int_{-\delta}^{\delta}\rho_s(u+v)dv\right)^2 G_s(u)duds\Bigg| > \epsilon\Bigg) = -\infty \tag{3.9}$$

for any $\epsilon,\delta>0$. Note that

$$\frac{1}{2\delta}\int_{-\delta}^{\delta}\rho_s(u+v)dv = \mu_s(\frac{1}{2\delta}\tau_u 1_{[-\delta,\delta]}).$$

Since

$$\lim_{N\to+\infty}\frac{1}{N}\sum_{x\in\mathbb{T}^N}\int_0^T\left(\mu_s\left(\frac{1}{2\delta}\tau_{x/N}1_{[-\delta,\delta]}\right)\right)^2 G_s\left(\frac{x}{N}\right)ds = \int_0^T\int_{\mathbb{T}}\left(\mu_s\left(\frac{1}{2\delta}\tau_u 1_{[-\delta,\delta]}\right)\right)^2 G_s(u)dsdu,$$

to prove Equation (3.9) we only need to show that

$$\limsup_{N\to+\infty}\frac{N}{a_N^2}\log P\Bigg(\Bigg|\int_0^T\frac{1}{N}\sum_{x\in\mathbb{T}^N}\mathcal{R}_s^N(\eta^N,x,\delta)G_s\left(\frac{x}{N}\right)ds\Bigg| > \epsilon\Bigg) = -\infty \tag{3.10}$$

for any $\epsilon,\delta>0$, where

$$\mathcal{R}_s^N(\eta^N,x,\delta) = \left|\left(\mu_s^N\left(\frac{1}{2\delta}\tau_{x/N}1_{[-\delta,\delta]}\right)\right)^2 - \left(\mu_s\left(\frac{1}{2\delta}\tau_{x/N}1_{[-\delta,\delta]}\right)\right)^2\right|.$$

For sufficiently large integer $m\ge 1$, let $g_m\in C(\mathbb{T})$ be defined as

$$g_m(u) = \begin{cases} \frac{1}{2\delta} & \text{if } |u|\le\delta, \\ \frac{m}{2\delta}(u+\delta+\frac{1}{m}) & \text{if } -\delta-\frac{1}{m}\le u\le -\delta, \\ -\frac{m}{2\delta}(u-\delta-\frac{1}{m}) & \text{if } \delta\le u\le \delta+\frac{1}{m}, \\ 0 & \text{else,} \end{cases}$$

then

$$\left|\mu_s^N\left(\frac{1}{2\delta}\tau_{x/N}1_{[-\delta,\delta]}\right) - \mu_s^N(\tau_{x/N}g_m)\right| \le \frac{1}{\delta m} \text{ and } \left|\mu_s\left(\frac{1}{2\delta}\tau_{x/N}1_{[-\delta,\delta]}\right) - \mu_s(\tau_{x/N}g_m)\right| \le \frac{1}{\delta m}.$$

Hence, to prove Equation (3.10) we only need to show that

$$\limsup_{N\to+\infty}\frac{N}{a_N^2}\log P\left(\left|\int_0^T\frac{1}{N}\sum_{x\in\mathbb{T}^N}\hat{\mathcal{R}}_s^{N,m}(x)G_s\left(\frac{x}{N}\right)ds\right|>\epsilon\right)=-\infty \tag{3.11}$$

for any $\epsilon>0$ and $m\geq 1$, where

$$\hat{\mathcal{R}}_s^{N,m}(x)=\left|\left(\mu_s^N(\tau_{x/N}g_m)\right)^2-\left(\mu_s(\tau_{x/N}g_m)\right)^2\right|.$$

Since $\mu_s^N(\mathbb{T}),\mu_s(\mathbb{T})\leq 1$, to prove Equation (3.11) we only need to show that

$$\limsup_{N\to+\infty}\frac{N}{a_N^2}\log P\left(\mu^N\in\mathcal{C}_{\epsilon,g_m}\right)=-\infty \tag{3.12}$$

for any $\epsilon>0$ and $m\geq 1$. Since $\mu\notin\mathcal{C}_{\epsilon,g_m}$, Equation (3.12) follows from Lemmas 3.4 and 3.5. Consequently, the proof is complete. □

At last, we prove Lemmas 3.4 and 3.5.

Proof of Lemma 3.4. Since $\lim_{N\to+\infty}\frac{N}{a_N}=+\infty$, to prove Lemma 3.4, we only need to show that

$$\limsup_{N\to+\infty}\frac{1}{N}\log P(\mu^N\in C)<0. \tag{3.13}$$

According to Proposition 3.3, to prove Equation (3.13), we only need to show that

$$\inf_{W\in C}\left(J_{ini}(W_0)+J_{dyn}(W)\right)>0. \tag{3.14}$$

It is shown in [4] that $J_{ini}(\cdot_0)+J_{dyn}(\cdot)$ is a good rate function. Hence, to prove Equation (3.14), we only need to show that $J_{ini}(W_0)=J_{dyn}(W)=0$ implies that $W=\mu$. For W making $J_{ini}(W_0)=J_{dyn}(W)=0$, we define

$$\begin{aligned} l_1(\epsilon,F)&=W_T(\epsilon F_T)-W_0(\epsilon F_0)-\int_0^T W_s((\partial_s+\partial_{uu}^2)(\epsilon F_s))ds\\ &\quad-\int_0^T\int_{\mathbb{T}}\frac{dW_s}{du}(u)\left(1-\frac{dW_s}{du}(u)\right)(\partial_u(\epsilon F_s)(u))^2dsdu \end{aligned}$$

for any $\epsilon\in\mathbb{R}$ and $F\in C^{1,2}([0,T]\times\mathbb{T})$ and

$$\begin{aligned} &l_2(\epsilon_1,\epsilon_2,f_1,f_2)\\ &=W_0(\epsilon_1 f_1-\epsilon_2 f_2)+\int_{\mathbb{T}}\epsilon_2 f_2(u)du-\int_{\mathbb{T}}\log\left(\phi(u)e^{\epsilon_1 f_1(u)}+(1-\phi(u))e^{\epsilon_2 f_2(u)}\right)du \end{aligned}$$

for any $\epsilon_1,\epsilon_2\in\mathbb{R}$ and $f_1,f_2\in C(\mathbb{T})$. Note that $l_1(0,F)=l_2(0,0,f_1,f_2)=0$ for any $F\in C^{1,2}([0,T]\times\mathbb{T})$ and $f_1,f_2\in C(\mathbb{T})$ by direct calculation. For any $f_1,f_2\in C(\mathbb{T})$, since

$$\sup_{\epsilon_1,\epsilon_2}l_2(\epsilon_1,\epsilon_2,f_1,f_2)\leq J_{ini}(W_0)=0=l_2(0,0,f_1,f_2),$$

we have $\sup_{\epsilon_1,\epsilon_2}l_2(\epsilon_1,\epsilon_2,f_1,f_2)=l_2(0,0,f_1,f_2)$ and hence

$$\partial_{\epsilon_1}l_2(0,0,f_1,f_2)=\partial_{\epsilon_2}l_2(0,0,f_1,f_2)=0.$$

Therefore, $W_0(f_1)=\int_{\mathbb{T}}\phi(u)f_1(u)du$ for any $f_1\in C(\mathbb{T})$. Consequently, $W_0(du)=\phi(u)du=\mu_0(du)$. Similarly, $J_{dyn}(W)=0$ implies that $\sup_\epsilon l_1(\epsilon,F)=l_1(0,F)$ and hence

$$\partial_\epsilon l_1(0,F)=0$$

for any $F \in C^{1,2}([0,T] \times \mathbb{T})$. We choose F with the form $F(s,u) = h(s)f(u)$ for some $h \in C^1([0,T])$ and $f \in C^\infty(\mathbb{T})$, then we have

$$h(T)W_T(f) - h(0)W_0(f) - \int_0^T \partial_s h(s) W_s(f) ds = \int_0^T h(s) W_s(\partial_{uu}^2 f) ds.$$

Since h is arbitrary, $\{W_t(f)\}_{0 \leq t \leq T}$ is absolutely continuous and

$$\partial_t W_t(f) = W_t(\partial_{uu}^2 f) \tag{3.15}$$

for any $f \in C(\mathbb{T})$. Let $e_n(u) = \cos(2n\pi u)$ for $n \geq 0$ and $e_{-n}(u) = \sin(2n\pi u)$ for $n \geq 1$ defined as in Subsection 3.1, then Equation (3.15) and the fact $W_0 = \mu_0$ implies that

$$W_t(e_m) = \mu_0(e_m) e^{-(2m\pi)^2 t}$$

for any integer m. Since the span of $\{e_m\}_{-\infty < m < +\infty}$ is dense in $C(\mathbb{T})$, the solution to Equation (3.15) with initial condition $W_0 = \mu_0$ is unique. Since μ is also a solution to Equation (3.15), we have $W = \mu$ and the proof is complete. □

Proof of Lemma 3.5. Assuming that $W^n \in \mathcal{C}_{\epsilon,f}$ for $n \geq 1$ and $W^n \to W$ in $\mathcal{D}([0,T], \mathbb{M})$, then we only need to show that $W \in \mathcal{C}_{\epsilon,f}$. Since $W^n \in \mathcal{C}_{\epsilon,f}$ for $n \geq 1$, there exist a sequence $\{v^n\}_{n \geq 1}$ in $\mathbb{T}$ such that

$$\liminf_{n \to +\infty} \sup_{0 \leq s \leq T} |W_s^n(\tau_{v^n} f) - \mu_s(\tau_{v^n} f)| \geq \epsilon.$$

Since $\mathbb{T}$ is compact, there exists $v \in \mathbb{T}$ such that v is the limit of a subsequence of $\{v^n\}_{n \geq 1}$ in $\mathbb{T}$. For simplicity, we still write this subsequence as $\{v^n\}_{n \geq 1}$. Since $W^n \to W$ in $\mathcal{D}([0,T], \mathbb{M})$ as $n \to +\infty$, there exist a sequence of increasing continuous functions $\{\varphi^n\}_{n \geq 1}$ from $[0,T]$ to $[0,T]$ such that $\varphi^n(0) = 0, \varphi^n(T) = T$ for all n, $\varphi^n(s) \to s$ uniformly for $s \in [0,T]$ and $W^n_{\varphi^n(s)} \to W_s$ uniformly for $s \in [0,T]$. According to the triangle inequality, for any $n \geq 1$,

$$\begin{aligned} \sup_{0 \leq s \leq T} |W_s^n(\tau_{v^n} f) - \mu_s(\tau_{v^n} f)| &= \sup_{0 \leq s \leq T} \left| W^n_{\varphi^n(s)}(\tau_{v^n} f) - \mu_{\varphi^n(s)}(\tau_{v^n} f) \right| \\ &\leq \sup_{0 \leq s \leq T} \left| W^n_{\varphi^n(s)}(\tau_v f) - W^n_{\varphi^n(s)}(\tau_{v^n} f) \right| \\ &\quad + \sup_{0 \leq s \leq T} \left| W_s(\tau_v f) - W^n_{\varphi^n(s)}(\tau_v f) \right| \\ &\quad + \sup_{0 \leq s \leq T} |W_s(\tau_v f) - \mu_s(\tau_v f)| \\ &\quad + \sup_{0 \leq s \leq T} \left| \mu_{\varphi^n(s)}(\tau_{v^n} f) - \mu_s(\tau_v f) \right|. \end{aligned} \tag{3.16}$$

According to the definition of μ, μ_t is continuous in t and hence $\mu_{\varphi^n(s)}(\tau_v f) \to \mu_s(\tau_v f)$ uniformly for $s \in [0,T]$. According to the uniform continuity of f, $\tau_{v^n} f(u) \to \tau_v f(u)$ uniformly for $u \in \mathbb{T}$. As a result, according to facts that $\mu_{\varphi^n(s)} \in \mathbb{M}$, $W^n_{\varphi^n(s)} \in \mathbb{M}$ and $W^n \to W$ in $\mathcal{D}([0,T], \mathbb{M})$, we have $\sup_{0 \leq s \leq T} \left| W_s(\tau_v f) - W^n_{\varphi^n(s)}(\tau_v f) \right| \to 0$,

$$\begin{aligned} &\sup_{0 \leq s \leq T} \left| \mu_{\varphi^n(s)}(\tau_{v^n} f) - \mu_s(\tau_v f) \right| \\ &\leq \sup_{0 \leq s \leq T} \left| \mu_{\varphi^n(s)}(\tau_{v^n} f) - \mu_{\varphi^n(s)}(\tau_v f) \right| + \sup_{0 \leq s \leq T} \left| \mu_{\varphi^n(s)}(\tau_v f) - \mu_s(\tau_v f) \right| \\ &\leq \sup_{u \in \mathbb{T}} |\tau_{v^n} f(u) - \tau_v f(u)| + \sup_{0 \leq s \leq T} \left| \mu_{\varphi^n(s)}(\tau_v f) - \mu_s(\tau_v f) \right| \to 0 \end{aligned}$$

and

$$\sup_{0\le s\le T}\left|W^n_{\varphi^n(s)}(\tau_v f)-W^n_{\varphi^n(s)}(\tau_{v^n} f)\right|\le \sup_{u\in\mathbb{T}}|\tau_{v^n}f(u)-\tau_v f(u)|\to 0$$

as $n\to+\infty$. As a result, let $n\to+\infty$ in Equation (3.16), we have

$$\sup_{0\le s\le T}|W_s(\tau_v f)-\mu_s(\tau_v f)|\ge \liminf_{n\to+\infty}\sup_{0\le s\le T}|W^n_s(\tau_{v^n} f)-\mu_s(\tau_{v^n} f)|\ge\epsilon$$

and hence $W\in\mathcal{C}_{\epsilon,f}$, which completes the proof. □

4 Proof of Theorem 2.1

In this section, we prove our main result Theorem 2.1. With Lemmas 3.1 and 3.2, the strategy introduced in [1] applies to cases discussed in this paper. So we only give outlines of the proof of Equations (2.3) and (2.4) to avoid repeating many similar details with those in [1]. For later use, we define

$$Y_t^N(F)=\exp\left(\frac{a_N^2}{N}\theta_t^N(F_t)\right)$$

for any $0\le t\le T$ and $F\in C^{1,+\infty}([0,T]\times\mathbb{T})$. Furthermore, we define

$$\mathcal{M}_t^N(F)=\frac{Y_t^N(F)}{Y_0^N(F)}\exp\left(-\int_0^t\frac{(\partial_s+\mathcal{L}_N)Y_s^N(F)}{Y_s^N(F)}ds\right)$$

for $0\le t\le T$, then $\{\mathcal{M}_t^N(F)\}_{0\le t\le T}$ is a martingale according to Feynman-Kac formula. Note that in the above definition of $\mathcal{M}_t^N(F)$, we denote $\partial_t V_F^N(t,\eta)\Big|_{\eta=\eta_t^N}$ by $\partial_t Y_t^N(F)$, where

$$V_F^N(t,\eta)=\exp\left\{\frac{a_N}{N}\sum_{x\in\mathbb{T}^N}\left(\eta(x)-\mathbb{E}\eta_t^N(x)\right)F_t(x/N)\right\}.$$

For any $g\in C(\mathbb{T})$, let P_g^N be the probability measure of our simple symmetric exclusion process with initial condition $\eta_0^N=\{\eta_0^N(x)\}_{x\in\mathbb{T}^N}$ where $\{\eta_0^N(x)\}_{x\in\mathbb{T}^N}$ are independent and

$$P\left(\eta_0^N(x)=1\right)=\phi(x/N)+\frac{a_N}{N}g(x/N)$$

for all $x\in\mathbb{T}^N$. Then, for any $F\in C^{1,\infty}([0,T]\times\mathbb{T})$, we define $\hat{P}_g^{N,F}$ as the probability measure such that

$$\frac{d\hat{P}_g^{N,F}}{dP_g^N}=\mathcal{M}_T^N(F).$$

Now we prove Equation (2.3).

Proof of Equation (2.3). According to the definition of $\mathcal{L}_N$ and Chapman-Kolmogorov equation, we have

$$\begin{aligned}\frac{\partial_s Y_s^N(F)}{Y_s^N(F)}=&\frac{a_N}{N}\sum_{x\in\mathbb{T}^N}\left(\eta_s^N(x)-\mathbb{E}\eta_s^N(x)\right)\partial_s F_s(x/N)\\&-N^2\frac{a_N}{N}\sum_{x\in\mathbb{T}^N}\left(\mathbb{E}\eta_s^N(x+1)+\mathbb{E}\eta_s^N(x-1)-2\mathbb{E}\eta_s^N(x)\right)F_s(x/N)\\=&\frac{a_N^2}{N}\theta_s^N(\partial_s F_s)-\frac{a_N^2}{N}\frac{1}{a_N}\sum_{x\in\mathbb{T}^N}\mathbb{E}\eta_s^N(x)(\Delta^N F_s(x/N))\end{aligned}$$

and

$$\frac{\mathcal{L}_N Y_s^N(F)}{Y_s^N(F)}$$
$$= N^2 \sum_{x\in\mathbb{T}^N} \left(\exp\left(\frac{a_N}{N}\left(\eta_t^N(x)-\eta_t^N(x+1)\right)\left(F_t^N((x+1)/N)-F_t^N(x/N)\right)\right)-1\right).$$

Then, according to the fact that $e^x-1=x+\frac{x^2}{2}+O(x^3)$, we have

$$\mathcal{M}_T^N(F)=\exp\left(\frac{a_N^2}{N}\left(Z_T^N(F)-\varepsilon_{3,T}^N(F)+o(1)\right)\right), \tag{4.1}$$

where

$$Z_T^N(F)=\theta_T^N(F_T)-\theta_0(F_0)-\int_0^T \theta_s^N((\partial_s+\partial_{uu}^2)F_s)ds$$

and

$$\begin{aligned}\varepsilon_{3,T}^N(F) &= \frac{1}{2}\int_0^T \frac{1}{N}\sum_{x\in\mathbb{T}^N}(\partial_u F_s(x/N))^2\left(\eta_s^N(x)+\eta_s^N(x+1)-2\eta_s^N(x)\eta_s^N(x+1)\right)ds\\ &= \int_0^T \mu_s^N((\partial_u F_s)^2)-\frac{1}{N}\sum_{x\in\mathbb{T}^N}(\partial_u F_s(x/N))^2\eta_s^N(x)\eta_s^N(x+1)ds+o(1).\end{aligned}$$

According to Lemma 3.4, we have

$$\limsup_{N\to+\infty}\frac{1}{N}\log P\left(\left|\int_0^T \mu_s^N((\partial_u F_s)^2)ds-\int_0^T \mu_s((\partial_u F_s)^2)ds\right|>\epsilon\right)<0$$

and hence

$$\limsup_{N\to+\infty}\frac{N}{a_N^2}\log P\left(\left|\int_0^T \mu_s^N((\partial_u F_s)^2)ds-\int_0^T \mu_s((\partial_u F_s)^2)ds\right|>\epsilon\right)=-\infty$$

for any $\epsilon>0$. According to Lemma 3.2, we have

$$\begin{aligned}\limsup_{N\to+\infty}\frac{N}{a_N^2}\log P\Bigg(\Bigg|\int_0^T \frac{1}{N}\sum_{x\in\mathbb{T}^N}\eta_s^N(x)\eta_s^N(x+1)(\partial_u F_s(x/N))^2 ds\\ -\int_0^T\int_{\mathbb{T}}\rho_s^2(u)(\partial_u F_s(u))^2 duds\Bigg|>\epsilon\Bigg)=-\infty.\end{aligned}$$

In conclusion,

$$\begin{aligned}\varepsilon_{3,T}^N(F) &= \int_0^T \mu_s((\partial_u F_s)^2)ds-\int_0^T\int_{\mathbb{T}}\rho_s^2(u)(\partial_u F_s(u))^2 duds+\varepsilon_{4,T}^N(F)\\ &= \int_0^T \rho_s(u)(1-\rho_s(u))(\partial_u F_s(u))^2 ds+\varepsilon_{4,T}^N(F),\end{aligned}$$

where

$$\limsup_{N\to+\infty}\frac{N}{a_N^2}\log P\left(|\varepsilon_{4,T}^N(F)|>\epsilon\right)=-\infty \tag{4.2}$$

for any $\epsilon>0$. As a result, by Equation (4.1), we have

$$\mathcal{M}_T^N(F)=\exp\left(\frac{a_N^2}{N}\left(\mathcal{I}(\theta^N,F)+\varepsilon_{4,T}^N(F)+o(1)\right)\right), \tag{4.3}$$

where

$$\mathcal{I}(W,F) = W_T(F_T) - W_0(F_0) - \int_0^T W_s((\partial_s + \partial_{uu}^2)F_s)ds \\ - \int_0^T \int_{\mathbb{T}} \rho_s(u)(1-\rho_s(u))(\partial_u F_s(u))^2 dsdu$$

for any $W \in \mathcal{D}([0,T],\mathcal{S})$. According to Assumption (A), it is easy to check that

$$\mathbb{E}\exp\left(\frac{a_N^2}{N}\theta_0^N(f)\right) = \exp\left(\frac{a_N^2}{2N}\left(\int_{\mathbb{T}} f^2(u)\phi(u)(1-\phi(u))du + o(1)\right)\right) \tag{4.4}$$

for any $f \in C^\infty(\mathbb{T})$. By utilizing Markov's inequality and the minimax theorem given in [8], Equation (2.3) holds for all compact $C \subseteq \mathcal{D}([0,T],\mathcal{S})$ according to Equations (4.3) and (4.4). To show that Equation (2.3) holds for all closed C, we only need to show that $\{\theta^N\}_{N\geq 1}$ are exponential tight and hence the proof is complete according to Lemma 3.1. □

To prove Equation (2.4), we need following two lemmas.

Lemma 4.1. *If W makes $I_{ini}(W_0) + I_{dyn}(W) < +\infty$, then there exist g, F such that $W_t(du) = \rho^{F,g}(t,u)du$ for all $0 \leq t \leq T$, where*

$$\begin{cases} \frac{d}{dt}\rho^{F,g}(t,u) = \partial_{uu}^2\rho^{F,g}(t,u) - 2\frac{\partial}{\partial u}\left(\rho_t(u)(1-\rho_t(u))\partial_u F_t(u)\right), \\ \rho_0^{F,g} = g. \end{cases}$$

Furthermore, $I_{ini}(W_0) = \frac{1}{2}\int_{\mathbb{T}} \frac{g^2(u)}{\phi(u)(1-\phi(u))}du$ and

$$I_{dyn}(W) = \mathcal{I}(W,F) = \int_0^T \int_{\mathbb{T}} \rho_s(u)(1-\rho_s(u))(\partial_u F_s(u))^2 dsdu.$$

Lemma 4.2. *As $N \to +\infty$, θ^N converges in $\hat{P}_g^{N,F}$-probability to $\{\rho^{F,g}(t,u)(du)\}_{0\leq t\leq T}$.*

Lemmas 4.1 and 4.2 are analogues of Lemma 5.1 and Theorem 4.1 of [1] respectively. With Lemmas 3.1 and 3.2, analyses given in proofs of Lemma 5.1 and Theorem 4.1 of [1] apply to Lemmas 4.1 and 4.2 respectively. Hence we omit proofs of Lemmas 4.1 and 4.2 here. At last, we prove Equation (2.4).

Proof of Equation (2.4). We only deal with the case where $\inf_{W\in O}(I_{ini}(W_0) + I_{dyn}(W)) < +\infty$. Otherwise, Equation (2.4) is trivial. For any $\epsilon > 0$, according to Lemma 4.1, there exists $W^\epsilon \in O$ such that

$$I_{ini}(W_0^\epsilon) + I_{dyn}(W^\epsilon) < \inf_{W\in O}(I_{ini}(W_0) + I_{dyn}(W)) + \epsilon$$

and $W_t^\epsilon(du) = \rho^{F^\epsilon,g^\epsilon}(t,u)du$ for some F^ϵ, g^ϵ and $0 \leq t \leq T$. According to Lemma 4.2, θ^N converges in $\hat{P}_{g^\epsilon}^{N,F^\epsilon}$-probability to $\{\rho^{F^\epsilon,g^\epsilon}(t,u)du\}_{0\leq t\leq T} = W^\epsilon$. Then, we have

$$\lim_{N\to+\infty} \hat{P}_{g^\epsilon}^{N,F^\epsilon}\left(\theta^N \in O\right) = 1 \tag{4.5}$$

and

$$\lim_{N\to+\infty} \hat{P}_{g^\epsilon}^{N,F^\epsilon}\left(|\mathcal{I}(\theta^N,F^\epsilon) - \mathcal{I}(W^\epsilon,F^\epsilon)| \leq \epsilon\right) = 1. \tag{4.6}$$

According to Equation (4.2), Assumption (A) and Cauchy-Schwarz inequality, it is easy to check that

$$\lim_{N\to+\infty} \hat{P}_{g^\epsilon}^{N,F^\epsilon}\left(|\varepsilon_{4,T}^N(F^\epsilon)| \leq \epsilon\right) = 1.$$

Then, by Equations (4.3), (4.5), (4.6) and Lemma 4.1, we have

$$\lim_{N\to+\infty} \hat{P}_{g^\epsilon}^{N,F^\epsilon}\left(\theta^N \in O, \mathcal{M}_T^N(F^\epsilon) \le e^{\frac{a_N^2}{N}(I_{dyn}(W^\epsilon)+3\epsilon)}\right) = 1. \tag{4.7}$$

Under Assumption (A), it is easy to check that $\frac{N}{a_N^2}\log\frac{dP}{dP_{g^\epsilon}^N}$ converges in $\hat{P}_{g^\epsilon}^{N,F^\epsilon}$-probability to

$$-\frac{1}{2}\int_{\mathbb{T}} \frac{(g^\epsilon(u))^2}{\phi(u)(1-\phi(u))}du = -I_{ini}(W_0^\epsilon)$$

as $N \to +\infty$. Hence, by Equation (4.7),

$$\begin{aligned}\lim_{N\to+\infty} \hat{P}_{g^\epsilon}^{N,F^\epsilon}\Big(\theta^N \in O, &\mathcal{M}_T^N(F^\epsilon) \le e^{\frac{a_N^2}{N}(I_{dyn}(W^\epsilon)+3\epsilon)}, \\ &\frac{dP}{dP_{g^\epsilon}^N} \ge \exp\left(-\frac{a_N^2}{N}(I_{ini}(W_0^\epsilon)+\epsilon)\right)\Big) = 1.\end{aligned} \tag{4.8}$$

According to the definition of $\hat{P}_{g^\epsilon}^{N,F^\epsilon}$, by Equation (4.8),

$$\begin{aligned}P\left(\theta^N \in O\right) &= \mathbb{E}_{\hat{P}_{g^\epsilon}^{N,F^\epsilon}}\left(\frac{dP}{dP_{g^\epsilon}^N}\left(\mathcal{M}_T^N(F^\epsilon)\right)^{-1} 1_{\{\theta^N\in O\}}\right) \\ &\ge \exp\left(-\frac{a_N^2}{N}\left(I_{ini}(W_0^\epsilon) + I_{dyn}(W^\epsilon) + 4\epsilon\right)\right)(1+o(1)).\end{aligned}$$

Hence,

$$\begin{aligned}\liminf_{N\to+\infty}\frac{N}{a_N^2}\log P\left(\theta^N \in O\right) &\ge -(I_{ini}(W_0^\epsilon) + I_{dyn}(W^\epsilon)) - 4\epsilon \\ &= -\inf_{W\in O}\left(I_{ini}(W_0) + I_{dyn}(W)\right) - 5\epsilon.\end{aligned}$$

Since ϵ is arbitrary, let $\epsilon \to 0$ and the proof is complete. □

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