

EXTENDING THE VALIDITY OF FREQUENCY DOMAIN BOOTSTRAP METHODS TO GENERAL STATIONARY PROCESSES

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Existing frequency domain methods for bootstrapping time series have a limited range. Essentially, these procedures cover the case of linear time series with independent innovations, and some even require the time series to be Gaussian. In this paper we propose a new frequency domain bootstrap method—the hybrid periodogram bootstrap (HPB)—which is consistent for a much wider range of stationary, even nonlinear, processes and which can be applied to a large class of periodogram-based statistics. The HPB is designed to combine desirable features of different frequency domain techniques while overcoming their respective limitations. It is capable to imitate the weak dependence structure of the periodogram by invoking the concept of convolved subsampling in a novel way that is tailor-made for periodograms. We show consistency for the HPB procedure for a general class of stationary time series, ranging clearly beyond linear processes, and for spectral means and ratio statistics on which we mainly focus. The finite sample performance of the new bootstrap procedure is illustrated via simulations.

1. Introduction. Frequency domain bootstrap methods for time series are quite attractive because in many situations they can be successful without imitating the complete (potentially complicated) temporal dependence structure of the underlying stochastic process as is the case for time domain bootstrap methods. Frequency domain methods mainly focus on bootstrapping the periodogram which is defined for any time series X_1, \dots, X_n by

$$(1.1) \quad I_n(\lambda) = \frac{1}{2\pi n} \left| \sum_{t=1}^n X_t e^{-i\lambda t} \right|^2, \quad \lambda \in [-\pi, \pi].$$

The periodogram is an important frequency domain statistic, and many statistics of interest in time series analysis can be written as functions of the periodogram. Furthermore, it obeys some useful nonparametric properties for a wide class of stationary processes which makes frequency domain bootstrap methods appealing. In particular, periodogram ordinates rescaled by the spectral density are asymptotically standard exponential distributed, and, moreover, periodogram ordinates corresponding to different frequencies in the interval $(0, \pi)$ are asymptotically independent. This asymptotic independence essentially means that the classical i.i.d. bootstrap of drawing with replacement, as has been introduced by Efron (1979), can potentially be applied to bootstrap the periodogram, in particular the properly rescaled periodogram ordinates. Motivated by these considerations, many researchers have developed bootstrap methods in frequency domain which generate pseudo periodogram ordinates with the intent to mimic the stochastic behavior of the ordinary periodogram.

A multiplicative bootstrap approach for the periodogram has been investigated by Hurvich and Zeger (1987), Franke and Härdle (1992) and Dahlhaus and Janas (1996). The main idea is to exploit the (asymptotic) independence of the periodogram at different frequencies and

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to generate new pseudo periodogram ordinates by multiplying an estimator of the spectral density at the frequencies of interest with pseudo innovations obtained by an i.i.d. resampling of appropriately defined frequency domain residuals. Franke and Härdle (1992) proved the validity of such an approach for estimating the distribution of nonparametric spectral density estimators for linear processes of the form

$$(1.2) \quad X_t = \sum_{j=-\infty}^{\infty} a_j \varepsilon_{t-j}, \quad t \in \mathbb{Z},$$

where $(\varepsilon_t)_{t \in \mathbb{Z}}$ denotes an i.i.d. white noise process. Shao and Wu (2007) established validity of this procedure for the same statistic but for a much wider class of stochastic processes. However, beyond nonparametric spectral density estimators, the range of validity of this bootstrap approach is limited. In the following we will explain this in more detail.

It should be emphasized at this point that whenever we use the term *linear process* in this work, we refer to a process as given by (1.2) *including* the i.i.d. assumption on the innovations, as it is done in many standard references for time series; cf. Brockwell and Davis (1991) among others. This notion is particularly important in the ensuing discussion on the range of validity of established bootstrap methods because most of the classical methods are valid exclusively for linear processes in this strict sense. To put this into perspective, processes with a Wold representation similar to (1.2) but with only uncorrelated (not i.i.d.) innovations are considered nonlinear. The same goes for popular models such as, for instance, bilinear processes and autoregressive, moving average or ARMA processes driven by ARCH noise or any other non-i.i.d. noise. Moreover, all linear processes are obviously strictly stationary which implies that any weakly but not strictly stationary process is nonlinear.

Even in the case of linear processes in the strict sense, Dahlhaus and Janas (1996) showed that the multiplicative approach fails to consistently estimate the limiting distribution of very basic statistics like sample autocovariances. This failure is due to the following: For many periodogram-based statistics of interest consistency is achieved by letting the number of frequencies at which the periodogram is evaluated increase with increasing sample size. The dependence between periodogram ordinates at different frequencies vanishes asymptotically, but the rate of this decay typically just compensates the increasing number of frequencies. This leads to the fact that the dependence structure of the periodogram actually shows up in the limiting distribution of many statistics of interest. Since the bootstrap pseudo periodogram ordinates generated by the multiplicative approach are independent, this approach fails to imitate the aforementioned dependence structure of periodogram ordinates. As a consequence, validity of this frequency domain bootstrap approach can be established only for a restricted class of processes and statistics. To be precise, even in the special case of linear processes (i.e., with representation (1.2) driven by i.i.d. white noise) the approach works only under additional assumptions, such as Gaussianity of the time series or for specific statistics such as ratio statistics. For the wide range of nonlinear processes, including all processes which are not strictly stationary, the approach fails for most classes of statistics.

Notice that the aforementioned limitations of the multiplicative periodogram bootstrap are common to other frequency domain bootstrap methods which generate independent pseudo periodogram ordinates. The local periodogram bootstrap introduced by Paparoditis and Politis (1999) is such an example. Thus, for a frequency domain bootstrap procedure to be successful for a wider class of statistics and/or a wider class of stationary processes, it also has to reflect the dependence structure of the ordinary periodogram at different frequencies.

Attempts in this direction include the approach proposed by Janas and Dahlhaus (1994), the autoregressive-aided periodogram bootstrap (AAPB) by Kreiss and Paparoditis (2003) and the hybrid wild bootstrap; cf. Kreiss and Paparoditis (2012). Although the hybrid wild bootstrap extends the validity of frequency domain bootstrap to a wider class of statistics

compared to the multiplicative periodogram bootstrap, its limitation lies, as for the AAPB and the approach proposed by Janas and Dahlhaus (1994), in the fact that its applicability is also restricted to linear processes. On a side note, many popular bootstrap methods in the time domain, such as for instance the autoregressive sieve bootstrap, share the same limitations since—for the class of statistics considered here—they are only valid for *linear* processes, cf. Kreiss, Paparoditis and Politis (2011) for a detailed discussion.

The above discussion demonstrates that a frequency domain bootstrap procedure which is valid for a wide range of stationary stochastic processes and for a rich class of periodogram-based statistics is missing. This paper attempts to fill this gap. Our main contribution is the proposal of the *hybrid periodogram bootstrap* (HPB). This is a hybrid procedure which combines elements from the aforementioned multiplicative approach and from a modified *convolved subsampling* approach. The concept of convolved subsampling has recently been introduced by Tewes, Politis and Nordman (2019). It approximates the distribution of a statistic of interest by evaluating the same statistic on smaller samples of the original time series, drawing independently from these subsample statistics and calculating a scaled sum of the obtained values. Tewes, Politis and Nordman (2019) showed how this procedure is connected to the moving block bootstrap, and they were able to show that the procedure is asymptotically valid in a very general setup as long as certain assumptions on the asymptotic behaviour of the underlying time series are fulfilled.

Our modified convolved subsampling approach will be called *convolved bootstrapped periodograms of subsamples* (CBP), and it will be discussed in detail for spectral means and ratio statistics. The CBP differs from the general convolved subsampling concept since it allows the user to plug in any consistent spectral density estimator, based on the entire observed time series, instead of implicitly using the particular subsampling estimator. In the CBP procedure “frequency domain residuals” are defined which are obtained by appropriately rescaling periodograms calculated at the Fourier frequencies of subsamples of the observed time series. These residuals together with a consistent estimator of the spectral density are used to generate pseudo periodograms of subsamples which mimic correctly the weak dependence structure of the periodogram. As we will see, this modification of the standard convolved subsampling procedure leads to certain advantages in finite samples; see Remark 3.2(iv) below and Section 5.2.

In our main contribution, the HPB procedure, we will combine desirable features from both the multiplicative approach discussed earlier and the CBP procedure. The HPB is designed in the following way: It employs the multiplicative approach to mimic those features of the statistic of interest that this approach is able to approximate well. This concerns the distributional shape of spectral means and ratio statistics and the part of the variance of these statistics that is not affected by the weak dependence structure of the periodogram. At the same time the HPB makes use of the CBP exclusively to imitate the part of the variance that is due to the dependence structure of periodogram ordinates at different frequencies (which can not be mimicked by the multiplicative approach). We will argue why this hybrid approach is preferable compared to the pure CBP for spectral means and ratio statistics. These arguments will be supported by a simulation study in which the HPB outperforms the pure CBP, the standard convolved subsampling procedure and the multiplicative bootstrap approach.

The paper is organized as follows. Section 2 reviews some asymptotic results concerning spectral means and ratio statistics and clarifies the limitations in approximating the distribution of such statistics by classical frequency domain bootstrap methods which generate independent periodogram ordinates. Section 3 introduces the new CBP and HPB procedures. In Section 4 we will establish asymptotic validity of both approaches for the entire class of spectral means and ratio statistics and for a very wide class of stationary processes. Section 5 discusses the issue of selecting the bootstrap parameters in practice and presents some

simulations demonstrating the finite sample performance of the new procedures. Finally, all proofs are deferred to the [Appendix](#) of the paper and to the [Supplementary Material \(Meyer, Paparoditis and Kreiss \(2020\)\)](#).

In the following P^* will denote conditional probability given the data X_1, \dots, X_n , and E^* , and Var^* will denote the corresponding expectations and variances.

2. Spectral means and ratio statistics. We consider a weakly stationary real-valued stochastic process $(X_t)_{t \in \mathbb{Z}}$ with mean zero, absolutely summable autocovariance function γ and spectral density $f : [-\pi, \pi] \rightarrow [0, \infty)$; the detailed conditions can be found in [Assumption 1](#) below. The periodogram of a sample X_1, \dots, X_n from this process is defined according to [\(1.1\)](#). While the periodogram is well known to be an inconsistent estimator of the spectral density $f(\lambda)$, integrated periodograms form an important class of estimators which are consistent under suitable regularity conditions. For some integrable function $\varphi : [-\pi, \pi] \rightarrow \mathbb{R}$ the integrated periodogram is defined as

$$M(\varphi, I_n) = \int_{-\pi}^{\pi} \varphi(\lambda) I_n(\lambda) d\lambda,$$

which is an estimator for the so-called *spectral mean* $M(\varphi, f)$. A further interesting class of statistics is obtained by scaling a spectral mean statistic by the quantity $M(1, I_n)$. In particular, this class of statistics, also known as *ratio statistics*, is defined by

$$R(\varphi, I_n) = \frac{M(\varphi, I_n)}{M(1, I_n)}.$$

For practical calculations the integral in $M(\varphi, I_n)$ is commonly replaced by a Riemann sum. This is usually done using the Fourier frequencies based on sample size n , which are given by $\lambda_{j,n} = 2\pi j/n$, for $j \in \mathcal{G}(n)$, with

$$(2.1) \quad \mathcal{G}(n) := \{j \in \mathbb{Z} : 1 \leq |j| \leq [n/2]\}.$$

The approximation of $M(\varphi, I_n)$, and, respectively, of $R(\varphi, I_n)$ via Riemann sum is then given by

$$M_{\mathcal{G}(n)}(\varphi, I_n) = \frac{2\pi}{n} \sum_{j \in \mathcal{G}(n)} \varphi(\lambda_{j,n}) I_n(\lambda_{j,n})$$

and

$$R_{\mathcal{G}(n)}(\varphi, I_n) = \frac{\sum_{j \in \mathcal{G}(n)} \varphi(\lambda_{j,n}) I_n(\lambda_{j,n})}{\sum_{j \in \mathcal{G}(n)} I_n(\lambda_{j,n})}.$$

Various statistics commonly used in time series analysis belong to the class of spectral means or ratio statistics. We give some examples.

EXAMPLE 2.1.

(i) The autocovariance $\gamma(h)$ of (X_t) at lag $0 \leq h < n$ can be estimated by the empirical autocovariance $\hat{\gamma}(h) = n^{-1} \sum_{t=1}^{n-h} X_t X_{t+h}$ which is an integrated periodogram statistic. This is due to the fact that choosing $\varphi(\lambda) = \cos(h\lambda)$ it follows from straightforward calculations that $\hat{\gamma}(h) = M(\varphi, I_n)$ as well as $\gamma(h) = M(\varphi, f)$.

(ii) The spectral distribution function evaluated at point $x \in (0, \pi]$ is defined as $\int_0^x f(\lambda) d\lambda = M(\varphi, f)$ for $\varphi(\lambda) = \mathbb{1}_{(0,x]}(\lambda)$. The corresponding integrated periodogram estimator is given by $M(\varphi, I_n) = \int_0^x I_n(\lambda) d\lambda$.

(iii) The autocorrelation $\rho(h) = \gamma(h)/\gamma(0)$ of (X_t) at lag $0 \leq h < n$ can be estimated by the empirical autocorrelation $\hat{\rho}(h) = \hat{\gamma}(h)/\hat{\gamma}(0)$ which in view of (i) is a ratio statistic, that is, $\hat{\rho}(h) = R(\cos(h \cdot), I_n)$.

We specify the assumptions imposed on $(X_t)_{t \in \mathbb{Z}}$ and the function φ :

ASSUMPTION 1.

(i) Assumptions on $(X_t)_{t \in \mathbb{Z}}$: The stochastic process $(X_t)_{t \in \mathbb{Z}}$ is real-valued with finite eighth moments, and it is eighth-order stationary, that is, the joint cumulants of the process up to the eighth order

$$\text{cum}(X_t, X_{t+h_1}, \dots, X_{t+h_7})$$

do not depend on $t \in \mathbb{Z}$ for any $h_1, \dots, h_7 \in \mathbb{Z}$. The process (X_t) has mean zero, autocovariance function $\gamma : \mathbb{Z} \rightarrow \mathbb{R}$ fulfilling $\sum_{h \in \mathbb{Z}} |h| |\gamma(h)| < \infty$ and spectral density f satisfying $\inf_{\lambda \in [0, \pi]} f(\lambda) > 0$. Furthermore, the fourth-order joint cumulants of the process fulfil

$$\sum_{h_1, h_2, h_3 \in \mathbb{Z}} (|h_1| + |h_2| + |h_3|) |\text{cum}(X_0, X_{h_1}, X_{h_2}, X_{h_3})| < \infty,$$

and the eighth-order cumulants are absolutely summable, that is,

$$\sum_{h_1, \dots, h_7 \in \mathbb{Z}} |\text{cum}(X_0, X_{h_1}, \dots, X_{h_7})| < \infty.$$

(ii) Assumptions on φ : The function $\varphi : [-\pi, \pi] \rightarrow \mathbb{R}$ is square integrable and of bounded variation.

Notice that the summability conditions imposed on the autocovariance function γ imply boundedness and differentiability of f , as well as boundedness of the derivative f' of f . Under the conditions of Assumption 1 and some additional weak dependence conditions, it is known that $M(\varphi, I_n)$ is a consistent estimator for $M(\varphi, f)$ and that the following central limit theorem holds true:

$$(2.2) \quad L_n = \sqrt{n}(M(\varphi, I_n) - M(\varphi, f)) \xrightarrow{d} \mathcal{N}(0, \tau^2),$$

with $\tau^2 = \tau_1^2 + \tau_2$, where

$$(2.3) \quad \tau_1^2 = 2\pi \int_{-\pi}^{\pi} \varphi(\lambda)(\varphi(\lambda) + \varphi(-\lambda))f(\lambda)^2 d\lambda,$$

$$(2.4) \quad \tau_2 = 2\pi \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \varphi(\lambda_1)\varphi(\lambda_2) f_4(\lambda_1, \lambda_2, -\lambda_2) d\lambda_1 d\lambda_2,$$

and where

$$f_4(\lambda, \mu, \eta) = \frac{1}{(2\pi)^3} \sum_{h_1, h_2, h_3 \in \mathbb{Z}} \text{cum}(X_0, X_{h_1}, X_{h_2}, X_{h_3}) e^{-i(h_1\lambda + h_2\mu + h_3\eta)}$$

is the fourth-order cumulant spectral density; cf. Rosenblatt (1985), Chapter III, Corollary 2.

REMARK 2.2. (i) In the literature the function φ is sometimes assumed to be even, that is, $\varphi(\lambda) = \varphi(-\lambda)$ for all λ . In this case the first part τ_1^2 of the limiting variance takes the form $4\pi \int_{-\pi}^{\pi} \varphi(\lambda)^2 f(\lambda)^2 d\lambda = 8\pi \int_0^{\pi} \varphi(\lambda)^2 f(\lambda)^2 d\lambda$. However, we allow for general functions φ since this restriction to even functions is not necessary.

(ii) Nonparametric approaches to directly estimate the integral involving the fourth-order cumulant spectral density and which can potentially be used to estimate the variance τ^2 have

been proposed by some authors; see Taniguchi (1982) and Keenan (1987). However, simulation results reported in the Supplementary Material (Meyer, Paparoditis and Kreiss (2020)) indicate that such methods are outperformed by the bootstrap procedure proposed in this paper. Alternatively, Shao (2009) proposed an approach to construct confidence intervals for $M(\varphi, f)$ based on selfnormalization which bypasses the problem of estimating the variance term τ^2 and does not deliver an estimation of the distribution of the statistics of interest, as is the case with the bootstrap methods proposed in this paper.

(iii) The second summand τ_2 of the limiting variance τ^2 simplifies in the case of a linear process, that is, if (X_t) admits the representation (1.2) for some square-summable sequence of coefficients $(a_j)_{j \in \mathbb{Z}}$ and some i.i.d. white noise process $(\varepsilon_t)_{t \in \mathbb{Z}}$ with finite fourth moments. Denoting $\sigma^2 := E(\varepsilon_1^2)$ and $\eta := E(\varepsilon_1^4)/\sigma^4$, it then follows by standard calculations that $2\pi f_4(\lambda_1, \lambda_2, -\lambda_2) = (\eta - 3)f(\lambda_1)f(\lambda_2)$. In this case the second summand τ_2 of the limiting variance from (2.2) can be written as

$$(2.5) \quad \tau_{2,\text{lin}} = (\eta - 3) \left(\int_{-\pi}^{\pi} \varphi(\lambda) f(\lambda) d\lambda \right)^2.$$

Regarding the class of ratio statistics, the situation is somehow different. Notice first that

$$L_{n,R} := \sqrt{n}(R(\varphi, I_n) - R(\varphi, f)) = \frac{1}{M(1, I_n)M(1, f)} \sqrt{n} \int_{-\pi}^{\pi} w(\lambda) I_n(\lambda) d\lambda,$$

where $w(\lambda) = \varphi(\lambda) \int_{-\pi}^{\pi} f(\alpha) d\alpha - \int_{-\pi}^{\pi} \varphi(\alpha) f(\alpha) d\alpha$. In view of (2.2) and the fact that $M(1, I_n) \xrightarrow{P} M(1, f)$, we then have that

$$(2.6) \quad L_{n,R} \xrightarrow{d} \mathcal{N}(0, \tau_R^2),$$

where $\tau_R^2 = \tau_{1,R}^2 + \tau_{2,R}$ with

$$(2.7) \quad \tau_{1,R}^2 = \frac{2\pi}{M^4(1, f)} \int_{-\pi}^{\pi} w(\lambda)(w(\lambda) + w(-\lambda))f(\lambda)^2 d\lambda \quad \text{and}$$

$$(2.8) \quad \tau_{2,R} = \frac{2\pi}{M^4(1, f)} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} w(\lambda_1)w(\lambda_2) f_4(\lambda_1, \lambda_2, -\lambda_2) d\lambda_1 d\lambda_2.$$

It can be easily verified that $\int_{-\pi}^{\pi} w(\lambda) f(\lambda) d\lambda = 0$. This implies that we get $\tau_{2,R} = 0$ for the class of linear processes considered in Remark 2.2(iii) and by the same arguments to those used there. That is, the variance of the limiting Gaussian distribution (2.6) simplifies for linear processes to $\tau_R^2 = \tau_{1,R}^2$ and it is, therefore, not affected by the fourth order structure of the process. Note that this simplification for ratio statistics no longer holds true when considering nonlinear processes.

3. The hybrid bootstrap procedure. In this section we will present our main contribution, the *hybrid periodogram bootstrap* (HPB). The HPB is a hybrid of two other procedures. The first ingredient is the well-established multiplicative periodogram bootstrap (MPB) discussed by Franke and Härdle (1992) and Dahlhaus and Janas (1996). The second ingredient will be a new algorithm which we will call *convolved bootstrapped periodograms of subsamples* (CBP). The CBP is closely related to the concept of convolved subsampling which was introduced and discussed in detail by Tewes, Politis and Nordman (2019), but CBP contains an important modification that we will explicate in Section 3.2 and in Remark 3.2 of Section 3.4.

This section will be organized as follows. In order to state the hybrid periodogram bootstrap (HPB), we will first formulate the two algorithms that will be parts of the hybrid procedure, the MPB and the CBP, in Sections 3.1 and 3.2, respectively. The HPB, which combines

the strengths of both approaches, will be introduced in Section 3.3. Section 3.4 discusses several features of the bootstrap algorithms considered. It may be surprising that we take such a close look especially at the well-established MPB. But this is necessary to clarify why the range of validity of this method and other existing frequency domain bootstraps is so restricted. We will see exactly which features of the distribution of interest the MPB is not able to mimic so that they have to be accounted for in the hybrid procedure by the CBP.

3.1. *The classical MPB approach.* The MPB procedure (as discussed by Franke and Härdle (1992) and Dahlhaus and Janas (1996)) approximates the distribution $\mathcal{L}(L_n)$ of $L_n = \sqrt{n}(M(\varphi, I_n) - M(\varphi, f))$ based on a given sample X_1, \dots, X_n by the distribution of

$$(3.1) \quad V_n^* = \sqrt{n} \left(\frac{2\pi}{n} \sum_{j \in \mathcal{G}(n)} \varphi(\lambda_{j,n}) (T^*(\lambda_{j,n}) - \widehat{f}_n(\lambda_{j,n})) \right),$$

where \widehat{f}_n is a consistent (e.g., kernel-type) estimator of f and

$$(3.2) \quad T^*(\lambda_{j,n}) := \widehat{f}_n(\lambda_{j,n}) U_j^*.$$

The bootstrap random variables $(U_j^*)_{j \in \mathcal{G}(n)}$ are constructed as follows: For $j > 0$, generate $U_1^*, \dots, U_{[n/2]}^*$ i.i.d. with a standard exponential distribution. Then, for $j < 0$, set $U_j^* := U_{-j}^*$. The construction with $U_j^* = U_{-j}^*$ ensures that $T^*(\cdot)$ is an even function which preserves an important property of the periodogram $I_n(\cdot)$. To see the motivation for approach (3.1), notice that V_n^* is a Riemann approximation for $\sqrt{n}(M(\varphi, T^*) - M(\varphi, \widehat{f}_n))$. Moreover, the bootstrap random variables $T^*(\lambda_{j,n})$ are supposed to mimic the behavior of the periodogram ordinates $I_n(\lambda_{j,n})$, based on the fact that $I_n(\lambda)/f(\lambda)$ has an asymptotic standard exponential distribution for $\lambda \in (-\pi, \pi) \setminus \{0\}$. Note that, alternatively, the U_j^* may be generated using so-called frequency domain residuals (cf. Dahlhaus and Janas (1996)) which is second-order equivalent to the approach described above.

Under mild conditions V_n^* has a limiting normal distribution with mean zero and variance τ_1^2 as defined in (2.3). The proof is given in Proposition 4.3 without the assumption of a linear process. Hence, for a quite general class of stationary processes, the bootstrap approach (3.1) correctly captures the first summand τ_1^2 but fails to mimic the second summand τ_2 of the limiting variance τ^2 from (2.2). Consequently, the approach asymptotically only works in those special cases where τ_2 vanishes. This happens, for example, if the underlying time series is Gaussian or if the underlying process is linear and one is looking at ratio statistics; see Dahlhaus and Janas (1996) and the discussion in the previous section. But except for these special cases the classical approach via V_n^* fails in general.

3.2. *The CBP approach.* In the following we propose the CBP approach denoted by L_n^* which will be the second building block of the hybrid periodogram bootstrap.

The CBP approach is based on bootstrapped periodograms of subsamples, and it is especially tailored for spectral means and ratio statistics. As already mentioned, this procedure is closely related to convolved subsampling; cf. Tewes, Politis and Nordman (2019). However, our CBP algorithm contains a modification that leads to the fact that the spectral density is not necessarily estimated by the subsampling estimator but instead by any consistent estimator that the user may choose. The CBP cannot, therefore, be formulated as a special case of the general convolved subsampling approach. Moreover, Tewes, Politis and Nordman (2019) show for the general convolved subsampling approach that asymptotic consistency is equivalent to certain high-level assumptions being met. With our CBP procedure being specifically tailored for periodogram-based statistics, we show in Theorem 4.4 asymptotic consistency for spectral means and ratio statistics under very mild conditions on the underlying stochastic process (even without assuming strict stationarity). In Section 1 of the Supplementary

Material (Meyer, Paparoditis and Kreiss (2020)), we also show how consistency of the CBP can be extended to more general statistics which are functionals of the periodogram. Furthermore, simulations reported in Section 5.2 and in the Supplementary Material (Meyer, Paparoditis and Kreiss (2020)) show that the CBP modification improves the behavior of the standard convolved subsampling for the class of statistics considered in this paper and makes this procedure, especially for ratio statistics, less sensitive with respect to the choice of the subsampling parameter.

CBP for spectral means and ratio statistics

(1) Choose a block length $b < n$. For $t = 1, \dots, N$, with $N := n - b + 1$, let

$$(3.3) \quad I_{t,b}(\lambda_{j,b}) = \frac{1}{2\pi b} \left| \sum_{s=1}^b X_{t+s-1} e^{-i\lambda_{j,b}s} \right|^2$$

be the subsample periodograms evaluated at the Fourier frequencies $\lambda_{j,b}$ (associated with length b time series, i.e., $\lambda_{j,b} = 2\pi j/b$, with $j \in \mathcal{G}(b)$; cf. (2.1) for the definition of $\mathcal{G}(\cdot)$).

(2) Define the rescaled frequency domain residuals of the subsample periodogram $I_{t,b}(\lambda_{j,b})$ as,

$$U_{t,b}(\lambda_{j,b}) = \frac{I_{t,b}(\lambda_{j,b})}{\widehat{f}_b(\lambda_{j,b})}, \quad j \in \mathcal{G}(b),$$

where $\widehat{f}_b(\lambda_{j,b}) = N^{-1} \sum_{t=1}^N I_{t,b}(\lambda_{j,b})$.

(3) Let \widehat{f}_n be the spectral density estimator used in (3.2). Set $k := [n/b]$ and generate random variables i_1^*, \dots, i_k^* i.i.d. with a discrete uniform distribution on the set $\{1, \dots, N\}$. For $l = 1, 2, \dots, k$, define

$$I_b^{(l)}(\lambda_{j,b}) = \widehat{f}_n(\lambda_{j,b}) \cdot U_{i_l^*,b}(\lambda_{j,b}),$$

and let $I_{j,b}^* = k^{-1} \sum_{l=1}^k I_b^{(l)}(\lambda_{j,b})$.

(4) For *spectral means* approximate the distribution of $L_n = \sqrt{n}(M(\varphi, I_n) - M(\varphi, f))$ by the distribution of the bootstrap quantity

$$(3.4) \quad \begin{aligned} L_n^* &:= \sqrt{kb} \left(\frac{1}{k} \sum_{l=1}^k \frac{2\pi}{b} \sum_{j \in \mathcal{G}(b)} \varphi(\lambda_{j,b}) (I_b^{(l)}(\lambda_{j,b}) - \widehat{f}_n(\lambda_{j,b})) \right) \\ &= \sqrt{kb} \frac{2\pi}{b} \sum_{j \in \mathcal{G}(b)} \varphi(\lambda_{j,b}) (I_{j,b}^* - \widehat{f}_n(\lambda_{j,b})). \end{aligned}$$

Furthermore, for *ratio statistics* approximate the distribution of $L_{n,R} = \sqrt{n}(R(\varphi, I_n) - R(\varphi, f))$ by that of

$$(3.5) \quad L_{n,R}^* := \sqrt{kb} \left(\frac{\sum_{j \in \mathcal{G}(b)} \varphi(\lambda_{j,b}) I_{j,b}^*}{\sum_{j \in \mathcal{G}(b)} I_{j,b}^*} - \frac{\sum_{j \in \mathcal{G}(b)} \varphi(\lambda_{j,b}) \widehat{f}_n(\lambda_{j,b})}{\sum_{j \in \mathcal{G}(b)} \widehat{f}_n(\lambda_{j,b})} \right).$$

Before we discuss some details of the CBP algorithm, we will first turn to the main contribution of this paper, the hybrid periodogram bootstrap.

3.3. *The hybrid periodogram bootstrap (HPB).* The hybrid periodogram bootstrap uses both the MPB and the CBP to imitate different features of the distribution of interest. We will state the HPB algorithm in two different versions; the first is supposed to be used for spectral means and the second for ratio statistics.

(HPB) Hybrid periodogram bootstrap (for spectral means)

(1) Let L_n^* be generated according to (3.4), that is, as in the CBP procedure, and let V_n^* be defined as in (3.1) where the i.i.d. random variables U_j^* , $j = 1, 2, \dots, [n/2]$, and $i_1^*, i_2^*, \dots, i_k^*$ are independent.

(2) Approximate the distribution of $L_n = \sqrt{n}(M(\varphi, I_n) - M(\varphi, f))$ by the empirical distribution of the rescaled bootstrap quantity

$$\tilde{V}_n^* := \sqrt{1 + \frac{\hat{\tau}^2 - c_n}{\hat{\tau}_1^2}} \cdot V_n^*,$$

where $\hat{\tau}^2 := \text{Var}^*(L_n^*)$, $\hat{\tau}_1^2 := \text{Var}^*(V_n^*)$ and

$$c_n = \frac{4\pi^2}{b} \sum_{j \in \mathcal{G}(b)} \varphi(\lambda_{j,b})(\varphi(\lambda_{j,b}) + \varphi(-\lambda_{j,b})) \hat{f}_n(\lambda_{j,n})^2 \left(\frac{1}{N} \sum_{t=1}^N \frac{I_{t,b}(\lambda_{j,b})^2}{\hat{f}_b(\lambda_{j,b})^2} - 1 \right).$$

It is important to emphasize that the HPB presented above does *not* simply use the variance estimate of the CBP as a correction factor for the MPB. Instead, only those features that are connected to the weak dependence structure of the periodogram (which the MPB is not able to mimic) are imitated by the CBP. We will give more details on this in Remark 3.1.

The HPB for ratio statistics reads a bit more tedious but is based on the very same idea as its counterpart for spectral means. The modification is necessary due to the normalizing term $M(1, I_n)M(1, f)$ (see the definition of $L_{n,R}$) whose stochastic behaviour has to be taken into account.

(HPB) Hybrid periodogram bootstrap (for ratio statistics)

(1) Let

$$\begin{aligned} V_{1,n}^* &= \frac{2\pi}{\sqrt{n}} \sum_{j \in \mathcal{G}(n)} \hat{w}(\lambda_{j,n}) T^*(\lambda_{j,n}), \\ V_{2,n}^* &= \sqrt{kb} \frac{2\pi}{b} \sum_{j \in \mathcal{G}(b)} \tilde{w}(\lambda_{j,b}) I_{j,b}^*, \end{aligned} \tag{3.6}$$

$$V_{n,R}^* = (D_n^*)^{-1} V_{1,n}^*, \text{ where } D_n^* = 4\pi^2 n^{-2} \sum_{j \in \mathcal{G}(n)} T^*(\lambda_{j,n}) \sum_{j \in \mathcal{G}(n)} \hat{f}_n(\lambda_{j,n}),$$

$$\hat{w}(\lambda) = \varphi(\lambda) \frac{2\pi}{n} \sum_{j \in \mathcal{G}(n)} \hat{f}_n(\lambda_{j,n}) - \frac{2\pi}{n} \sum_{j \in \mathcal{G}(n)} \varphi(\lambda_{j,n}) \hat{f}_n(\lambda_{j,n})$$

and

$$\tilde{w}(\lambda) = \varphi(\lambda) \frac{2\pi}{b} \sum_{j \in \mathcal{G}(b)} \hat{f}_n(\lambda_{j,b}) - \frac{2\pi}{b} \sum_{j \in \mathcal{G}(b)} \varphi(\lambda_{j,b}) \hat{f}_n(\lambda_{j,b}).$$

(2) Approximate the distribution of $L_{n,R} = \sqrt{n}(R(\varphi, I_n) - R(\varphi, f))$ by the distribution of $\tilde{V}_{n,R}^*$, where

$$\tilde{V}_{n,R}^* = \sqrt{1 + \frac{\hat{\sigma}_{2,R}^2 - c_{n,R}}{\hat{\sigma}_{1,R}^2}} \cdot V_{n,R}^*$$

with $\hat{\sigma}_{1,R}^2 = \text{Var}^*(V_{1,n}^*)$, $\hat{\sigma}_{2,R}^2 = \text{Var}^*(V_{2,n}^*)$ and

$$c_{n,R} = \frac{4\pi^2}{b} \sum_{j \in \mathcal{G}(b)} \tilde{w}(\lambda_{j,b}) (\tilde{w}(\lambda_{j,b}) + \tilde{w}(-\lambda_{j,b})) \hat{f}_n(\lambda_{j,b})^2 \times \left(\frac{1}{N} \sum_{t=1}^N \frac{I_{t,b}(\lambda_{j,b})^2}{\tilde{f}_b(\lambda_{j,b})^2} - 1 \right).$$

3.4. *Remarks on the proposed bootstrap algorithms.* Remark 3.1 looks at the HPB procedure and specifically sheds more light on how its ingredients imitate different features of the distribution of interest. Remark 3.2 deals with the CBP algorithm in more detail while Remark 3.3 addresses questions related to the practical implementation of the HPB.

REMARK 3.1. As we have seen, the MPB performs well in approximating the distribution of L_n in cases where the fourth-order cumulant term τ_2 vanishes. However, it is not consistent in cases where $\tau_2 \neq 0$. In the new HPB algorithm the factor $\sqrt{1 + (\hat{\tau}^2 - c_n)/\hat{\tau}_1^2}$ corrects for this shortcoming. It is important to note that this correction factor does *not* simply replace the variance of the MPB by that of the CBP. Instead, only the part of the variance that is due to the dependence structure of the periodogram ordinates is imitated by the CBP, that is, the term τ_2 only. To be precise, observe that according to the proof of Theorem 4.4(i) the variance of the CBP quantity L_n^* separates into

$$\hat{\tau}^2 = \text{Var}^*(L_n^*) = R1^* + R2^*,$$

where $R1^* \rightarrow \tau_1^2$ and $R2^* \rightarrow \tau_2$, in probability. Further observe that we have chosen $c_n = R1^*$. Therefore, the quantity $\hat{\tau}^2 - c_n$ in the correction factor equals $R2^*$ which represents (even for finite sample sizes) exactly the contribution of the CBP variance that accounts for τ_2 in the limit. Thus, the correction factor converges toward $\sqrt{1 + (\tau_2/\tau_1^2)}$, and the HPB approach has the correct asymptotic variance $\tau_1^2 + \tau_2$. Due to this construction, the first summand of the limiting variance τ_1^2 is established by the classical MPB approach while the τ_2 part is imitated by the CBP. Note that the distributional shape as approximated by the MPB pseudo random variable V_n^* , is not affected by this correction.

As we will see, this modification makes the proposed HPB procedure consistent for a vast class of stochastic processes. Furthermore, it leads to certain advantages in finite sample situations which can be seen from the simulation results reported in Section 5.2 and in the Supplementary Material (Meyer, Paparoditis and Kreiss (2020)). For instance, convolved subsampling in its different versions may have a tendency (due to averaging) to move subsampling distributions closer to normality which the MPB does not. It appears that the hybrid procedure HPB outperforms the CBP and the standard convolved subsampling procedure, and it is less sensitive with respect to the choice of the tuning parameter b .

REMARK 3.2. (i) The CBP approach differs from classical subsampling and convolved subsampling since we do not calculate L_n^* (respectively $L_{n,R}^*$) solely based on subsamples of the observed time series. In fact, our procedure generates new pseudo periodograms of subsamples $I_b^{(l)}(\lambda_{j,b})$, $l = 1, 2, \dots, k$, using one's favourite spectral density estimator \hat{f}_n based on the entire time series X_1, X_2, \dots, X_n . This also allows for the use of the same spectral density estimator \hat{f}_n in both ingredients of the HPB (the MPB and the CBP).

(ii) Similar to the MPB, the rescaling in step (2) of the CBP ensures that $E^*(U_{i^*,b}(\lambda_{j,b})) = 1$, that is, $E^*(I_b^{(l)}(\lambda_{j,b})) = \hat{f}_n(\lambda_{j,b})$ which avoids an unnecessary bias at the resampling step.

(iii) Notice that L_n^* can be written as $L_n^* = k^{-1/2} \sum_{l=1}^k L_{l,n}^*$, where

$$L_{l,n}^* = \sqrt{b} \frac{2\pi}{b} \sum_{j \in \mathcal{G}(b)} \varphi(\lambda_{j,b}) \widehat{f}_n(\lambda_{j,b}) (U_{i_l^*,b}(\lambda_{j,b}) - 1).$$

Comparing the above expression with that of the MPB, given by

$$V_n^* = \sqrt{n} \frac{2\pi}{n} \sum_{j \in \mathcal{G}(n)} \varphi(\lambda_{j,n}) \widehat{f}_n(\lambda_{j,n}) (U_j^* - 1),$$

clarifies that there are two essential differences between the two. The first lies in the way the pseudo innovations in the two approaches are generated. In particular, while the U_j^* 's are independent the pseudo random variables $U_{i_l^*,b}(\lambda_{j,b})$ are (for each fixed l) *not independent*. Due to using the *same* bootstrap sample i_1^*, \dots, i_k^* for all $I_{j,b}^*$, $j \in \mathcal{G}(b)$ and therefore for all Fourier frequencies, the weak dependence structure of the periodogram ordinates is preserved by $I_{j,b}^*$, $j \in \mathcal{G}(b)$.

The second difference is the respective number of frequencies on which $L_{l,n}^*$ and V_n^* are based. The subsample periodograms used in $L_{l,n}^*$ have to be evaluated at the Fourier frequencies for subsample size $b < n$, since we will exploit certain asymptotic properties of the periodogram that do not hold if the periodograms are evaluated at other frequencies than $\lambda_{j,b}$, $j \in \mathcal{G}(b)$. While this step allows for appropriately mimicking the limiting variance in general situations, as we will show, the price to pay here is obviously that a new tuning parameter b is introduced.

(iv) Note that the rescaling quantity \widetilde{f}_b used in step (2) of the CBP is actually a spectral density estimator itself which is based on averaging periodograms calculated over subsamples. Such estimators have been thoroughly investigated by many authors in the literature, for example, Bartlett (1948, 1950) and Welch (1967); see also Dahlhaus (1985). In step (2) it is important to use \widetilde{f}_b because this is the appropriate rescaling quantity; see part (ii) of this remark. In principle one could of course set $\widehat{f}_n := \widetilde{f}_b$ in step (3) of the CBP procedure. However, it has to be noted that \widehat{f}_n used in (3.2) and in step (3) is the quantity that approximates f and which is identical in both corresponding algorithms. Hence, we do not have to use \widetilde{f}_b at this point but can rather allow, as in the MPB, for the use of any consistent spectral density estimator \widehat{f}_n which may approximate f much better than \widetilde{f}_b and which makes the proposed bootstrap approach much more flexible. Depending on the data sample at hand, either appropriate parametric or nonparametric estimators may be favoured and chosen from case to case. Due to this modification, the CBP differs from the general convolved subsampling approach of Tewes, Politis and Nordman (2019). Clearly, if one chooses $\widehat{f}_n = \widetilde{f}_b$, then both approaches are identical. However, as our simulation results show, using \widehat{f}_n instead of \widetilde{f}_b has certain advantages, especially for ratio statistics, cf Figure 1 in Section 5.2 and Figure II of the Supplementary Material (Meyer, Papanoditis and Kreiss (2020)).

(v) One might ask why in step (3) of the CBP algorithm the mean of exactly k bootstrap random variables $I_b^{(1)}(\lambda_{j,b}), \dots, I_b^{(k)}(\lambda_{j,b})$ is taken in order for $I_{j,b}^*$ to mimic the periodogram appropriately. If one is only interested in $\text{Var}(L_n^*)$, then the representation $L_n^* = k^{-1/2} \sum_{l=1}^k L_{l,n}^*$ from (iii) immediately yields that one could in principle replace k by any natural number, here (even $k = 1$ is admissible), and get a modified version of L_n^* , say L_n^+ . By the same arguments, as in the proof of Theorem 4.4(i), we would still get $\text{Var}^*(L_n^+) = R1^* + R2^*$, that is, L_n^+ has the correct variance. However, if k is chosen non-increasing in n , for example, replacing k by 1 which yields $L_n^+ = L_{1,n}^*$, this would not be a consistent approximation of the distribution of L_n . An increasing sequence k is necessary in order to achieve asymptotic normality of L_n^* . The choice $k = \lfloor n/b \rfloor$ is somewhat natural in the context of convolved subsampling and block bootstrap techniques, and it also avoids

to introduce an additional tuning parameter k . Of course, the user may choose a different sequence that increases to infinity if this seems appropriate in the specific situation.

REMARK 3.3. In step (2) of the HPB for spectral means, the bootstrap variances $\hat{\tau}^2 := \text{Var}^*(L_n^*)$ and $\hat{\tau}_1^2 := \text{Var}^*(V_n^*)$ are used. In practice, these values may, for example, be obtained via Monte Carlo. As for V_n^* , repeat step (1) multiple times to generate M replications $V_n^*(1), \dots, V_n^*(M)$ of V_n^* . Then use the estimate

$$\hat{\tau}_1^2 \approx \frac{1}{M-1} \sum_{j=1}^M (V_n^*(j) - \bar{V}_n^*)^2,$$

where $\bar{V}_n^* = M^{-1} \sum_{j=1}^M V_n^*(j)$. Proceed analogously with $\hat{\tau}^2$ and, regarding the HPB for ratio statistics, with the quantities $\hat{\sigma}_{1,R}^2$ and $\hat{\sigma}_{2,R}^2$ used in step (2) of that algorithm. As for the HPB algorithm for ratio statistics, the term $c_{n,R}$ is an estimator of $\tau_{1,R}^2$ appearing in the asymptotic variance τ_R^2 of $L_{n,R}$; cf. (2.6). Moreover, notice that $2\pi n^{-1} \sum_{j \in \mathcal{G}(n)} \hat{w}(\lambda_{j,n}) \hat{f}_n(\lambda_{j,n}) = 0$ and $2\pi b^{-1} \sum_{j \in \mathcal{G}(b)} \hat{w}(\lambda_{j,b}) \hat{f}_n(\lambda_{j,b}) = 0$ which make the centering of $V_{1,n}^*$ and $V_{2,n}^*$ in (3.6) obsolete.

4. Bootstrap validity. In order to establish consistency for the aforementioned bootstrap approaches, we impose the following assumption on the asymptotic behavior of the subsample block length b :

ASSUMPTION 2.

- (i) For block length b and $k = [n/b]$, it holds $b = b(n)$ and $k = k(n)$, such that $b \rightarrow \infty$ and $k \rightarrow \infty$, as $n \rightarrow \infty$.
- (ii) It holds $b^3/n \rightarrow 0$ and $\ln(n-b+1)/b \rightarrow 0$, as $n \rightarrow \infty$.

Note that under the conditions of this assumption, especially with b increasing at a slower rate than n , we have for all asymptotic considerations $n \approx bk$ or, to be precise, $bk = n(1 + o(b^{-2}))$. To keep the presentation of the proofs of the upcoming results concise, we will in those proofs often replace n by bk or vice versa. It can be easily seen that this does not change the asymptotics.

For the bootstrap approaches considered, we assume uniform consistency for the spectral density estimator \hat{f}_n used, that is:

ASSUMPTION 3. The spectral density estimator \hat{f}_n fulfils

$$\sup_{\lambda \in [-\pi, \pi]} |\hat{f}_n(\lambda) - f(\lambda)| = o_P(1).$$

This is a common and rather weak assumption in spectral analysis of time series. The following two preliminary results will be useful for the proof of consistency results of the CBP approach L_n^* .

LEMMA 4.1. Under the conditions of Assumptions 1 and 2(i), it holds for the Fourier frequencies $\lambda_{j,b} = 2\pi j/b$, $j \in \mathcal{G}(b)$:

$$(i) \quad \sum_{j \in \mathcal{G}(b)} |\tilde{f}_b(\lambda_{j,b}) - EI_{1,b}(\lambda_{j,b})| = \mathcal{O}_P(\sqrt{b^3/N}),$$

$$(ii) \quad \sum_{j,s \in \mathcal{G}(b)} \left| \frac{1}{N} \sum_{t=1}^N I_{t,b}(\lambda_{j,b}) I_{t,b}(\lambda_{s,b}) - E(I_{1,b}(\lambda_{j,b}) I_{1,b}(\lambda_{s,b})) \right| = \mathcal{O}_P(\sqrt{b^5/N}).$$

LEMMA 4.2. *Let Assumptions 1 and 2 as well as Assertion (2.2) hold. Then, for*

$$(4.1) \quad W_{t,b} := \frac{2\pi}{\sqrt{b}} \sum_{j \in \mathcal{G}(b)} \varphi(\lambda_{j,b})(I_{t,b}(\lambda_{j,b}) - \tilde{f}_b(\lambda_{j,b})),$$

it holds, as $n \rightarrow \infty$:

- (i) $E(W_{1,b}^2) \rightarrow \tau^2$, with τ^2 as in (2.2).
- (ii) $W_{1,b} \xrightarrow{d} \mathcal{N}(0, \tau^2)$.

The upcoming proposition states an asymptotic result for the MPB approach V_n^* . Dahlhaus and Janas (1996) proved this result for the special case of linear processes as given by (1.2). However, since this restriction is not necessary for the bootstrap quantities, we derive the limiting distribution for general stationary processes satisfying Assumption 1(i).

PROPOSITION 4.3. *Let Assumptions 1, 2 and 3 be fulfilled. Moreover, let Φ denote the cdf of the standard normal distribution. Then, with τ_1^2 as defined in (2.3), it holds:*

- (i) $\text{Var}^*(V_n^*) \xrightarrow{P} \tau_1^2$,
- (ii) $\sup_{x \in \mathbb{R}} |P^*(V_n^* \leq x) - \Phi(x/\tau_1)| = o_P(1)$.

The next theorem shows that the CBP approaches L_n^* and $L_{n,R}^*$ consistently estimate the variance and the distribution of the statistics of interest.

THEOREM 4.4. *Let the assumptions of Proposition 4.3 as well as Assertion (2.2) be fulfilled. Then, with τ^2 as below (2.2), it holds:*

- (i) $\text{Var}^*(L_n^*) \xrightarrow{P} \tau^2$,
- (ii) $\sup_{x \in \mathbb{R}} |P^*(L_n^* \leq x) - P(L_n \leq x)| = o_P(1)$,
- (iii) $\sup_{x \in \mathbb{R}} |P^*(L_{n,R}^* \leq x) - P(L_{n,R} \leq x)| = o_P(1)$.

Our last theorem establishes consistency of the HPB approaches \tilde{V}_n^* and $\tilde{V}_{n,R}^*$.

THEOREM 4.5. *Let the assumptions of Proposition 4.3 as well as Assertion (2.2) be fulfilled. Then, with τ^2 as in Theorem 4.4, it holds:*

- (i) $\text{Var}^*(\tilde{V}_n^*) \xrightarrow{P} \tau^2$,
- (ii) $\sup_{x \in \mathbb{R}} |P^*(\tilde{V}_n^* \leq x) - P(L_n \leq x)| = o_P(1)$,
- (iii) $\sup_{x \in \mathbb{R}} |P^*(\tilde{V}_{n,R}^* \leq x) - P(L_{n,R} \leq x)| = o_P(1)$.

Notice that the above theorem allows for the use of the distribution of \tilde{V}_n^* (respective of $\tilde{V}_{n,R}^*$) in order to construct an asymptotic $(1 - 2\alpha)$ level confidence interval for $M(\varphi, f)$ (respectively $R(\varphi, f)$). More details are given in Section 3 of the Supplementary Material (Meyer, Papanoditis and Kreiss (2020)).

The proofs of some of these results can be found in the Appendix of this paper, while the rather technical proofs of Lemma 4.1, Lemma 4.2, Proposition 4.3 and Theorem 4.5(iii) have been deferred to the Supplementary Material (Meyer, Papanoditis and Kreiss (2020)).

5. Practical issues and numerical results.

5.1. *Some remarks on choosing the subsampling parameter.* Implementation of the proposed bootstrap procedure requires the choice of the subsampling parameter b . Recall that, for the HPB procedure, the choice of b affects only the estimation of the second variance term τ_2 in (2.4).

Notice that the rate condition $b^3/n \rightarrow 0$ as $n \rightarrow \infty$ required for our asymptotic considerations, does not provide any guidance on how to choose the parameter b in practice and for a time series X_1, X_2, \dots, X_n at hand. For this issue additional investigations are required. For instance, the choice of this parameter can be based on some optimality considerations, like for instance on minimizing the mean square error $E(\widehat{\tau}_2(b) - \tau_2)^2$. Here, $\widehat{\tau}_2(b)$ denotes the bootstrap estimator of τ_2 . Since such an approach faces the problem that the target τ_2 is unknown, different possibilities can be explored. For instance, one can develop a cross-validation type criterion to select b , or one can search for approximations of the bias and of the variance of $\widehat{\tau}_2(b)$ and use these approximations together with some plug-in procedure in order to select b . These are interesting venues of future research which require additional investigations that go beyond the scope of the current paper. However, the following observations can be made based on our extensive simulations discussed in the next section. The HPB procedure seems not to be very sensitive with respect to the choice of b , provided this parameter is not chosen too small with respect to the sample size n ; see, in particular, Figure 1, Figure 2 and Figure 3 of the next section. This motivated us to suggest the following practical rule to select b : Choose b as the smallest integer which is greater than or equal to $b = 4 \cdot n^{0.30}$. On the one hand, this ad hoc rule delivers a value of b which is large enough and performs satisfactory for all estimation problems and for all models considered, and on the other hand, it satisfies the asymptotic rate condition $b^3 = o(n)$.

5.2. *Simulation results.* We investigate the finite sample behavior of the new bootstrap procedures proposed in this paper and compare their performance with that of existing methods. We will focus in the following on the case of the first-order sample autocovariance $\widehat{\gamma}(1)$ and sample autocorrelation $\widehat{\rho}(1)$ and compare the performance of the hybrid periodogram bootstrap approach (HPB) with that of the convolved bootstrapped periodograms of subsamples (CBP), the multiplicative periodogram bootstrap (MPB) and the moving block bootstrap (MBB). A comparison of the HPB with the standard convolved subsampling procedure is presented in Figure II of the Supplementary Material (Meyer, Paparoditis and Kreiss (2020)). Notice that the CBP and the MBB are essentially the only competitors of the HPB since they are the only methods which have been proven to be consistent for all models considered in our simulation experiment. Furthermore, although the autoregressive sieve bootstrap and the autoregressive aided periodogram bootstrap are not consistent for the three nonlinear models considered in the sequel, some comparisons of these methods with the HPB can be found in the Supplementary Material (Meyer, Paparoditis and Kreiss (2020)). The following four time series are considered:

Model I: $X_t = 0.8X_{t-1} + \varepsilon_t$ and i.i.d. innovations $\varepsilon_t \sim \mathcal{N}(0, 1)$.

Model II: $X_t = 0.3X_{t-1} - 3.5X_{t-1} \cdot \varepsilon_{t-1} + \varepsilon_t$ and i.i.d. innovations $\varepsilon_t \sim \mathcal{N}(0, 0.1^2)$.

Model III: $X_t = v_t + 0.8v_{t-1}$, where $v_t = \varepsilon_t \sqrt{1 + 0.25v_{t-1}^2}$ and i.i.d. $\varepsilon_t \sim \mathcal{N}(0, 1)$.

Model IV:

$$X_t = \begin{cases} -0.3X_{t-1} + \varepsilon_t & \text{if } X_{t-1} \leq 0, \\ 0.8X_{t-1} + \varepsilon_t & \text{if } X_{t-1} > 0, \end{cases}$$

where ε_t are i.i.d. innovations $\varepsilon_t \sim \mathcal{N}(0, 1)$.

The above choice covers a wide range of linear and nonlinear models commonly used in time series analysis. In particular, model I is a linear AR(1) model with Gaussian i.i.d. innovations, while models II–IV are nonlinear and have been considered in Auestad and Tjøstheim (1990) and Shao (2009). Model II is a bilinear model, model III is a MA(1) model with ARCH(1) innovations and model IV is a threshold autoregressive model.

We demonstrate the capability of the bootstrap methods compared to estimate the distribution of the first-order sample autocovariance $L_n = \sqrt{n}(\hat{\gamma}(1) - \gamma(1))$ and of the first-order sample autocorrelation $R_n = \sqrt{n}(\hat{\rho}(1) - \rho(1))$, and we investigate the sensitivity of the HPB, the CBP and the MBB procedures with respect to the choice of the block size b . For this purpose time series of length $n = 150$ and of $n = 2000$ are considered, where the latter sample size has been chosen in order to clearly see the differences in the consistency behavior of the bootstrap procedures considered. In order to measure the distance of the bootstrap distribution to the exact distribution of interest, we calculated the d_1 -distance between distributions defined as $d_1(F, G) = \int_0^1 |F^{-1}(x) - G^{-1}(x)| dx$ for F and G distribution functions. We present mean distances calculated over $R = 200$ repetitions using a wide range of different subsampling sizes b . Notice that for the sample size of $n = 150$, respectively, $n = 2000$, the values of b chosen according to the rule of thumb discussed in Section 5.1 are $b = 18$, respectively, $b = 40$. To evaluate the exact distributions of interest, 10,000 replications have been used while all bootstrap approximations are based on 1000 replications. To estimate the spectral density f used in the multiplicative as well as in the HPB and CBP procedures, the Parzen lag window estimator, see Priestley (1981), has been used with truncation lag $M_n = 15$ for $n = 150$ and $M_n = 25$ for $n = 2000$. The MBB procedure has been implemented in its standard form, that is, by choosing with replacement $k = \lceil n/b \rceil$ blocks of length b and joining them together in the order selected to form a new pseudo sample. The results obtained are shown in Figure 1, Figure 2 and Figure 3.

In Figure 1 the performance of the HPB is compared with that of the CBP. As it can be seen, the HPB clearly outperforms the CBP for both statistics and for almost all choices of the subsampling parameter b considered. Furthermore, the HPB is much less sensitive with respect to the choice of this parameter than the CBP procedure. As Figure 2 and Figure 3 show, the differences between the MPB and the HPB method are small for the sample size of $n = 150$ observations, and both methods clearly outperform the MBB procedure for all four models and for all values of the block size, respectively, subsampling parameter b considered. The differences between the MPB and the HBP are clearly seen for the sample size of $n = 2000$ observations (right columns of Figure 2 and Figure 3). Observe that the d_1 -distances of the MPB estimates are for $n = 2000$ more or less the same as those for the case of $n = 150$ observations. As these exhibits show, apart from small values of b , the HPB method performs very well for a wide range of values of the subsampling parameter b . The same observation regarding the sensitivity of the HPB with respect to the choice of b can be made for the case of $n = 150$ observations. Notice that for the Gaussian AR(1) model considered, the MPB is consistent for estimating the distributions of L_n and R_n which is clearly seen in the first rows of Figure 2 and Figure 3. For the same model and for the case of $n = 2000$ observations, the HPB and the MPB procedures behave very similar and both outperform the MBB procedure. For the same sample size and for the nonlinear models considered, the HPB (apart for small values of b) and the MBB procedure behave very similarly while the MPB procedure behaves worse due to its inconsistency for this class of time series models. Summarizing our findings, the HPB behaves good for all models and for both sample sizes used in our simulation study which is not the case for the other three bootstrap procedures considered.

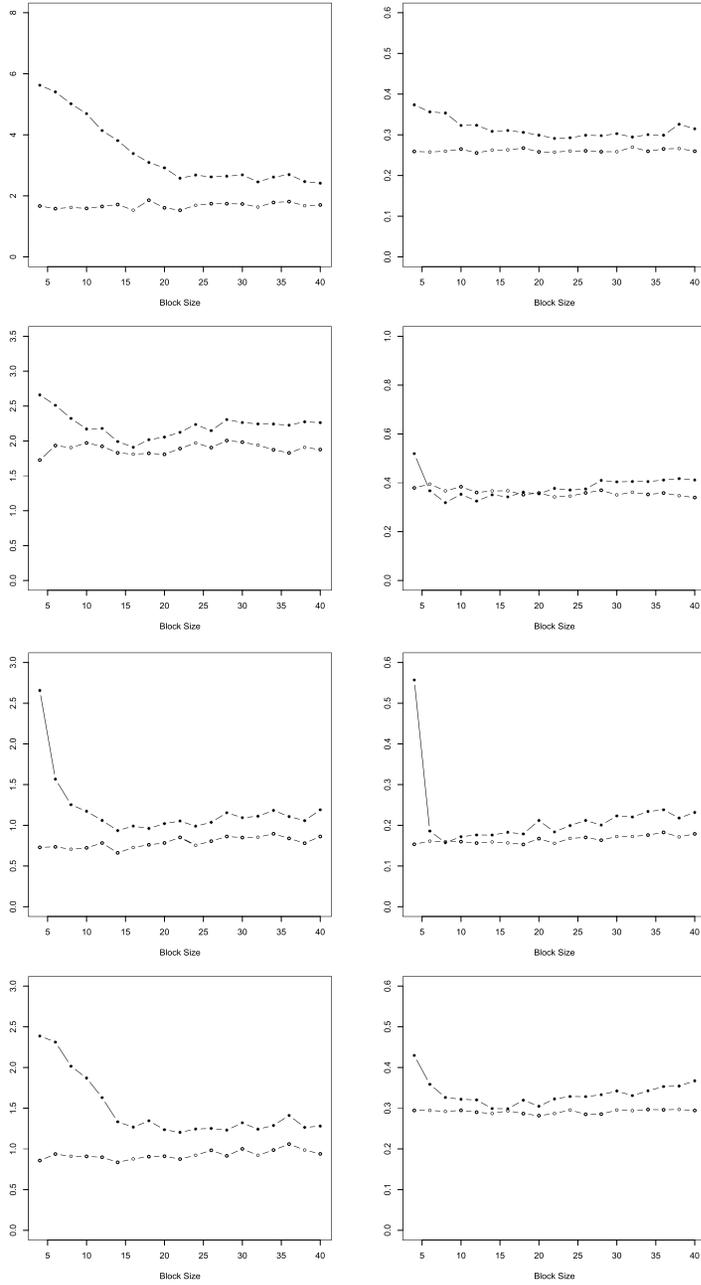


FIG. 1. Average d_1 -distances between the exact and the bootstrap approximation of the convolved bootstrapped periodograms (CBP) and of the hybrid periodogram bootstrap (HPB) for $n = 150$ and for various block sizes. Model I (first row), Model II (second row), Model III (third row) and Model IV (last row). The left panels refer to the distribution of $\sqrt{n}(\hat{\gamma}(1) - \gamma(1))$ and the right panels to the distribution of $\sqrt{n}(\hat{\rho}(1) - \rho(1))$. The dots denote the d_1 -distance of the CPB, the circles the one of the HPB.

APPENDIX

PROOF OF THEOREM 4.4(i). For each Fourier frequency $\lambda_{j,b}$, $j \in \mathcal{G}(b)$, the random variables $I_b^{(1)}(\lambda_{j,b}), \dots, I_b^{(k)}(\lambda_{j,b})$ are conditionally independent and each possesses a dis-

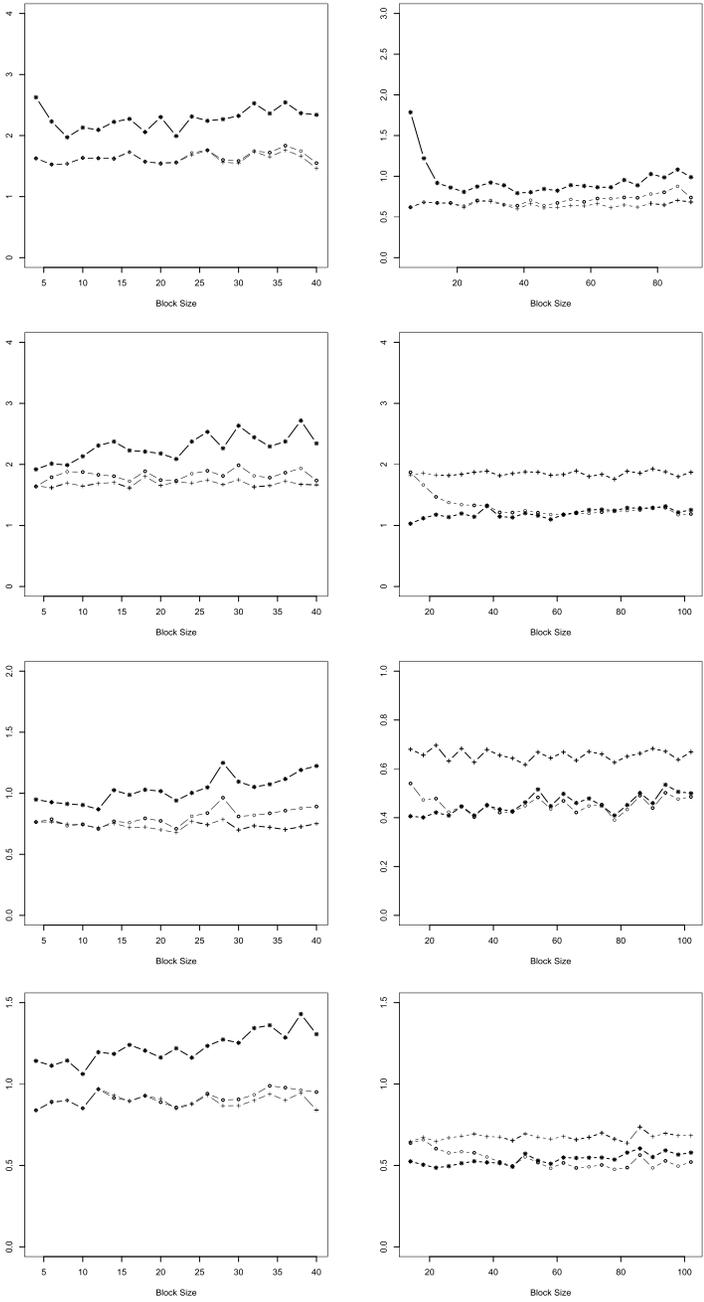


FIG. 2. Average d_1 -distances between the exact and the bootstrap distribution of $\sqrt{n}(\hat{\gamma}(1) - \gamma(1))$ for various block sizes and Model I (first row), Model II (second row), Model III (third row) and Model IV (last row). The left panels refer to $n = 150$ and the right panels to $n = 2000$. The crosses denote the d_1 -distance of the multiplicative periodogram bootstrap (MPB), the circles of the hybrid periodogram bootstrap (HPB) and the stars of the block bootstrap (MBB) estimates.

crete uniform distribution on the set

$$\left\{ \frac{\hat{f}_n(\lambda_{j,b})}{\hat{f}_b(\lambda_{j,b})} I_{1,b}(\lambda_{j,b}), \dots, \frac{\hat{f}_n(\lambda_{j,b})}{\hat{f}_b(\lambda_{j,b})} I_{N,b}(\lambda_{j,b}) \right\}.$$

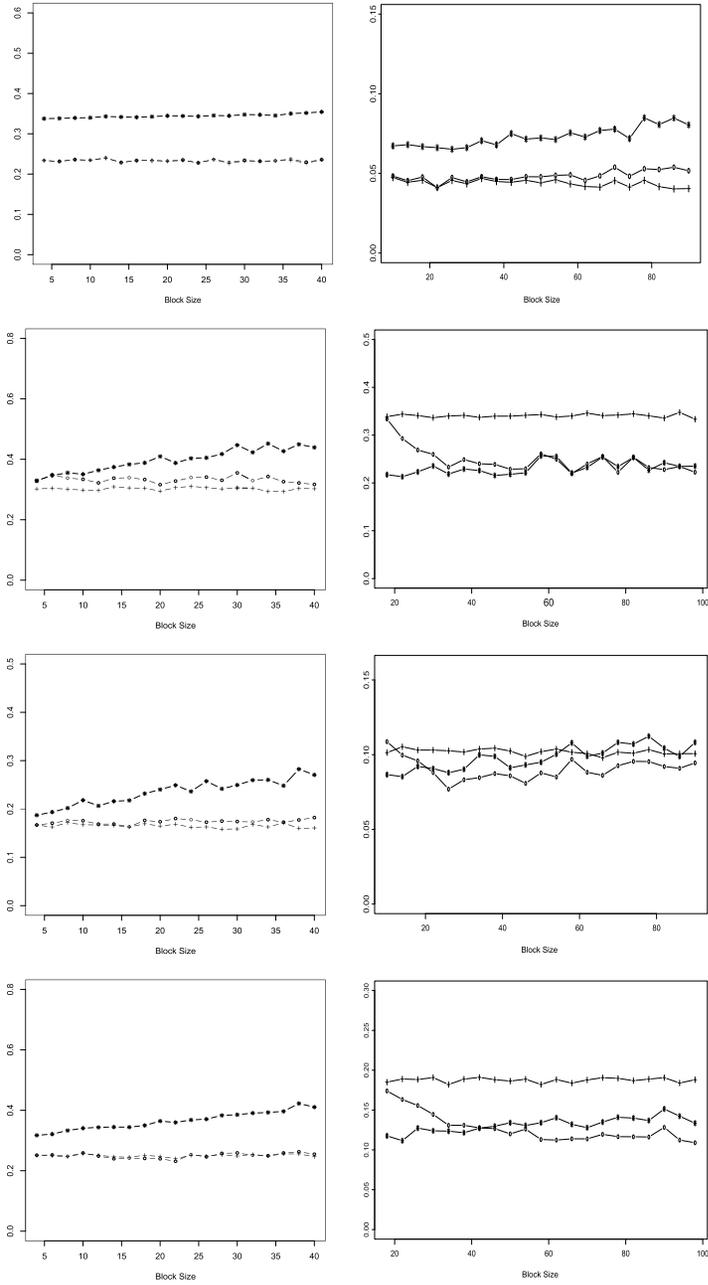


FIG. 3. Average d_1 -distances between the exact and the bootstrap distribution of $\sqrt{n}(\hat{\rho}(1) - \rho(1))$ for various block sizes and Model I (first row), Model II (second row), Model III (third row) and Model IV (last row). The left panels refer to $n = 150$ and the right panels to $n = 2000$. The crosses denote the d_1 -distance of the multiplicative periodogram bootstrap (MPB), the circles of the hybrid periodogram bootstrap (HPB) and the stars of the block bootstrap (MBB) estimates.

Hence, we get as an auxiliary consideration

$$\begin{aligned}
 & \text{Cov}^*(I_b^{(1)}(\lambda_{j_1,b}), I_b^{(1)}(\lambda_{j_2,b})) \\
 \text{(A.1)} \quad &= \frac{\widehat{f}_n(\lambda_{j_1,b}) \widehat{f}_n(\lambda_{j_2,b})}{\widetilde{f}_b(\lambda_{j_1,b}) \widetilde{f}_b(\lambda_{j_2,b})} \left[\frac{1}{N} \sum_{t=1}^N I_{t,b}(\lambda_{j_1,b}) I_{t,b}(\lambda_{j_2,b}) - \widetilde{f}_b(\lambda_{j_1,b}) \widetilde{f}_b(\lambda_{j_2,b}) \right] \\
 &=: Q_{j_1,j_2,n} \cdot H_{j_1,j_2,n}.
 \end{aligned}$$

Next, note that, due to the assumption $b^3/n = o(1)$, the expression appearing in Lemma 4.1(ii) has rate $o_P(b)$, which will be used in the ensuing calculation. Lemma 4.1 now yields

$$\begin{aligned}
 & \sum_{j_1, j_2 \in \mathcal{G}(b)} |H_{j_1, j_2, n} - \text{Cov}(I_{1, b}(\lambda_{j_1, b}), I_{1, b}(\lambda_{j_2, b}))| \\
 \text{(A.2)} \quad & \leq \sum_{j_1, j_2 \in \mathcal{G}(b)} \left| \frac{1}{N} \sum_{t=1}^N I_{t, b}(\lambda_{j_1, b}) I_{t, b}(\lambda_{j_2, b}) - E(I_{1, b}(\lambda_{j_1, b}) I_{1, b}(\lambda_{j_2, b})) \right| \\
 & \quad + \sum_{j_1, j_2 \in \mathcal{G}(b)} |\tilde{f}_b(\lambda_{j_1, b}) \tilde{f}_b(\lambda_{j_2, b}) - E I_{1, b}(\lambda_{j_1, b}) E I_{1, b}(\lambda_{j_2, b})| = o_P(b),
 \end{aligned}$$

the bound for the second sum expression can be deduced from Lemma 4.1(i) using $\sum_{j \in \mathcal{G}(b)} |\tilde{f}_b(\lambda_{j, b})| = \mathcal{O}_P(b)$ and $\sum_{j \in \mathcal{G}(b)} |E I_{1, b}(\lambda_{j, b})| = \mathcal{O}(b)$. The previous calculation (A.2) together with equation (2.8) of the Supplementary Material (Meyer, Paparoditis and Kreiss (2020)), can be used to derive $\sum_{j_1, j_2 \in \mathcal{G}(b)} |H_{j_1, j_2, n}| = \mathcal{O}_P(b)$. Moreover, notice that by Assumption 3 and by equation (2.12) of the Supplementary Material (Meyer, Paparoditis and Kreiss (2020)), which implies $\sup_{j \in \mathcal{G}(b)} |\tilde{f}_b(\lambda_{j, b}) - f(\lambda_{j, b})| = o_P(1)$ as well as the assumption that $f(\lambda)$ is bounded away from zero in the interval $[-\pi, \pi]$, we get that

$$\text{(A.3)} \quad \sup_{j \in \mathcal{G}(b)} \left| \frac{\widehat{f}_n(\lambda_{j, b})}{\tilde{f}_b(\lambda_{j, b})} - 1 \right| \xrightarrow{P} 0 \quad \text{and} \quad \sup_{j_1, j_2 \in \mathcal{G}(b)} \left| \frac{\widehat{f}_n(\lambda_{j_1, b}) \widehat{f}_n(\lambda_{j_2, b})}{\tilde{f}_b(\lambda_{j_1, b}) \tilde{f}_b(\lambda_{j_2, b})} - 1 \right| \xrightarrow{P} 0.$$

The results from (A.1) through (A.3) can be combined to replace bootstrap covariances by their respective nonbootstrap counterparts via

$$\begin{aligned}
 & \sum_{j_1, j_2 \in \mathcal{G}(b)} |\text{Cov}^*(I_b^{(l_1)}(\lambda_{j_1, b}), I_b^{(l_2)}(\lambda_{j_2, b})) - \text{Cov}(I_{1, b}(\lambda_{j_1, b}), I_{1, b}(\lambda_{j_2, b}))| \\
 \text{(A.4)} \quad & \leq \sup_{j_1, j_2 \in \mathcal{G}(b)} |Q_{j_1, j_2, n} - 1| \sum_{j_1, j_2 \in \mathcal{G}(b)} |H_{j_1, j_2, n}| \\
 & \quad + \sum_{j_1, j_2 \in \mathcal{G}(b)} |H_{j_1, j_2, n} - \text{Cov}(I_{1, b}(\lambda_{j_1, b}), I_{1, b}(\lambda_{j_2, b}))| = o_P(b).
 \end{aligned}$$

After these preliminaries, using conditional independence of $I_b^{(l_1)}(\lambda_{j_1, b})$ and $I_b^{(l_2)}(\lambda_{j_2, b})$ for $l_1 \neq l_2$, we get

$$\begin{aligned}
 \text{Var}^*(L_n^*) &= \frac{(2\pi)^2 k b}{b^2} \sum_{j_1, j_2 \in \mathcal{G}(b)} \varphi(\lambda_{j_1, b}) \varphi(\lambda_{j_2, b}) \text{Cov}^*(I_{j_1, b}^*, I_{j_2, b}^*) \\
 &= \frac{(2\pi)^2 k}{b} \sum_{j_1, j_2 \in \mathcal{G}(b)} \varphi(\lambda_{j_1, b}) \varphi(\lambda_{j_2, b}) \frac{1}{k} \text{Cov}^*(I_b^{(l_1)}(\lambda_{j_1, b}), I_b^{(l_2)}(\lambda_{j_2, b})) \\
 &= R1^* + R2^*,
 \end{aligned}$$

where R_1^* and R_2^* are defined as

$$\begin{aligned}
 R1^* &:= \frac{(2\pi)^2}{b} \sum_{j \in \mathcal{G}(b)} \varphi(\lambda_{j, b}) (\varphi(\lambda_{j, b}) + \varphi(-\lambda_{j, b})) \text{Var}^*(I_b^{(l_1)}(\lambda_{j, b})), \\
 \text{(A.5)} \quad R2^* &:= \frac{(2\pi)^2}{b} \sum_{j_1 \in \mathcal{G}(b)} \sum_{j_2 \in \mathcal{G}(b) \setminus \{j_1, -j_1\}} \varphi(\lambda_{j_1, b}) \varphi(\lambda_{j_2, b}) \\
 & \quad \times \text{Cov}^*(I_b^{(l_1)}(\lambda_{j_1, b}), I_b^{(l_2)}(\lambda_{j_2, b})).
 \end{aligned}$$

As for $R1^*$ and $R2^*$, note that $I_b^{(1)}(\lambda_{j_1,b}) = I_b^{(1)}(\lambda_{j_2,b})$ whenever $j_2 = j_1$ or $j_2 = -j_1$ and that $\lambda_{-j,b} = -\lambda_{j,b}$. $R1^*$ and $R2^*$ are bootstrap analogues of $R1$ and $R2$ from equation (2.7) in the Supplementary Material (Meyer, Paparoditis and Kreiss (2020)). Now, using (A.4), $|\mathcal{G}(b)| = \mathcal{O}(b)$ and the boundedness of φ , it follows that $|R1^* - R1|$ and $|R2^* - R2|$ vanish asymptotically in probability. Moreover, invoking the limiting results for $R1$ and $R2$ from the proof of Lemma 4.2(i), we have

$$(A.6) \quad \begin{aligned} R1^* &= R1 + o_P(1) = \tau_1^2 + o_P(1), \\ R2^* &= R2 + o_P(1) = \tau_2 + o_P(1). \end{aligned}$$

With $\tau^2 = \tau_1^2 + \tau_2$, this completes the proof of (i). \square

PROOF OF THEOREM 4.4(ii). Since L_n is asymptotically normal according to (2.2), which implies $\sup_{x \in \mathbb{R}} |P(L_n \leq x) - \Phi(x/\tau)| = o(1)$, it suffices to show $\sup_{x \in \mathbb{R}} |P^*(L_n^* \leq x) - \Phi(x/\tau)| = o_P(1)$. Notice that by definition of L_n^* and recalling that i_1^*, \dots, i_k^* are (conditionally) i.i.d. with a discrete uniform distribution on $\{1, \dots, N\}$, we have

$$\begin{aligned} L_n^* &= \frac{1}{\sqrt{k}} \sum_{l=1}^k \frac{2\pi}{\sqrt{b}} \sum_{j \in \mathcal{G}(b)} \varphi(\lambda_{j,b}) (I_{i_l^*,b}(\lambda_{j,b}) - \tilde{f}_b(\lambda_{j,b})) \\ &\quad + \frac{1}{\sqrt{k}} \sum_{l=1}^k \frac{2\pi}{\sqrt{b}} \sum_{j \in \mathcal{G}(b)} \varphi(\lambda_{j,b}) \left(\frac{\hat{f}_n(\lambda_{j,b})}{\tilde{f}_b(\lambda_{j,b})} - 1 \right) (I_{i_l^*,b}(\lambda_{j,b}) - \tilde{f}_b(\lambda_{j,b})) \\ &=: M1^* + M2^*. \end{aligned}$$

The strategy is to first show that $M2^*$ vanishes asymptotically in probability, and then to show that $M1^*$ is asymptotically normal with mean zero and with the proper variance τ^2 .

As for $M2^*$, we have, due to (conditional) independence of i_l^* and i_m^* for all $l \neq m$ and due to (A.1),

$$\begin{aligned} \text{Var}^*(M2^*) &= \frac{4\pi^2}{b} \sum_{j_1, j_2 \in \mathcal{G}(b)} \varphi(\lambda_{j_1,b}) \varphi(\lambda_{j_2,b}) \\ &\quad \times \left(\frac{\hat{f}_n(\lambda_{j_1,b})}{\tilde{f}_b(\lambda_{j_1,b})} - 1 \right) \left(\frac{\hat{f}_n(\lambda_{j_2,b})}{\tilde{f}_b(\lambda_{j_2,b})} - 1 \right) H_{j_1, j_2, n} \\ &\leq \frac{4\pi^2}{b} \left(\sup_{|\lambda| \leq \pi} |\varphi(\lambda)| \right)^2 \left(\sup_{j \in \mathcal{G}(b)} \left| \frac{\hat{f}_n(\lambda_{j,b})}{\tilde{f}_b(\lambda_{j,b})} - 1 \right| \right)^2 \sum_{j_1, j_2 \in \mathcal{G}(b)} |H_{j_1, j_2, n}| \\ &= o_P(1), \end{aligned}$$

where boundedness of φ , Assertion (A.3) and $\sum_{j_1, j_2 \in \mathcal{G}(b)} |H_{j_1, j_2, n}| = \mathcal{O}_P(b)$ from part (i) have been used. This is sufficient for $P^*(|M2^*| > \varepsilon) = o_P(1)$, $\forall \varepsilon > 0$, that is, $M2^*$ vanishes asymptotically in probability. As for $M1^*$, we can write $M1^* = \sum_{l=1}^k W_b^{(l)} / \sqrt{k}$ with $W_b^{(l)} := \frac{2\pi}{\sqrt{b}} \sum_{j \in \mathcal{G}(b)} \varphi(\lambda_{j,b}) (I_{i_l^*,b}(\lambda_{j,b}) - \tilde{f}_b(\lambda_{j,b}))$, and $W_b^{(1)}, \dots, W_b^{(k)}$ are, for each $n \in \mathbb{N}$ and conditionally on the original data sample X_1, \dots, X_n , i.i.d. random variables with a discrete uniform distribution on $\{W_{1,b}, \dots, W_{N,b}\}$, as defined in (4.1). Hence, $(W_b^{(1)}, \dots, W_b^{(k)})_{n \in \mathbb{N}}$ form a triangular array, and a conditional version of the Lindeberg–Feller CLT can be applied. From the above result and Assertion (i) of Theorem 4.4, we see that $\sum_{l=1}^k \text{Var}^*(W_b^{(l)} / \sqrt{k})$ converges to τ^2 in probability. Hence, it suffices to prove

$$(A.7) \quad \sum_{l=1}^k E^* \left((W_b^{(l)} / \sqrt{k})^2 \mathbb{1}_{\{|W_b^{(l)}| \geq \varepsilon \sqrt{k}\}} \right) = o_P(1) \quad \forall \varepsilon > 0,$$

in order for Lindeberg’s condition to be fulfilled and, therefore, the assertion to hold. In order to do this, we will first show that the sequence $(W_{1,b}^2)_{n \in \mathbb{N}}$ is uniformly integrable in the sense of Billingsley (1995), that is, that

$$(A.8) \quad \lim_{\alpha \rightarrow \infty} \sup_{n \in \mathbb{N}} E(W_{1,b}^2 \mathbb{1}_{\{W_{1,b}^2 \geq \alpha^2\}}) = 0.$$

When dealing with $(W_{1,b}^2)_{n \in \mathbb{N}}$ in the following, recall that $b = b(n)$ is suppressed in the notation for convenience reasons. From Lemma 4.2(ii) we already have $W_{1,b}^2 \xrightarrow{d} Z^2$, as $n \rightarrow \infty$, with $Z \sim \mathcal{N}(0, \tau^2)$. According to Theorem 25.6 of Billingsley (1995), there exists a probability space with random variables $(Y_n)_{n \in \mathbb{N}}$ and Y , with $Y \sim Z^2$ and $Y_n \sim W_{1,b}^2$ for all $n \in \mathbb{N}$, such that $Y_n \rightarrow Y$ almost surely. Moreover, Lemma 4.2(i) implies $E(Y_n) = E(W_{1,b}^2) \rightarrow \tau^2 = E(Z^2) = E(Y)$. Theorem 16.14(ii) of Billingsley (1995) states that this convergence of expectations together with almost sure convergence of Y_n yields uniform integrability of the nonnegative sequence $(Y_n)_{n \in \mathbb{N}}$, that is, $\lim_{\alpha \rightarrow \infty} \sup_{n \in \mathbb{N}} E(Y_n \mathbb{1}_{\{Y_n \geq \alpha^2\}}) = 0$. Due to $Y_n \sim W_{1,b}^2$, this assertion immediately gives (A.8). Now, we turn back to the Lindeberg condition. Since $\sqrt{k} \rightarrow \infty$, as $n \rightarrow \infty$, it can be verified in a straightforward way that (A.8) implies

$$(A.9) \quad E(W_{1,b}^2 \mathbb{1}_{\{|W_{1,b}| \geq \varepsilon \sqrt{k}\}}) = o(1) \quad \forall \varepsilon > 0.$$

It follows for arbitrary $\varepsilon > 0$, due to eighth-order stationarity of $(X_t)_{t \in \mathbb{Z}}$,

$$(A.10) \quad \begin{aligned} E\left(\sum_{l=1}^k E^*\left(\left(\frac{W_b^{(l)}}{\sqrt{k}}\right)^2 \mathbb{1}_{\{|W_b^{(l)}| \geq \varepsilon \sqrt{k}\}}\right)\right) &= E(E^*((W_b^{(1)})^2 \mathbb{1}_{\{|W_b^{(1)}| \geq \varepsilon \sqrt{k}\}})) \\ &= E\left(\frac{1}{N} \sum_{t=1}^N W_{t,b}^2 \mathbb{1}_{\{|W_{t,b}| \geq \varepsilon \sqrt{k}\}}\right) = o(1), \end{aligned}$$

due to (A.9). Assertion (A.10) is sufficient for (A.7) which completes the proof of (ii). \square

PROOF OF THEOREM 4.4(iii). Comparing the definitions of $L_{n,R}^*$ and $V_{2,n}^*$ from (3.5) and (3.6), respectively, and due to the structure of the function \tilde{w} defined in (3.8), $L_{n,R}^*$ can be written as

$$L_{n,R}^* = \frac{1}{\left(\frac{2\pi}{b} \sum_{j \in \mathcal{G}(b)} I_{j,b}^*\right) \left(\frac{2\pi}{b} \sum_{j \in \mathcal{G}(b)} \hat{f}_n(\lambda_{j,b})\right)} V_{2,n}^* =: \frac{1}{G_n^*} V_{2,n}^*.$$

In view of (2.6), it suffices to show $\sup_{x \in \mathbb{R}} |P^*(L_{n,R}^* \leq x) - \Phi(x/\tau_R)| = o_P(1)$, where Φ denotes the CDF of the standard normal distribution. Invoking a conditional version of Slutsky’s Lemma, this will be established by showing:

- (a) $\text{Var}^*(V_{2,n}^*) \xrightarrow{P} M(1, f)^4 \tau_R^2$,
- (b) $\sup_{x \in \mathbb{R}} |P^*(V_{2,n}^* \leq x) - \Phi(x/(M(1, f)^2 \tau_R))| = o_P(1)$,
- (c) $P^*(|G_n^* - M(1, f)^2| > \varepsilon) = o_P(1) \quad \forall \varepsilon > 0$.

Verify first that from Assumption 3 we have $\sup_{\lambda \in [-\pi, \pi]} |\tilde{w}(\lambda) - w(\lambda)| \xrightarrow{P} 0$. Moreover, the form of \tilde{w} yields $2\pi b^{-1} \sum_{j \in \mathcal{G}(b)} \tilde{w}(\lambda_{j,b}) \hat{f}_n(\lambda_{j,b}) = 0$. Hence,

$$(A.11) \quad V_{2,n}^* = \sqrt{kb} \frac{2\pi}{b} \sum_{j \in \mathcal{G}(b)} \tilde{w}(\lambda_{j,b}) (I_{j,b}^* - \hat{f}_n(\lambda_{j,b})),$$

which means that $V_{2,n}^*$ equals L_n^* if one replaces φ by its linear transformation \tilde{w} . Therefore, Assertion (a) follows with the same arguments as the result for $\text{Var}^*(L_n^*)$ in the proof of

Theorem 4.4(i), after replacing \tilde{w} by w via $\sup_{\lambda \in [-\pi, \pi]} |\tilde{w}(\lambda) - w(\lambda)| \xrightarrow{P} 0$. As for Assertion (c), Assumption 3 can be invoked to obtain

$$(A.12) \quad \begin{aligned} E^* \left(\frac{2\pi}{b} \sum_{j \in \mathcal{G}(b)} I_{j,b}^* \right) &= \frac{2\pi}{b} \sum_{j \in \mathcal{G}(b)} \hat{f}_n(\lambda_{j,b}) = \frac{2\pi}{b} \sum_{j \in \mathcal{G}(b)} f(\lambda_{j,b}) + o_P(1) \\ &= M(1, f) + o_P(1). \end{aligned}$$

Also, $\text{Var}^*(2\pi b^{-1} \sum_{j \in \mathcal{G}(b)} I_{j,b}^*) = (kb)^{-1} \text{Var}^*(L_n^*)$ if one uses the function $\varphi(\lambda) = 1$ in L_n^* . Therefore, $\text{Var}^*(2\pi b^{-1} \sum_{j \in \mathcal{G}(b)} I_{j,b}^*) = \mathcal{O}_P((kb)^{-1})$ by Theorem 4.4(i) which is sufficient for $P^*(|\frac{2\pi}{b} \sum_{j \in \mathcal{G}(b)} I_{j,b}^* - M(1, f)| > \varepsilon) = o_P(1) \forall \varepsilon > 0$. Since the second factor in G_n^* also converges to $M(1, f)$ in probability due to (A.12), Assertion (c) holds true.

Finally, to establish (b), note that from representation (A.11) $V_{2,n}^*$ can be decomposed as

$$(A.13) \quad \begin{aligned} V_{2,n}^* &= \frac{1}{\sqrt{k}} \sum_{l=1}^k \frac{2\pi}{\sqrt{b}} \sum_{j \in \mathcal{G}(b)} w(\lambda_{j,b}) (I_{l,b}^* - \tilde{f}_b(\lambda_{j,b})) \\ &+ \frac{1}{\sqrt{k}} \sum_{l=1}^k \frac{2\pi}{\sqrt{b}} \sum_{j \in \mathcal{G}(b)} (\tilde{w}(\lambda_{j,b}) - w(\lambda_{j,b})) (I_{l,b}^* - \tilde{f}_b(\lambda_{j,b})) \\ &+ \frac{1}{\sqrt{k}} \sum_{l=1}^k \frac{2\pi}{\sqrt{b}} \sum_{j \in \mathcal{G}(b)} \tilde{w}(\lambda_{j,b}) \left(\frac{\hat{f}_n(\lambda_{j,b})}{\tilde{f}_b(\lambda_{j,b})} - 1 \right) (I_{l,b}^* - \tilde{f}_b(\lambda_{j,b})). \end{aligned}$$

The first summand on the right-hand side equals expression $M1^*$ from the proof of Theorem 4.4(ii) if one replaces φ by w there. Hence, asymptotic normality of the first summand follows along these lines (and the proper variance has already been established in (a)). The third summand on the right-hand side of (A.13) can be treated similar to expression $M2^*$ from the proof of Theorem 4.4(ii) and, therefore, vanishes asymptotically in probability using $\sup_{\lambda \in [-\pi, \pi]} |\tilde{w}(\lambda) - w(\lambda)| \xrightarrow{P} 0$. With the same argument the second summand vanishes which establishes asymptotic normality of $V_{2,n}^*$ and completes the proof. \square

PROOF OF THEOREM 4.5(i). From the definition of $R1^*$ in (A.5) and from (A.1), it is obvious that actually $c_n = R1^*$. Hence, (A.6) implies $c_n = \tau_1^2 + o_P(1)$. In particular, the variance correction term c_n represents exactly the contribution of L_n^* to τ_1^2 of the limiting variance while $R2^*$ delivers the contribution to τ_2 . Now, since $\text{Var}^*(\tilde{V}_n^*) = \text{Var}^*(V_n^*) + \text{Var}^*(L_n^*) - c_n$, this completes the proof of (i) due to Proposition 4.3(i) and Theorem 4.4(i). \square

PROOF OF THEOREM 4.5(ii). Heuristically, the idea is that the (conditional) distribution of V_n^* converges to $\mathcal{N}(0, \tau_1^2)$ in probability, while

$$\hat{v}_n := \sqrt{1 + \frac{\hat{\tau}^2 - c_n}{\hat{\tau}_1^2}} = \frac{\sqrt{\hat{\tau}_1^2 + \hat{\tau}^2 - c_n}}{\hat{\tau}_1} = \frac{\tau}{\tau_1} + o_P(1),$$

and, therefore, the distribution of $\tilde{V}_n^* = \hat{v}_n \cdot V_n^*$ converges in probability to $\mathcal{N}(0, \tau^2)$. In the following we will prove this in a more rigorous way. Denote the CDF of the standard normal distribution by Φ . Observe that $\tau/\hat{v}_n = \tau_1 + o_P(1)$ yields with a standard argument that $\sup_{x \in \mathbb{R}} |\Phi(x/\tau_1) - \Phi(x\hat{v}_n/\tau)| = o_P(1)$. Using this assertion, we can proceed by

$$\begin{aligned} &\sup_{x \in \mathbb{R}} \left| P^*(\tilde{V}_n^* \leq x) - \Phi\left(\frac{x}{\tau}\right) \right| \\ &= \sup_{x \in \mathbb{R}} \left| P^*(V_n^* \leq x) - \Phi\left(\frac{x\hat{v}_n}{\tau}\right) \right| \end{aligned}$$

$$\leq \sup_{x \in \mathbb{R}} \left| P^*(V_n^* \leq x) - \Phi\left(\frac{x}{\tau_1}\right) \right| + \sup_{x \in \mathbb{R}} \left| \Phi\left(\frac{x}{\tau_1}\right) - \Phi\left(\frac{x}{(\tau/\hat{v}_n)}\right) \right| = o_P(1),$$

due to Proposition 4.3(ii). This completes the proof because (2.2) implies $\sup_{x \in \mathbb{R}} |P(L_n \leq x) - \Phi(x/\tau)| = o(1)$. \square

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SUPPLEMENTARY MATERIAL

Supplement to “Extending the validity of frequency domain bootstrap methods to general stationary processes” (DOI: [10.1214/19-AOS1892SUPP](https://doi.org/10.1214/19-AOS1892SUPP); .pdf). The online supplement contains theoretical results regarding the validity of the CBP, the proofs that were omitted in this paper and some additional numerical results.

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