The Annals of Probability
2018, Vol. 46, No. 3, 1597–1650
https://doi.org/10.1214/17-AOP1211

OPTIMAL SURVIVING STRATEGY FOR DRIFTED BROWNIAN MOTIONS WITH ABSORPTION

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We study the "Up the River" problem formulated by Aldous (2002), where a unit drift is distributed among a finite collection of Brownian particles on $\mathbb{R}_+$, which are annihilated once they reach the origin. Starting K particles at $x = 1$, we prove Aldous' conjecture [Aldous (2002)] that the "push-the-laggard" strategy of distributing the drift asymptotically (as $K \to \infty$) maximizes the total number of surviving particles, with approximately $\frac{4}{\sqrt{\pi}}\sqrt{K}$ surviving particles. We further establish the hydrodynamic limit of the particle density, in terms of a two-phase partial differential equation (PDE) with a moving boundary, by utilizing certain integral identities and coupling techniques.

CONTENTS

1. Introduction.

In this paper, we study the "Up the River" problem formulated by Aldous [1]. That is, we consider K independent Brownian particles, which all start at $x = 1$, and are absorbed (annihilated) once they hit $x = 0$. Granted a unit drift, we ask what is the optimal strategy of dividing and allocating the drift among all surviving particles in order to maximize the number of particles that survive forever. More precisely, letting $B_i(t)$, $i = 1, \dots, K$, denote independent standard Brownian motions, we define the model as an $\mathbb{R}_+^K$-valued diffusion

Received December 2015; revised July 2017.
[1]Supported in part by NSF Grant DMS-07-09248.
MSC2010 subject classifications. Primary 60K35; secondary 35Q70, 82C22.
Key words and phrases. Atlas model, competing Brownian particles, hydrodynamic limit, Stefan problems, moving boundary.

$(X_i(t); t \geq 0)_{i=1}^K$, satisfying

$$X_i(t) = 1 + B_i(t \wedge \tau_i) + \int_0^{t \wedge \tau_i} \phi_i(s)\, ds. \tag{1.1}$$

Here $\tau_i := \inf\{t > 0 : X_i(t) = 0\}$ denotes the absorption time of the ith particle, and the strategy is any $[0, 1]^K$-valued, $\{B_i(t)\}_{i=1}^K$-progressively measurable function $(\phi_i(t); t \geq 0)_{i=1}^K$ such that $\sum_{i=1}^K \phi_i(t) \leq 1$, $\forall t \geq 0$. Our goal is to maximize $\widetilde{U}(\infty)$, where

$$\widetilde{U}(\infty) := \lim_{t\to\infty} \widetilde{U}(t), \qquad \widetilde{U}(t) := \#\{i : X_i(t) > 0\}.$$

Here, $\widetilde{U}(t)$ actually depends on K, but we *suppress* the dependence in this notation, and reserve notation such as $\widetilde{U}_K(t)$ for *scaled* quantities. Inspired by the "Up the River: Race to the Harbor" board game, this simple model serves as a natural optimization problem for a random environment with limited resources. For $K = 2$, [17] obtains an explicit expression of the law of $\widetilde{U}(t)$, and for large K, numerical results are obtained in [11] for the discrete analog of (1.1).

Focusing on the asymptotic behavior as $K \to \infty$, we prove that the optimal strategy is the naïve *push-the-laggard strategy*

$$\phi_i(t) := \mathbf{1}_{\{X_i(t)=Z(t)\}} \qquad \text{where } Z(t) := \min\{X_i(t) : X_i(t) > 0\}, \tag{1.2}$$

which allocates all the unit drift on the *laggard* $Z(t)$.

REMARK 1.1. Due to the recursive nature of Brownian motions in one dimension, ties do occur in (1.2), namely $\mathbf{P}(\#\{i : X_i(s) = Z(s)\} > 1, \text{for some } s \leq t) > 0$, for all large enough t. Here, we break the ties in an *arbitrarily* fixed manner. That is, any strategy $(\phi_i(t))_{i=1}^K$ satisfying

$$\sum_{i:X_i(t)=Z(t)} \phi_i(t) = 1 \tag{1.3}$$

is regarding as a push-the-laggard strategy. As the analysis in this paper is independent of the exact choice of breaking the ties, hereafter we fix some arbitrary way of breaking the ties and refer to (1.3) as *the* push-the-laggard strategy.

Furthermore, we prove that, due to self-averaging, $\widetilde{U}(\infty)$ is in fact deterministic to the leading order, under the push-the-laggard strategy. More explicitly, $\widetilde{U}(\infty) \approx \frac{4}{\sqrt{\pi}} K^{1/2}$. Define the scaled process

$$\widetilde{U}_K(t) := \frac{1}{\sqrt{K}} \widetilde{U}(tK).$$

The following is our main result.

THEOREM 1.2. (a) *Regardless of the strategy, for any fixed* $n < \infty$ *and* $\gamma \in (0, \frac{1}{4})$, *we have*

$$\mathbf{P}\Big(\widetilde{U}_K(\infty) \le \frac{4}{\sqrt{\pi}} + K^{-\gamma}\Big) \ge 1 - CK^{-n} \qquad \forall K < \infty, \tag{1.4}$$

where $C = C(n, \gamma) < \infty$ *depends only on* n *and* γ, *not on the strategy.*

(b) *Under the push-the-laggard strategy, for any fixed* $\gamma \in (0, \frac{1}{96})$ *and* $n < \infty$, *we have*

$$\mathbf{P}\Big(\Big|\widetilde{U}_K(\infty) - \frac{4}{\sqrt{\pi}}\Big| \le K^{-\gamma}\Big) \ge 1 - CK^{-n} \qquad \forall K < \infty, \tag{1.5}$$

where $C = C(\gamma, n) < \infty$ *depends only on* γ *and* n.

REMARK 1.3. While the exponent $\frac{1}{4}^{-}$ of the error term in Theorem 1.2(a) (originating from the control on the relevant martingales) is optimal, the choice of exponent $\gamma \in (0, \frac{1}{96})$ in Theorem 1.2(b) is purely technical. The latter may be improved by establishing sharper estimates, which we do not pursue in this paper.

Theorem 1.2 resolves Aldous' conjecture ([1], Conjecture 2) in a slightly different form. The intuition leading to such a theorem, as well as the the main ingredient of proving it, is the hydrodynamic limit picture given in [1]. To be more precise, we consider the diffusively scaled process $X_i^K(t) := \frac{1}{\sqrt{K}} X_i(tK)$ and let $Z_K(t) := \frac{1}{\sqrt{K}} Z(tK)$ denote the scaled process of the laggard. Consider further the scaled complementary distribution function

$$\widetilde{U}_K(t, x) := \frac{1}{\sqrt{K}} \#\{X_i^K(t) > x\}, \tag{1.6}$$

and let $p(t, x) := \frac{1}{\sqrt{2\pi t}} \exp(-\frac{x^2}{2t})$ denote the standard heat kernel. Under the push-the-laggard strategy, we expect $(\widetilde{U}_K(t, x), Z_K(t))$ to be well approximated by $(\widetilde{U}_\star(t, x), z_\star(t))$. Here, $\widetilde{U}_\star(t, x)$ and $z_\star(t)$ are deterministic functions, which are defined in *two separated phases* as follows. For $t \le \frac{1}{2}$, the *absorption phase*, we define

$$\widetilde{U}_\star(t, x) := 2p(t, x) + \int_0^t 2p(t - s, x)\, ds \qquad \forall t \le \frac{1}{2}, x \ge 0, \tag{1.7}$$

$$z_\star(t) := 0 \qquad \forall t \le \frac{1}{2}. \tag{1.8}$$

For $t > \frac{1}{2}$, the *moving boundary phase*, letting $p^{\mathrm{N}}(t, y, x) := p(t, y - x) + p(t, y + x)$ denote the Neumann heat kernel, we define

$$\begin{aligned} \widetilde{U}_\star(t, x) &:= 2p(t, x) \\ &\quad + \int_0^t p^{\mathrm{N}}(t - s, z_\star(s), x)\, ds \qquad \forall t \ge \frac{1}{2}, x \ge z_\star(t), \end{aligned} \tag{1.9}$$

where $z_\star(t)$ is the unique solution to the following integral equation:

$$\begin{cases} z_\star\Big(\cdot+\dfrac{1}{2}\Big)\in\mathcal{C}(\mathbb{R}_+) \qquad \text{nondecreasing}\ , z_\star\Big(\dfrac{1}{2}\Big)=0, \\ \displaystyle\int_0^\infty p\Big(t-\frac{1}{2}, z_\star(t)-y\Big)\Big(\widetilde{U}_\star\Big(\frac{1}{2},0\Big)-\widetilde{U}_\star\Big(\frac{1}{2},y\Big)\Big)\,dy \\ \qquad = \displaystyle\int_{\frac{1}{2}}^t p\big(t-s, z_\star(t)-z_\star(s)\big)\,ds \qquad \forall t\in\Big(\frac{1}{2},\infty\Big). \end{cases} \tag{1.10}$$

As we show in Section 3, the integral equation (1.10) admits a unique solution.

The pair $(\widetilde{U}_\star, z_\star)$, defined by (1.7)–(1.10), is closely related to certain PDE problems, as follows. Let $\widetilde{\Phi}(t,y) := \mathbf{P}(B(t)>y)$ denote the Brownian tail distribution function. For $t\le\frac{1}{2}$, a straightforward calculation (see Remark 1.4) shows that the function $\widetilde{U}_\star(t,x)$ in (1.7) is written as the tail distribution function of $u_1(t,x)$:

$$\widetilde{U}_\star(t,x)=\int_x^\infty u_1(t,y)\,dy \qquad \forall t\le\frac{1}{2}, \tag{1.11}$$

where $u_1(t,x)$ is defined as

$$u_1(t,x) := -2\partial_x p(t,x)+4\widetilde{\Phi}(t,x). \tag{1.12}$$

It is straightforward to check that this density function u_1 solves the heat equation on $x>0$ with a boundary condition $u_1(t,0)=2$:

$$\partial_t u_1=\frac{1}{2}\partial_{xx}u_1 \qquad \forall 0<t<\frac{1}{2}, x>0, \tag{1.13a}$$

$$u_1(t,0)=2 \qquad \forall 0<t<\frac{1}{2}, \tag{1.13b}$$

$$\lim_{t\downarrow 0}\big(u_1(t,x)+2\partial_x p(t,x)\big)=0 \qquad \forall x\ge 0. \tag{1.13c}$$

For $t>\frac{1}{2}$, we consider the following *Stefan problem*, a PDE with a *moving boundary*:

$$z_2\in\mathcal{C}\Big([\frac{1}{2},\infty\Big)) \qquad \text{nondecreasing}, \qquad z_2\Big(\frac{1}{2}\Big)=0, \tag{1.14a}$$

$$\partial_t u_2=\frac{1}{2}\partial_{xx}u_2 \qquad \forall t>\frac{1}{2}, x>z_2(t) \tag{1.14b}$$

$$u_2\Big(\frac{1}{2},x\Big)=u_1\Big(\frac{1}{2},x\Big) \qquad \forall x\ge 0, \tag{1.14c}$$

$$u_2\big(t,z_2(t)\big)=2 \qquad \forall t\ge\frac{1}{2}, \tag{1.14d}$$

$$2\frac{d}{dt}z_2(t)+\frac{1}{2}\partial_x u_2\big(t,z_2(t)\big)=0 \qquad \forall t>\frac{1}{2}. \tag{1.14e}$$

As we show in Lemma 3.1, for each sufficiently smooth solution (u_2, z_2) to (1.14), the functions $\widetilde{U}_\star(t,x) := \int_x^\infty u_2(t,y)\mathbf{1}_{\{y\geq z_\star(t)\}}\,dy$ and $z_\star(t) := z_2(t)$ satisfy (1.9)–(1.10) for $t \geq \frac{1}{2}$.

REMARK 1.4. To see why (1.11) holds, differentiate (1.7) in x to obtain $\partial_x \widetilde{U}_\star(t,x) = 2\partial_x p(t,x) - 2\int_0^t \frac{x}{t-s} p(t-s,x)\,ds$. Within the last integral, performing the change of variable $y := \frac{x}{\sqrt{t-s}}$, we see that $\int_0^t \frac{x}{t-s} p(t-s,x)\,ds = 2\widetilde{\Phi}(t-s,x)$. From this, (1.11) follows.

REMARK 1.5. Note that for equation (1.14) to make sense classically, one needs $u_2(t,x)$ to be $\mathcal{C}^1$ *up to* the boundary $\{(t, z_2(t)) : t \geq 0\}$ and needs $z_2(t)$ to be $\mathcal{C}^1$. Here, instead of defining the hydrodynamic limit classically through (1.14), we take the integral identity and integral equation (1.9)–(1.10) as the *definition* of the hydrodynamic limit equation. This formulation is more convenient for our purpose, and in particular it requires neither the smoothness of $u_\star$ onto the boundary nor the smoothness of $z_\star$. We note that, however, it should be possible to establish classical solutions to (1.14), by converting (1.14) to a parabolic variational inequality; see, for example, [10]. We do not pursue this direction here.

Before stating the precise result on hydrodynamic limit, we explain the intuition of how (1.13)–(1.14) arise from the behavior of the particle system. Indeed, the heat equations (1.13a) and (1.14b) model the diffusive behavior of $(X_i^K(t))_i$ away from $Z_K(t)$. In view of the equilibrium measure of gaps of the infinite Atlas model [18], near $Z_K(t)$ we expect the particle density to be 2 to balance the drift exerted on $Z_K(t)$, yielding the boundary conditions (1.13b) and (1.14d). The function $-2\partial_x p(t,x)$ is the average density of the system without the drift. [The singularity of $-2\partial_x p(t,x)$ at $t = 0$ captures the overabundance of particles at $t = 0$ compared to the scaling $K^{1/2}$.] As the drift affects little of the particle density near $t = 0$, we expect the entrance law (1.13c). The absorption phase ($t \leq \frac{1}{2}$) describes the initial state of the particle system with a high density, where particles are constantly being absorbed, yielding a fixed boundary $Z_K(t) \approx 0$. Under the push-the-laggard strategy, the system enters a new phase at $t \approx \frac{1}{2}$, where the density of particles is low enough (≤ 2 everywhere) so that the drift carries all remaining particles away from 0. This results in a moving boundary $Z_K(t)$, with an additional boundary condition (1.14e), which simply paraphrases the conservation of particles $\frac{d}{dt}\int_{z_2(t)}^\infty u_2(t,y)\,dy = 0$.

The following is our result on the hydrodynamic limit of $(\widetilde{U}_K(t,x), Z_K(t))$.

THEOREM 1.6 (Hydrodynamic limit). *Under the push-the-laggard strategy, for any fixed $\gamma \in (0, \frac{1}{96})$ and $T, n < \infty$, there exists $C = C(T, \gamma, n) < \infty$ such* *that*

$$\mathbf{P}\Big(\sup_{t\in[0,T],x\in\mathbb{R}}\{|\widetilde{U}_K(t,x)-\widetilde{U}_\star(t,x)|t^{\frac{3}{4}}\} \le CK^{-\gamma}\Big)\ge 1-CK^{-n} \qquad \forall K<\infty, \tag{1.15}$$

$$\mathbf{P}\Big(\sup_{t\in[0,T]}|Z_K(t)-z_\star(t)|\le CK^{-\gamma}\Big) \ge 1-CK^{-n} \qquad \forall K<\infty. \tag{1.16}$$

REMARK 1.7. The factor of $t^{\frac{3}{4}}$ in (1.15) is in place to regulate the singularity of $\widetilde{U}_K(t,x)$ and $\widetilde{U}_\star(t,x)$ near $t=0$. Indeed, with $\widetilde{U}_\star(t,x)$ defined in (1.11) for $t\le\frac{1}{2}$, it is standard to verify that $\sup_{x\in\mathbb{R}}\widetilde{U}_\star(t,x)$ diverges as $\frac{2}{\sqrt{2\pi t}}$ as $t\downarrow 0$. With $\widetilde{U}_K(t,x)$ defined in (1.6), we have that $\widetilde{U}_K(0,x)=\sqrt{K}\,\mathbf{1}_{\{x<1/\sqrt{K}\}}$, which diverges at $x=0$ as $K\to\infty$. This singularity at $t=0$ of $\widetilde{U}_K(t,x)$ propagates into $t>0$, resulting in a power law singularity of the form $|t|^{-\frac{1}{2}}$.

The choice of the exponent $\frac{3}{4}$ in (1.15) is technical, and may be sharpened to $\frac{1}{2}$ as discussed in the preceding, but we do not pursue this direction here.

Under the push-the-laggard strategy (1.2), the process $(X_i^K(t))_i$ is closely related to the Atlas model [9]. The latter is a simple special case of diffusions with rank-dependent drift; see [2, 4, 5, 13–15], for their ergodicity and sample path properties, and [6, 19] for their large deviations properties as the dimension tends to infinity. In particular, the hydrodynamic limit and fluctuations of the Atlas-type model have been analyzed in [3, 7, 12].

Here, we take one step further and analyze the combined effect of rank-dependent drift and absorption, whereby demonstrating the two-phase behavior. With the absorption at $x=0$, previous methods of analyzing the large scale behaviors of diffusions with rank-dependent drift do not apply. In particular, the challenge of proving Theorem 1.6 originates from the lack of invariant measure (for the absorption phase) and the singularity at $t=0$, where a rapid transition from K particles to an order of $K^{1/2}$ particles occurs. Here, we solve the problem by adopting a *new* method of exploiting certain integral identities of the particle system that mimic (1.7)–(1.10). Even though here we mainly focus on the push-the-laggard strategy under the initial condition $X_i(0)=1,\forall i$, the integral identities apply to general rank-dependent drifts and initial conditions, and may be used for analyzing for general models with both rank-dependent drifts and absorption.

Outline. In Section 2, we develop certain integral identities of the particle system X that are crucial for our analysis, and in Section 3, we establish the necessary tools pertaining to the integral equation (1.10). Based on results obtained in Sections 2–3, in Sections 4 and 5 we prove Theorems 1.6 and 1.2, respectively.

2. Integral identities. Recall that $p^{\mathrm{N}}(t,x,y)$ denotes the Neumann heat kernel, and let $\Phi(t,x) := \mathbf{P}(B(t) \le x) = 1 - \widetilde{\Phi}(t,x)$ denote the Brownian distribution function. With $z_\star(t)$ as in (1.8) and (1.10), we unify the integral identities (1.7) and (1.9) into a single expression as

$$\begin{aligned} \widetilde{U}_\star(t,x) &= 2p(t,x) \\ &\quad + \int_0^t p^{\mathrm{N}}\bigl(t-s, z_\star(s), x\bigr)\,ds \qquad \forall t>0, x \ge z_\star(t). \end{aligned} \tag{2.1}$$

Essential to our proof of Theorems 1.2 and 1.6 are certain integral identities of the *particle system* $X = (X(t); t \ge 0)$ that mimic the integral identities (2.1). This section is devoted to deriving such identities of the particle system, particularly Proposition 2.6 in the following.

As it turns out, in addition to the particle system X, it is helpful to consider also the Atlas models. We say that $Y = (Y_i(t); t \ge 0)_{i=1}^m$ is an *Atlas model* with m particles if it evolves according to the following system of stochastic differential equations:

$$\begin{aligned} dY_i(t) &= \mathbf{1}_{\{Y_i(t)=W(t)\}}\,dt + dB_i(t) \qquad \text{for } 1 \le i \le m, \\ W(t) &:= \min\{Y_i(t)\}. \end{aligned} \tag{2.2}$$

We similarly define the scaled processes $Y_i^K(t) := \frac{1}{\sqrt{K}} Y_i(tK)$ and $W_K(t) := \frac{1}{\sqrt{K}} W(tK)$. Note that here K is just a scaling parameter, not necessarily related to the number of particles in Y.

To state the first result of this section, we first prepare some notation. Define the scaled empirical measures of X and Y as

$$\mu_t^K(\cdot) := \frac{1}{\sqrt{K}} \sum\nolimits_{\{i: X_i^K(t) > 0\}} \delta_{X_i^K(t)}(\cdot), \tag{2.3}$$

$$\nu_t^K(\cdot) := \frac{1}{\sqrt{K}} \sum_i \delta_{Y_i^K(t)}(\cdot). \tag{2.4}$$

For any fixed $x \ge 0$, consider the tail distribution function $\Psi(t,y,x) := \mathbf{P}(B_x^{\mathrm{ab}}(t) > y)$, $y > 0$, of a Brownian motion B_x^{ab}, starting at $B_x^{\mathrm{ab}}(0) = x$ and absorbed at 0. More explicitly,

$$\Psi(t,y,x) := \Phi(t, y-x) - \widetilde{\Phi}(t, y+x), \tag{2.5}$$

which is the unique solution to the following equation:

$$\partial_t \Psi(t,y,x) = \frac{1}{2}\partial_{yy}\Psi(t,y,x) \qquad \forall t, y > 0, \tag{2.6a}$$

$$\Psi(t,0,x) = 0 \qquad \forall t > 0, \tag{2.6b}$$

$$\Psi(0,y,x) = \mathbf{1}_{(x,\infty)}(y) \qquad \forall y > 0. \tag{2.6c}$$

Adopt the notation $t_K := t + \frac{1}{K}$, $\tau_i^K := K^{-1}\tau_i$ and $\phi_i^K(t) := \phi_i(Kt)$ hereafter.

LEMMA 2.1. (a) *For the particle system* $(X(t); t \geq 0)$, *under any strategy*, *we have the following integral identity*:

$$\begin{aligned}&\Big\langle \mu_t^K, \Psi\Big(\frac{1}{K}, \cdot, x\Big)\Big\rangle \\ &\quad = \widetilde{G}_K(t_K, x) + \sum_{i=1}^{K} \int_0^t \phi_i^K(s) p^N\big(t_K - s, X_i^K(s), x\big)\, ds \\ &\qquad + M_K(t, x) \qquad \forall t \in \mathbb{R}_+, x \geq 0,\end{aligned} \tag{2.7}$$

where

$$\widetilde{G}_K(t, x) := \sqrt{K}\Psi\Big(t, \frac{1}{\sqrt{K}}, x\Big), \tag{2.8}$$

$$M_K(t, x) := \frac{1}{\sqrt{K}} \sum_{i=1}^{K} \int_0^{t \wedge \tau_i^K} p^N\big(t_K - s, X_i^K(s), x\big)\, dB_i^K(s). \tag{2.9}$$

(b) *Let* $(Y_i(t); t \geq 0)_i$ *be an Altas model*. *We have the following integral identity*:

$$\begin{aligned}&\Big\langle \nu_t^K, \Phi\Big(\frac{1}{K}, x - \cdot\Big)\Big\rangle \\ &\quad = \big\langle \nu_0^K, \Phi(t_K, x - \cdot)\big\rangle - \int_0^t p\big(t_K - s, x - W_K(s)\big)\, ds \\ &\qquad - N_K(t, x) \qquad \forall t \in \mathbb{R}_+, x \in \mathbb{R},\end{aligned} \tag{2.10}$$

where

$$N_K(t, x) := \frac{1}{\sqrt{K}} \sum_i \int_0^t p\big(t_K - s, Y_i^K(s) - x\big)\, dB_i^K(s). \tag{2.11}$$

REMARK 2.2. To motivate our analysis in the following, here we explain the meaning of each term in the integral identity (2.7). From the definitions (1.6) and (2.3) of $\widetilde{U}_K(t, x)$ and μ_t^K, we have that $\lim_{\varepsilon \to 0} \langle \mu_t^K, \Psi(\varepsilon, \cdot, x)\rangle = \widetilde{U}_K(t, x)$, so it is reasonable to expect the term $\langle \mu_t^K, \Psi(\frac{1}{K}, \cdot, x)\rangle$ on the left-hand side to approximate $\widetilde{U}_K(t, x)$ as $K \to \infty$.

Next, consider a system $(X_i^{\mathrm{ab}}(t); t \geq 0)_{i=1}^K$ of independent Brownian particles starting at $x = 1$ and absorbed at $x = 0$, *without* drifts. Letting $X_i^{\mathrm{ab},K}(t) := \frac{1}{\sqrt{K}} X_i^{\mathrm{ab}}(Kt)$ denote the diffusively scaled process, with the corresponding scaled tailed distribution function

$$\widetilde{U}_K^{\mathrm{ab}}(t, x) := \frac{1}{\sqrt{K}} \#\big\{i : X_i^{\mathrm{ab},K}(t) > x\big\}, \tag{2.12}$$

it is standard to show that

$$\mathbf{E}\big(\widetilde{U}_K^{\mathrm{ab}}(t,x)\big) = \sqrt{K}\,\mathbf{P}\big(X_1^{\mathrm{ab}}(Kt) > \sqrt{K}x\big) \tag{2.13}$$
$$= \sqrt{K}\Psi(Kt, 1, \sqrt{K}x) = \widetilde{G}_K(t,x).$$

That is, the term $\widetilde{G}_K(t,x)$ on the right-hand side (2.7) accounts for the contribution (in expectation) of the *absorption*.

Subsequent, the time integral term $\sum_{i=1}^{K}\int_0^{\tau_i^K}(\dots)\,ds$ arises from the contribution of the drifts $(\phi_i(t))_{i=1}^K$ allocated to the particles, while the martingale term $M(t,x)$ encodes the random fluctuation due to the Brownian nature of the particles.

PROOF OF LEMMA 2.1. Under the diffusive scaling $X_i^K(t) := \frac{1}{\sqrt{K}}X_i(tK)$, we rewrite the SDE (1.1) as

$$dX_i^K(t) = \phi_i^K(t)\sqrt{K}d\big(t\wedge\tau_i^K\big) + dB_i^K\big(t\wedge\tau_i^K\big). \tag{2.14}$$

Fixing arbitrary $t<\infty$, $x\geq 0$, with Ψ solving (2.6a), we apply Itô's formula to $F_i(s) := \Psi(t_K - s, X_i^K(s), x)$ using (2.14) to obtain

$$F_i\big(t\wedge\tau_i^K\big) - F_i(0) \tag{2.15}$$
$$= \sqrt{K}\int_0^{t\wedge\tau_i^K}\phi_i^K(s)p^{\mathrm{N}}\big(t_K - s, X_i^K(s), x\big)\,ds + M_{i,K}(t,x),$$

where $M_{i,K}(t,x) := \int_0^{t\wedge\tau_i^K} p^{\mathrm{N}}(t_K - s, X_i^K(s), x)\,dB_i^K(s)$. With $\Psi(s,0,x)=0$, we have $F_i(t\wedge\tau_i^K) = \Psi(\frac{1}{K}, X_i^K(t), x)$. Using this in (2.15), summing the result over i, and dividing both sides by $\sqrt{K}$, we conclude the desired identity (2.7). Similarly, the identity (2.66) follows by applying Itô's formula with the test function $\Phi(t_K - s, y - x)$. $\square$

Based on the identities (2.7) and (2.10), we proceed to establish bounds on the empirical measures μ_t^K and ν_t^K. Hereafter, we use $C = C(\alpha,\beta,\dots)<\infty$ to denote a generic deterministic finite constant that may change from line to line, but depends only on the designated variables. In the following, we will use the estimates of the heat kernel $p(t,x)$. The proof is standard and we omit it here:

$$\big|p(t,x) - p\big(t,x'\big)\big| \leq C(\alpha)\big|x - x'\big|^{\alpha}t^{-\frac{1+\alpha}{2}}, \qquad \alpha\in(0,1], \tag{2.16}$$

$$\big|p(t,x) - p\big(t',x\big)\big| \tag{2.17}$$
$$\leq C(\alpha)\big|t - t'\big|^{\frac{\alpha}{2}}\big(t'\big)^{-\frac{1+\alpha}{2}}, \qquad \alpha\in(0,1], t' < t < \infty.$$

We adopt the standard notation $\|\xi\|_n := (\mathbf{E}|\xi|^n)^{\frac{1}{n}}$ for the L^n-norm of a give random variable ξ and $|f|_{L^\infty(\Omega)} := \sup_{x\in\Omega}|f(x)|$ for the uniform norm over the designated region Ω.

LEMMA 2.3. *Let $(Y_i(t); t \geq 0)_i$ be an Atlas model. The total number $\#\{Y_i(0)\}$ of particles may be random but is independent of $\sigma(Y_i(t) - Y_i(0); t \geq 0, i = 1, \dots)$. Let ν_t^K to be as in (2.4). Assume $(Y_i^K(0))_i$ satisfies the following initial condition: given any $\alpha \in (0,1)$ and $n < \infty$, there exist $D_*, D_{\alpha,n} < \infty$ such that*

$$\mathbf{P}\big(\#\{Y_i(0)\} \leq K\big) \geq 1 - \exp\Big(-\frac{1}{D_*} K^{\frac{1}{2}}\Big), \tag{2.18}$$

$$\big\|\langle \nu_0^K, \mathbf{1}_{[a,b]}\rangle\big\|_n \leq D_{\alpha,n}|b-a|^\alpha \qquad \forall |b-a| \geq \frac{1}{\sqrt{K}}. \tag{2.19}$$

For any given $T < \infty$, we have

$$\begin{aligned}&\big\|\langle \nu_s^K, \mathbf{1}_{[a,b]}\rangle\big\|_n \\ &\quad \leq C|b-a|^\alpha\Big(\Big(\frac{|b-a|}{\sqrt{s_K}}\Big)^{1-\alpha} + 1\Big) \qquad \forall \frac{1}{\sqrt{K}} \leq |b-a|, s \leq T,\end{aligned} \tag{2.20}$$

$$\begin{aligned}&\big\|\langle \nu_s^K, p(t_K, \cdot - x)\rangle\big\|_n \\ &\quad \leq C t_K^{\frac{\alpha-1}{2}}\Big(\Big(\frac{t_K}{s_K}\Big)^{\frac{1-\alpha}{2}} + 1\Big) \qquad \forall x \in \mathbb{R}, s, t < T,\end{aligned} \tag{2.21}$$

where $C = C(T, \alpha, n, D_*, D_{\alpha,n}) < \infty$.

PROOF. Fixing such T, α, n and $[a,b]$, throughout this proof we use $C = C(T, \alpha, n, D_*, D_{\alpha,n}) < \infty$ to denote a generic finite constant. To the end of showing (2.20), we begin by estimating $\|\langle \nu_s^K, \mathbf{1}_{[a,b]}\rangle\|_1 = \mathbf{E}(\langle \nu_s^K, \mathbf{1}_{[a,b]}\rangle)$. To this end, we set $x = b, a$ in (2.10), take the difference of the resulting equation, and take expectations of the result to obtain

$$\mathbf{E}\Big(\Big\langle \nu_s^K, \Phi\Big(\frac{1}{K}, b - \cdot\Big) - \Phi\Big(\frac{1}{K}, a - \cdot\Big)\Big\rangle\Big) = \mathbf{E}(J_1) + \mathbf{E}(J_2), \tag{2.22}$$

where

$$J_1 := \big\langle \nu_0^K, \Phi(s_K, b - \cdot) - \Phi(s_K, a - \cdot)\big\rangle,$$

$$J_2 := -\int_0^s \big(p\big(s_K - u, b - W_K(u)\big) - p\big(s_K - u, a - W_K(u)\big)\big)\,du.$$

Further, with $|b-a| \geq K^{-\frac{1}{2}}$, it is straightforward to verify that

$$\Phi\Big(\frac{1}{K}, b - y\Big) - \Phi\Big(\frac{1}{K}, a - y\Big) \geq \frac{1}{C}\mathbf{1}_{[a,b]}(y). \tag{2.23}$$

Combining (2.23) and (2.22) yields

$$\big\|\langle \nu_s^K, \mathbf{1}_{[a,b]}\rangle\big\|_1 \leq C\,\mathbf{E}(J_1) + C\,\mathbf{E}(J_2). \tag{2.24}$$

With (2.24), our next step is to bound $\mathbf{E}(J_1)$ and $\mathbf{E}(J_2)$. For the former, we use $\Phi(t_K, b-y) - \Phi(t_K, a-y) = \int_a^b p(t_K, z-y)\,dz$ to write $J_1 = \int_a^b \langle \nu_0^K, p(s_K, x-\cdot)\rangle\,dx$. Taking the L^m-norm of the last expression yields

$$\|J_1\|_m \le \int_a^b \big\|\langle \nu_0^K, p(s_K, x-\cdot)\rangle\big\|_m\,dx \qquad \forall m \in \mathbb{N}. \tag{2.25}$$

Further, as the heat kernel $p(t, y-x) = \frac{1}{\sqrt{t}} p(1, \frac{y-x}{\sqrt{t}})$ decreases in $|y-x|$, letting $I_j(t,x) := x + [j\sqrt{t}, (j+1)\sqrt{t}]$ and $j_* := |j| \wedge |j+1|$, we have

$$\begin{aligned}\big\|\langle \nu_s^K, p(t, \cdot - x)\rangle\big\|_m &\le \bigg\|\sum_{j\in\mathbb{Z}} \frac{1}{\sqrt{t}} p(1, j_*)\langle \nu_s^K, \mathbf{1}_{I_j(t,x)}\rangle\bigg\|_m \\ &\le \sum_{j\in\mathbb{Z}} \frac{1}{\sqrt{t}} p(1, j_*)\big\|\langle \nu_s^K, \mathbf{1}_{I_j(t,x)}\rangle\big\|_m.\end{aligned} \tag{2.26}$$

Set $m = n$, $s = 0$ and $t = s_K$ in (2.26). Then, for each jth term within the sum, use (2.19) to bound $\|\langle \nu_0^K, \mathbf{1}_{I_j(s_K,x)}\rangle\|_n \le C|\sqrt{s_K}|^\alpha$, followed by using $\sum_j p(1, j_*) < \infty$. This yields

$$\big\|\langle \nu_0^K, p(s_K, \cdot - x)\rangle\big\|_n \le C s_K^{\frac{\alpha-1}{2}}. \tag{2.27}$$

Inserting (2.27) into (2.25), we then obtain

$$\|J_1\|_n \le \int_a^b C s_K^{\frac{\alpha-1}{2}}\,dx \le C|b-a| s_K^{\frac{\alpha-1}{2}}. \tag{2.28}$$

As for J_2, by (2.16) we have

$$|J_2| \le C \int_0^s |b-a|^\alpha (u_K)^{-\frac{1+\alpha}{2}}\,du \le C|b-a|^\alpha. \tag{2.29}$$

Inserting (2.28)–(2.29) in (2.24), we see that (2.20) holds for $n = 1$.

To progress to $n > 1$, we use induction, and assume (2.20) has been established for an index $m \in [1, n)$. To set up the induction, similar to the proceeding, we set $x = b, a$ in (2.10), take the difference of the resulting equation, and take the L^{m+1}-norm of the result to obtain

$$\begin{aligned}&\bigg\|\bigg\langle \nu_s^K, \Phi\Big(\frac{1}{K}, b-\cdot\Big) - \Phi\Big(\frac{1}{K}, a-\cdot\Big)\bigg\rangle\bigg\|_{m+1} \\ &\quad\le \|J_1\|_{m+1} + \|J_2\|_{m+1} + \|J_3\|_{m+1},\end{aligned}$$

where $J_3 := N_K(s, b) - N_K(s, a)$. Further combining this with (2.23) yields

$$\big\|\langle \nu_s^K, \mathbf{1}_{[a,b]}\rangle\big\|_{m+1} \le C\|J_1\|_{m+1} + C\|J_2\|_{m+1} + C\|J_3\|_{m+1}. \tag{2.30}$$

For $\|J_1\|_{m+1}$ and $\|J_2\|_{m+1}$, we have already established the bounds (2.28)–(2.29), so it suffices to bound $\|J_3\|_{m+1}$. As J_3 is a martingale integral of quadratic variation $\frac{1}{\sqrt{K}}\int_0^s \langle \nu_u^K, \widehat{p}^2(u,\cdot)\rangle\, du$, where $\widehat{p}(u,y) := p(s_K-u, a-y) - p(s_K-u, b-y)$, we applying the Burkholder–Davis–Gundy (BDG) inequality to obtain

$$\|J_3\|_{m+1}^2 \le \frac{C}{\sqrt{K}}\int_0^s \big\|\langle \nu_u^K, \widehat{p}^2(u,\cdot)\rangle\big\|_{\frac{m+1}{2}}\, du. \tag{2.31}$$

The induction hypothesis asserts the bound (2.20) for $n=m$. With this in mind, within the integral in (2.31), we use $\frac{m+1}{2}\le m$ to bound the $\|\cdot\|_{\frac{m+1}{2}}$ norm by the $\|\cdot\|_m$ norm, and write

$$\begin{aligned}\big\|\langle \nu_u^K, \widehat{p}^2(u,\cdot)\rangle\big\|_{\frac{m+1}{2}} &\le \big\|\langle \nu_u^K, \widehat{p}^2(u,\cdot)\rangle\big\|_m\\ &\le \big|\widehat{p}(u,\cdot)\big|_{L^\infty(\mathbb{R})}\big\|\langle \nu_u^K, \widehat{p}(u,\cdot)\rangle\big\|_m.\end{aligned} \tag{2.32}$$

To bound the factor $|\widehat{p}(u,\cdot)|_{L^\infty(\mathbb{R})}$ on the right-hand side of (2.32), fixing $(2\alpha-1)_+ < \beta < \alpha$, we use (2.16) to write

$$\big|\widehat{p}(u,\cdot)\big|_{L^\infty(\mathbb{R})} \le C|b-a|^\beta (s_K-u)^{-\frac{1+\beta}{2}}. \tag{2.33}$$

Now, within the right-hand side of (2.32), using (2.33),

$$\big|\langle \nu_u^K, \widehat{p}(u,\cdot)\rangle\big| \le \langle \nu_u^K, p(s_K-u, b-\cdot)\rangle + \langle \nu_u^K, p(s_K-u, a-\cdot)\rangle$$

and (2.26), we obtain

$$\begin{aligned}&\big\|\langle \nu_u^K, \widehat{p}^2(u,\cdot)\rangle\big\|_{\frac{m+1}{2}}\\ &\quad\le C|b-a|^\beta (s_K-u)^{-\frac{1+\beta}{2}}\\ &\quad\quad\times\sum_{j\in\mathbb{Z}}\frac{1}{\sqrt{s_K-u}}p(1,j_*)\Big(\sum_{x\in a,b}\big\|\langle \nu_u^K, \mathbf{1}_{I_j(s_K-u,x)}\rangle\big\|_m\Big).\end{aligned} \tag{2.34}$$

By the induction hypothesis, $\|\langle \nu_u^K, \mathbf{1}_{I_j(s_K-u,x)}\rangle\|_m \le C(\sqrt{s_K-u})^\alpha \times ((\frac{\sqrt{s_K-u}}{\sqrt{u_K}})^{1-\alpha}+1)$. Using this for $x=a,b$ in (2.34), and combining the result with (2.31), followed by $\sum_{j\in\mathbb{Z}} p(1,j_*)\le C$, we obtain

$$\begin{aligned}&\big\|\langle \nu_u^K, \widehat{p}^2(u,\cdot)\rangle\big\|_{\frac{m+1}{2}}\\ &\quad\le C|b-a|^\beta\big((s_K-u)^{-\frac{1+\beta}{2}}u_K^{\frac{\alpha-1}{2}} + (s_K-u)^{-1+\frac{\alpha-\beta}{2}}\big).\end{aligned} \tag{2.35}$$

Inserting this bound (2.35) back into (2.31), followed by using $\frac{1}{\sqrt{K}} \le K^{-(2\alpha-\beta)/2} \le |b-a|^{2\alpha-\beta}$, we arrive at

$$\|J_3\|_{m+1}^2 \le C|b-a|^{2\alpha}\int_0^s \big((s_K-u)^{-\frac{1+\beta}{2}}u_K^{\frac{\alpha-1}{2}} + (s_K-u)^{-1+\frac{\alpha-\beta}{2}}\big)\, du.$$

Within the last expression, using the readily verifiable inequality

$$\int_0^s (s_K - u)^{-\delta_1} u_K{}^{-\delta_2}\, du \le C(\delta_1, \delta_2) s_K^{1-\delta_1-\delta_2} \qquad \forall \delta_1, \delta_2 < 1, \tag{2.36}$$

we obtain $\|J_3\|_{m+1}^2 \le C|b-a|^{2\alpha} s_K^{\frac{\alpha-\beta}{2}} \le C|b-a|^{2\alpha}$. Using this bound and the bounds (2.28)–(2.29) in (2.30), we see that (2.20) holds for the index $m+1$. This completes the induction and hence concludes (2.20).

The bound (2.21) follows by combining (2.26) and (2.20). □

Next, we establish bounds on μ_t^K.

LEMMA 2.4. *Let $n, T < \infty$ and $\alpha \in (0,1)$. Given any strategy,*

$$\begin{aligned} &\|\langle \mu_s^K, \mathbf{1}_{[a,b]}\rangle\|_n \le C|b-a|^\alpha s_K^{-\frac{1+\alpha}{2}} \\ &\qquad \forall [a,b] \subset \mathbb{R}_+ \textit{ with } |b-a| \ge \frac{1}{\sqrt{K}} \qquad \forall s \le T, \end{aligned} \tag{2.37}$$

$$\begin{aligned} &\|\langle \mu_s^K, p(t_K, \cdot - x)\rangle\|_n \le C t_K^{-\frac{1-\alpha}{2}} s_K^{-\frac{1+\alpha}{2}} \\ &\qquad \forall x \in \mathbb{R}, s, t \le T, \end{aligned} \tag{2.38}$$

where $C = C(T, \alpha, n) < \infty$, which, in particular, is independent of the strategy.

PROOF. With $\Psi(\frac{1}{K}, y, x)$ defined in the proceeding, it is straightforward to verify that

$$\begin{aligned} &\frac{1}{C}\mathbf{1}_{[a,b]}(y) \le \Psi\left(\frac{1}{K}, y, a\right) - \Psi\left(\frac{1}{K}, y, b\right) \\ &\qquad \forall [a,b] \subset \left[\frac{1}{K}, \infty\right), \text{satisfying } |b-a| \ge \frac{1}{\sqrt{K}}. \end{aligned} \tag{2.39}$$

The idea of the proof is to follow the same general strategy as in the proof of Lemma 2.3. However, unlike (2.23), here the inequality (2.39) does *not* hold for all desired interval $[a,b] \subset \mathbb{R}_+$, but only for $[a,b] \subset [\frac{1}{K}, \infty)$. This is due to the fact that $\Psi(t, 0, x) = 0$. To circumvent the problem, we consider the *shifted* process $(\mathcal{X}_i^m(t); t \ge 0)_{i=1}^K$

$$\begin{aligned} \mathcal{X}_i^m(t) &= 1 + n + B_i(t \wedge \sigma_i^n) \\ &\quad + \int_0^{t\wedge\sigma_i^n} \phi_i(s)\, ds \qquad \text{where } \sigma_i^n := \inf\{t : \mathcal{X}_i^m(t) = 0\}. \end{aligned}$$

That is, $\mathcal{X}_i^m(t)$, $i = 1, \dots, K$, are driven by the same Brownian motion as $X(t)$, drifted under *the same strategy* $\phi(t)$ as $X(t)$, and absorbed at $x = 0$, but started at

$x = m+1$ instead of $x = 1$. Define the analogous scaled variables as $\mathcal{X}_i^{K,m}(t) := \frac{1}{\sqrt{K}}\mathcal{X}_i^m(Kt)$, $\sigma_i^{K,m} := K^{-1}\sigma_i^m$,

$$\mu_t^{K,m}(\cdot) := \frac{1}{\sqrt{K}} \sum_{\mathcal{X}_i^{K,m}(t)>0} \delta_{\mathcal{X}^{K,m}(t)}(\cdot).$$

We adopt the convention that $\mathcal{X}_i^{K,0}(t) = X_i^K(t)$ and $\mu_t^{K,0} = \mu_t^K$. Under the proceeding construction, we clearly have that $\mathcal{X}_i^{m-1}(t) = \mathcal{X}_i^m(t) - 1$, $\forall t \le \sigma_i^{m-1}$, so in particular

$$\mu_t^{K,m-1}\Big(\Big[a - \frac{1}{\sqrt{K}}, b - \frac{1}{\sqrt{K}}\Big]\Big) \le \mu_t^{K,m}([a,b]) \qquad \forall [a,b] \subset \mathbb{R}_+. \tag{2.40}$$

For the shifted process $\mathcal{X}^{K,m}(t) = (\mathcal{X}_i^{K,m}(t))_{i=1}^K$, by the same procedure of deriving (2.7), we have the following integral identity:

$$\begin{aligned}
&\Big\langle \mu_t^{K,m}, \Psi\Big(\frac{1}{K}, \cdot, x\Big)\Big\rangle \\
&\quad = \widetilde{G}_{K,m}(t_K, x) + \sum_{i=1}^K \int_0^t \phi_i^K(s) p^{\mathrm{N}}\big(t_K - s, \mathcal{X}_i^{K,m}(s), x\big)\, ds \\
&\qquad + M_{K,n}(t,x) \qquad \forall t \in \mathbb{R}_+, x \ge 0,
\end{aligned} \tag{2.41}$$

where

$$\widetilde{G}_{K,m}(t,x) := \sqrt{K}\Psi\Big(t, \frac{1+m}{\sqrt{K}}, x\Big), \tag{2.42}$$

$$M_{K,m}(t,x) := \frac{1}{\sqrt{K}} \sum_{i=1}^K \int_0^{t\wedge\tau_i^K} p^{\mathrm{N}}\big(t_K - s, \mathcal{X}_i^{K,m}(s), x\big)\, dB_i^K(s). \tag{2.43}$$

Having prepared the necessary notation, we now begin the proof of (2.37). Instead of proving (2.37) directly, we show

$$\begin{aligned}
&\|\langle \mu_s^{K,n-m+1}, \mathbf{1}_{[a,b]}\rangle\|_m \le C|b-a|^\alpha s_K^{-\frac{1+\alpha}{2}} \\
&\qquad \forall [a,b] \subset \Big[\frac{1}{\sqrt{K}}, \infty\Big) \text{ with } |b-a| \ge \frac{1}{\sqrt{K}}\ \forall s \le T,
\end{aligned} \tag{2.44}$$

for all $m = 1, \ldots, n$. Once this is established, combining (2.44) for $m = n$ and (2.40) for $m = 1$, the desired result (2.37) follows.

Fixing $[a,b] \subset [\frac{1}{\sqrt{K}}, \infty)$ with $|b-a| \ge \frac{1}{\sqrt{K}}$. We begin by settling (2.44) for $m = 1$. Similar to the procedure for obtaining (2.24), using (2.41) for $m = n$ and (2.39) in place of (2.10) and (2.23), respectively, here we have

$$\|\langle \mu_s^{K,n}, \mathbf{1}_{[a,b]}\rangle\|_1 \le C\mathcal{J}_1^n + C\,\mathbf{E}(\mathcal{J}_2^n), \tag{2.45}$$

where $\mathcal{J}_1^m$, $\mathcal{J}_2^m$ is defined for $m = 1, \dots, n$ as

$$\mathcal{J}_1^m := \widetilde{G}_{K,m}(s_K, a) - \widetilde{G}_{K,m}(s_K, b),$$

$$\begin{aligned}\mathcal{J}_2^m := \sum_{i=1}^{K} \int_0^{\sigma_i^{K,n}\wedge s} \phi_i^K(u)\big(p^{\mathrm{N}}\big(s_K - u, \mathcal{X}^{K,m}(u), a\big) \\ - p^{\mathrm{N}}\big(s_K - u, \mathcal{X}_i^{K,m}(u), b\big)\big)\, du.\end{aligned}$$

As noted in Remark 2.2, expressions of the type $\mathcal{J}_1^m$ account for the contribution of the system with only absorption, while $\mathcal{J}_2^m$ encodes the contribution of the drifts $\phi_i^K(s)\, ds$. The singular behavior of the empirical measure $\mu_s^{K,m}$ at $s = 0$ (due to having $K \gg \sqrt{K}$ particles) is entirely encoded in $\mathcal{J}_1^m$. In particular, recalling from (2.12) the notation $\widetilde{U}_K^{\mathrm{ab}}(t, x)$, by (2.13) we have

$$\begin{aligned}\widetilde{G}_{K,m}(0, a) - \widetilde{G}_{K,m}(0, b) &= \frac{1}{\sqrt{K}} \#\big\{X_i^{\mathrm{ab},K}(0) + m \in (a, b]\big\} \\ &= \sqrt{K}\, \mathbf{1}_{\frac{1+m}{\sqrt{K}} \in (a,b]}.\end{aligned}$$

While this expression diverges (for $a < \frac{1+m}{\sqrt{K}}$) as $K \to \infty$, for any fixed $s > 0$ the absorption mechanism remedies the divergence, resulting in converging expression for the fixed $s > 0$. To see this, with $\widetilde{G}_K(t, x)$ defined in (2.8), we use $\partial_y \Psi(s, y, x) = p^{\mathrm{N}}(s, y, x)$ and $\Psi(s, 0, x) = 0$ to write

$$\widetilde{G}_{K,m}(t, x) = \sqrt{K} \int_0^{\frac{1+m}{\sqrt{K}}} p^{\mathrm{N}}(s, y, x)\, dy. \tag{2.46}$$

Letting $x = a, b$ in (2.82), taking the difference of the resulting equations, followed by applying the estimate (2.16), we obtain the following bound of $\mathcal{J}_1^m$, which stays bounded as $K \to \infty$ for any fixed $s > 0$:

$$\begin{aligned}\mathcal{J}_1^m &= \sqrt{K} \int_0^{\frac{1+m}{\sqrt{K}}} \big(p^{\mathrm{N}}(s_K, y, a) - p^{\mathrm{N}}(s_K, y, b)\big)\, dy \\ &\le C\sqrt{K} \int_0^{\frac{1+m}{\sqrt{K}}} s_K^{-\frac{1+\alpha}{2}} |b - a|^\alpha\, dy \\ &\le C s_K^{-\frac{1+\alpha}{2}} |b - a|^\alpha \qquad \forall m = 1, \dots, n.\end{aligned} \tag{2.47}$$

As for $\mathcal{J}_2^m$, similar to (2.29), by using (2.16) and $\sum_{i=1}^K \phi_i^K(s) \le 1$ here we have

$$\begin{aligned}\big|\mathcal{J}_2^m\big| &\le \sum_{i=1}^{K} \int_0^{\sigma_i^{K,n}\wedge s} \phi_i^K(u) C |b - a|^\alpha (u_K)^{-\frac{1+\alpha}{2}}\, du \\ &\le C|b - a|^\alpha \qquad \forall m = 1, \dots, n.\end{aligned} \tag{2.48}$$

Combining (2.47)–(2.48) with (2.45), we conclude (2.44) for $m = 1$.

Having establishing (2.44) for $m=1$, we use induction to progress, and assume (2.44) has been established for some index $m \in [1,n)$. Similar to (2.30), here we have

$$\begin{aligned} &\|\langle \mu_s^{K,n-m}, \mathbf{1}_{[a,b]}\rangle\|_{m+1} \\ &\quad \leq C\mathcal{J}_1^{n-m} + C\|\mathcal{J}_2^{n-m}\|_{n-m} + C\|\mathcal{J}_3^{n-m}\|_{m+1}, \end{aligned} \tag{2.49}$$

where $\mathcal{J}_3^m := M_{K,m}(s,b) - M_{K,m}(s,a)$. To bound $\|\mathcal{J}_3^{n-m}\|_{m+1}$, similar to (2.31)–(2.32), by using the BDG inequality here we have

$$\|\mathcal{J}_3^{n-m}\|_{m+1}^2 \leq \frac{C}{\sqrt{K}} \int_0^s \|\langle \mu_u^{K,n-m}, \widehat{p}^{\mathrm{N}}(u,\cdot)^2\rangle\|_{\frac{m+1}{2}}\, du, \tag{2.50}$$

where $\widehat{p}^{\mathrm{N}}(u,y) := p^{\mathrm{N}}(s_K - u, a, y) - p^{\mathrm{N}}(s_K - u, a, y)$. For the expression $\|\langle \mu_u^{K,m+1}, \widehat{p}^{\mathrm{N}}(u,\cdot)^2\rangle\|_{\frac{m+1}{2}}$, following the same calculations in (2.32)–(2.34), here we have

$$\begin{aligned} &\|\langle \mu_u^{K,n-m}, \widehat{p}^{\mathrm{N}}(u,\cdot)^2\rangle\|_{\frac{m+1}{2}} \\ &\quad \leq C|b-a|^{\beta}(s_K-u)^{-\frac{1+\beta}{2}} \\ &\qquad \times \sum_{j\in\mathbb{Z}} \frac{1}{\sqrt{s_K-u}} p(1,j_*)\Bigg(\sum_{x=\pm a,\pm b} \|\langle \mu_u^{K,n-m}, \mathbf{1}_{I_j'(x)}\rangle\|_m\Bigg), \end{aligned} \tag{2.51}$$

where $\beta \in ((2\alpha-1)_+, \alpha)$ is fixed, $j_* := |j| \wedge |j+1|$ and

$$I_j'(x) := I_j(\sqrt{s_K-u}, x) = \big[x + j\sqrt{s_K-u}, x + (j+1)\sqrt{s_K-u}\big].$$

Since the empirical measure $\mu_t^{K,m+1}$ is supported on $\mathbb{R}_+$, letting $I_j''(x) := [(x + j\sqrt{s_K-u})_+, (x+j\sqrt{s_K-u})_+ + \sqrt{s_K-u}]$, we have

$$\langle \mu_u^{K,n-m}, \mathbf{1}_{I_j'(x)}\rangle = \langle \mu^{K,n-m}, \mathbf{1}_{I_j'(x)\cap\mathbb{R}_+}\rangle \leq \langle \mu_u^{K,n-m}, \mathbf{1}_{I_j''(x)}\rangle.$$

Further using (2.40) yields

$$\langle \mu_u^{K,n-m}, \mathbf{1}_{I_j'(x)}\rangle \leq \langle \mu_u^{K,n-m+1}, \mathbf{1}_{\frac{1}{\sqrt{K}}+I_j''(x)}\rangle. \tag{2.52}$$

By the induction hypothesis,

$$\|\langle \mu_u^{K,n-m+1}, \mathbf{1}_{\frac{1}{\sqrt{K}}+I_j''(x)}\rangle\|_m \leq C(\sqrt{s_K-u})^{\alpha}(u_K)^{-\frac{1+\alpha}{2}}. \tag{2.53}$$

Using (2.53) for $x=\pm a, \pm b$ in (2.51), and combining the result with (2.50), we arrive at

$$\|\mathcal{J}_3^{n-m}\|_{m+1}^2 \leq C\frac{|b-a|^{\beta}}{\sqrt{K}} \int_0^s (s_K-u)^{\frac{-2-\beta+\alpha}{2}} u_K^{\frac{-1-\alpha}{2}}\, du.$$

Within the last expression, further using $\frac{1}{\sqrt{K}} \le K^{-(2\alpha-\beta)/2} \le |b-a|^{2\alpha-\beta}$ and (2.36), we obtain $\|\mathcal{J}_3^{n-m}\|_{m+1}^2 \le C|b-a|^{2\alpha} s_K^{-\frac{1+\beta}{2}} \le C|b-a|^{2\alpha} s_K^{-\frac{1+\alpha}{2}}$. Using this bound and the bounds (2.47)–(2.48) in (2.49), we see that (2.37) holds for the index $m+1$. This completes the induction, and hence concludes (2.37).

The bound (2.38) follows by (2.37) and (2.51). □

Lemmas 2.3–2.4 establish bounds that are "pointwise" in time, in the sense that they hold at a fixed time s within the relevant interval. We next improve these pointwise bounds to bounds that hold for *all* time within a relevant interval.

LEMMA 2.5. *Let $T, L, n < \infty$ and*

$$I_{x,\alpha} := [-K^{-\alpha} + x, x + K^{-\alpha}]. \tag{2.54}$$

(a) *For any given $\gamma \in (0, \frac{1}{4})$, $\alpha \in (2\gamma, \frac{1}{2}]$ and any strategy,*

$$\mathbf{P}(\langle \mu_t^K, \mathbf{1}_{I_{x,\alpha}} \rangle \le t^{-\frac{3}{4}} K^{-\gamma}, \forall t \le T, |x| \le L) \ge 1 - CK^{-n}, \tag{2.55}$$

where $C = C(T, L, \alpha, \gamma, n) < \infty$, which, in particular, is independent of the strategy.

(b) *Letting ν_t^K be as in Lemma 2.3, we have, for any $\alpha \in (\frac{1}{4}, \frac{1}{2}]$,*

$$\mathbf{P}(\langle \nu_t^K, \mathbf{1}_{I_{x,\alpha}} \rangle \le K^{-\frac{1}{4}}, \forall t \le T, |x| \le L) \ge 1 - CK^{-n}, \tag{2.56}$$

where $C = C(T, L, \alpha, n, D_*, D_{\alpha,n}) < \infty$, for D_*, $D_{\alpha,n}$ as in (2.18)–(2.19).

PROOF. We first prove Part (b). Fixing $L, T, n < \infty$ and $\alpha \in (\frac{1}{4}, \frac{1}{2}]$, to simplify notation we use, $C(a_1, a_2, \ldots) < \infty$ to denote generic finite constants that may depend on L, T, α, n, D_*, $D_{\alpha,j}$ and the designated variable $a_1, a_2, \ldots$. To the end of proving (2.56), we cover $[-L, L]$ by intervals I_j of length $K^{-\alpha}$:

$$I_j := [jK^{-\alpha}, (j+1)K^{-\alpha}], \qquad |j| \le LK^{\alpha}.$$

Indeed, each $I_{x,\alpha}$ is contained in the union of three consecutive such intervals I_j, so it suffices to prove

$$\mathbf{P}\left(\langle \nu_t^K, \mathbf{1}_{I_j} \rangle \le \frac{1}{3} K^{-\frac{1}{4}}, \forall |j| \le LK^{\alpha}, t \le T\right) \ge 1 - CK^{-n}. \tag{2.57}$$

By (2.20) we have, for any $t \in [\frac{1}{K}, T]$, $\beta \in (0,1)$ and $k < \infty$,

$$\begin{aligned} \|\langle \nu_t^K, \mathbf{1}_{I_j} \rangle\|_k &\le C(k,\beta)|I_j|^{\beta}\big((|I_j|K^{\frac{1}{2}})^{1-\beta} + 1\big) \\ &\le C(k,\beta)\big(K^{-\alpha+\frac{1-\beta}{2}} + K^{-\alpha\beta}\big). \end{aligned} \tag{2.58}$$

With $\alpha > \frac{1}{4}$, fixing β close enough to 1 we have $\|\langle \nu_t^K, \mathbf{1}_{I_{x,\alpha}}\rangle\|_k \leq C(k)K^{-\frac{1}{4}-\varepsilon}$, for some fixed $\varepsilon > 0$. With this, applying Markov's inequality we obtain

$$\mathbf{P}\Big(\langle \nu_t^K, \mathbf{1}_{I_j}\rangle \geq \frac{1}{9}K^{-\frac{1}{4}}\Big) \leq C(k)K^{-k\varepsilon}. \tag{2.59}$$

Now, fixing $k \geq (n+\alpha+2)\varepsilon^{-1}$ and taking the union bound of (2.59) over $|j| \leq LK^{\alpha}$ and $t = t_\ell := \ell K^{-2}$, $1 \leq \ell \leq TK^2$, we arrive at

$$\mathbf{P}\Big(\langle \nu_{t_\ell}^K, \mathbf{1}_{I_j}\rangle \leq \frac{1}{9}K^{-\frac{1}{4}}, \forall |j| \leq LK^{\alpha}, 1 \leq \ell \leq TK^2\Big) \geq 1 - CK^{-n}. \tag{2.60}$$

To move from the "discrete time" t_ℓ to "continuous time" $t \in [0,T]$, we need to control $\nu_s^K(I_j)$ within each time interval $s \in [t_{\ell-1}, t_\ell] := J_\ell$. Within each J_ℓ, since each $Y_i^K(s)$ evolves as a drifted Brownian motion with drift $\leq \sqrt{K}$, we have that

$$\begin{aligned} \mathbf{P}\big(\big|Y_i^K(s) - Y_i^K(t_\ell)\big| \leq K^{-\alpha}, \forall s \in J_\ell\big) &\geq 1 - \exp\Big(-\frac{1}{C}K^{1-\alpha}\Big) \\ &\geq 1 - CK^{-n-3}. \end{aligned} \tag{2.61}$$

By (2.18), we assume without lost of generality the total number of Y-particles is at most K. Hence, taking the union bound of (2.61) over $\ell \leq TK^{-2}$ and over all particles $i = 1, 2, \ldots \leq K$, we obtain

$$\mathbf{P}\Big(\sup_{s\in J_\ell}\big|Y_i^K(s) - Y_i^K(t_\ell)\big| \leq K^{-\alpha}, \forall i, \forall 1 \leq \ell \leq TK^2\Big) \geq 1 - CK^{-n}.$$

That is, with high probability, no particle travels farther than distance $|I_j|$ within each time interval J_ℓ. Therefore,

$$\mathbf{P}\Big(\sup_{s\in J_\ell}\langle \nu_s^K, \mathbf{1}_{I_j}\rangle \leq \langle \nu_{t_\ell}^K, \mathbf{1}_{I_{j-1}\cup I_j\cup I_{j+1}}\rangle, \forall 1 \leq \ell \leq TK^2\Big) \geq 1 - CK^{-n}.$$

Combining this with (2.60) yields the desired result (2.57).

Part (a) is proven by a similar argument as in the preceding. The only difference is that, instead of a moment bound of the form (2.58), we have from (2.37) the moment bound

$$\big\|\langle \mu_t^K, I_j\rangle\big\|_k \leq C(k)|I_j|^{\frac{1}{2}}t^{-\frac{3}{4}} \leq C(k)K^{-\frac{\alpha}{2}}t^{-\frac{3}{4}}, \tag{2.62}$$

for all $k < \infty$. With $\frac{\alpha}{2} > \gamma$, (2.62) yields (2.55) by same argument we obtained (2.58). □

Equipped with Lemmas 2.3–2.5, we proceed to the main goal of this section: to develop integral identities [in different forms from (2.7) and (2.10)] that are convenient for proving the hydrodynamic limits. Recall from (2.2) that $W(t)$ is the analogous laggard of the Atlas model $(Y_i(t); t \geq 0)_i$ and that $W_K(t)$ denotes the scaled process. For any fixed t, we define the scaled distribution function of Y as

$$V_K(t,x) := \frac{1}{\sqrt{K}}\#\big\{Y_i^K(t) \leq x\big\} = \langle \nu_t^K, \mathbf{1}_{(-\infty,x]}\rangle. \tag{2.63}$$

PROPOSITION 2.6. (a) *Let* $(\phi_i(t); t \geq 0)_{i=1}^K$ *be any given strategy. The following integral identity holds for all* $t < \infty$ *and* $x \geq 0$:

$$\widetilde{U}_K(t,x) = \widetilde{G}_K(t,x) + \sum_{i=1}^{K} \int_0^{t\wedge\tau_i^K} \phi_i^K(s) p^N\bigl(t-s, X_i^K(s), x\bigr)\, ds + R_K(t,x). \tag{2.64}$$

Here, $R_K(t,x)$ *is a remainder term such that*, *for given any* $T, n < \infty$ *and* $\gamma \in (0, \frac{1}{4})$,

$$\mathbf{P}\bigl(\bigl|R_K(t,x)\bigr| \leq K^{-\gamma} t^{-\frac{3}{4}}, \forall t \leq T, x \in \mathbb{R}\bigr) \geq 1 - CK^{-n}, \tag{2.65}$$

where $C = C(T,\gamma,n) < \infty$, *and is in particular independent of the strategy.*

(b) *Let* $(Y_i(t); t \geq 0)_i$ *be an Atlas model*, *and let* $W_K(t)$ *and* $V_K(t,x)$ *be as in the preceding*, *and assume* $(Y_i^K(0))_i$ *satisfies the conditions* (2.18)–(2.19). *Then the following integral identity holds for all* $t < \infty$ *and* $x \in \mathbb{R}$:

$$V_K(t,x) = \int_0^\infty p(t, x-y) V_K(0,y)\, dy - \int_0^t p\bigl(t-s, W_K(s), x\bigr)\, ds + R_K'(t,x). \tag{2.66}$$

Here, $R_K'(t,x)$ *is a remainder term such that*, *given any* $T, n < \infty$ *and* $\gamma \in (0, \frac{1}{4})$,

$$\mathbf{P}\bigl(\bigl|R_K'(t,x)\bigr| \leq K^{-\gamma}, \forall t \leq T, x \in \mathbb{R}\bigr) \geq 1 - CK^{-n}, \tag{2.67}$$

where $C < \infty$ *depends only on* T, n *and* D_*, $D_{\alpha,n}$.

The proof of Proposition 2.6 requires a Kolmogorov-type estimate, which we recall from [16] as follows.

LEMMA 2.7 ([16], Theorem 1.4.1). *Let* $T < \infty$, $a \in \mathbb{R}$, *and let* F *be a* $\mathcal{C}([0,\infty) \times \mathbb{R})$-*valued process. If*, *for some* α_1, α_2, $k \in \mathbb{N}$ *and* $C_1 < \infty$ *with* $\frac{1}{k\alpha_1} + \frac{1}{k\alpha_2} < 1$,

$$\bigl\|F(0,0)\bigr\|_k \leq C_1, \tag{2.68}$$

$$\bigl\|F(t,x) - F\bigl(t',x'\bigr)\bigr\|_k \leq C_1\bigl(\bigl|t-t'\bigr|^{\alpha_1} + \bigl|x-x'\bigr|^{\alpha_2}\bigr), \tag{2.69}$$

$\forall t, t' \in [0,T]$, $x, x' \in [a, a+1]$, *then* $\||F|_{L^\infty([0,T]\times[a,a+1])}\|_k \leq C_2 = C_2(C_1, T, \alpha_1, \alpha_2) < \infty$.

Note that, although the dependence of C_2 is not explicitly designated in [16], Theorem 1.4.1, under the present setting, it is clear from the proof of Lemma 1.4.2, Lemma 1.4.3 from [16] that $C_2 = C_2(C_1, T, \alpha_1, \alpha_2, k)$.

PROOF OF PROPOSITION 2.6. The first step of the proof is to rewrite (2.7) and (2.10) in a form similar to (2.64) and (2.66). To motivate this step, recall from Remark 2.2 that the term $\langle \mu_t^K, \Psi(\frac{1}{K}, x, \cdot)\rangle$ should approximate $\widetilde{U}_K(t,x)$ as $K \to \infty$. In view of this, we write

$$\Big\langle \mu_t^K, \Psi\Big(\frac{1}{K}, x, \cdot\Big)\Big\rangle = \widetilde{U}_K(t,x) + E_K(t,x)$$

$$\text{where } E_K(t,x) := \Big\langle \mu_t^K, \Psi\Big(\frac{1}{K}, x, \cdot\Big) - \Psi(0, x, \cdot)\Big\rangle \tag{2.70}$$

$$= \Big\langle \mu_t^K, \Psi\Big(\frac{1}{K}, x, \cdot\Big) - \mathbf{1}_{(x,\infty)}\Big\rangle.$$

Similarly, for the the the first two terms on the right-hand side of (2.7), we write

$$\begin{aligned}
&\widetilde{G}_K(t_K, x)\\
&\quad = \widetilde{G}_K(t,x)\\
&\qquad + \big(\widetilde{G}_K(t_K,x) - \widetilde{G}_K(t,x)\big) \sum_{i=1}^{K} \int_0^t \phi_i^K(s) p^{\mathrm{N}}\big(t_K - s, X_i^K(s), x\big)\, ds\\
&\quad = \sum_{i=1}^{K} \int_0^t \phi_i^K(s) p^{\mathrm{N}}\big(t - s, X_i^K(s), x\big)\, ds + Q_N(t,x),
\end{aligned}$$

where

$$\begin{aligned}
Q_K(t,x) := \sum_{i=1}^{K} \int_0^t \phi_i^K(s) \big(&p^{\mathrm{N}}\big(t - s, X_i^K(s), x\big)\\
&- p^{\mathrm{N}}\big(t_K - s, X_i^K(s), x\big)\big)\, ds.
\end{aligned} \tag{2.71}$$

Under this notation, we rewrite (2.7) as

$$\begin{aligned}
\widetilde{U}_K(t,x) = \widetilde{G}_K(t,x) &+ \sum_{i=1}^{K} \int_0^{\tau_i^K \wedge t} \phi_i^K(s) p^{\mathrm{N}}\big(t - s, X_i^K(s), x\big)\, ds\\
&+ R_K(t,x),
\end{aligned} \tag{2.72}$$

where

$$\begin{aligned}
R_K(t,x) := \big(\widetilde{G}_K(t_K,x) - \widetilde{G}_K(t,x)\big) - E_K(t,x) + Q_K(t,x)\\
+ M_K(t,x).
\end{aligned} \tag{2.73}$$

Equation (2.72) gives the desired identity (2.64) with the explicit remainders $R_K(t,x)$. Similarly, for the Atlas model Y, we define

$$E'_K(u,t,x) := \langle \nu_u^K, \Phi(t_K, x - \cdot) - \Phi(t, x - \cdot)\rangle, \tag{2.74}$$

$$Q'_K(t,x) := \int_0^t \big(p(t-s, x-W_K(s)) - p(t_K-s, x-W_K(s))\big)\,ds, \tag{2.75}$$

$$R'_K(t,x) := E'_K(0,t,x) - E'_K(t,0,x) - Q'_K(t,x) + N_K(t,x), \tag{2.76}$$

and rewrite (2.10) as

$$V_K(t,x) = \langle \nu_0^K, \Phi(t, x-\cdot)\rangle - \int_0^t p(t-s, x-W_K(s))\,ds + R'_K(t,x).$$

Further using integration by parts:

$$\begin{aligned}\langle \nu_0^K, \Phi(t, x-\cdot)\rangle &= \int_{\mathbb{R}} \Phi(t, x-y)\,dV_K(0,y)\\ &= -\int_{\mathbb{R}} V_K(0,y)\partial_y \Phi(t, x-y)\,dy,\end{aligned}$$

we write

$$\begin{aligned}V_K(t,x) &= \int_{\mathbb{R}} p(t, x-y) V_K(0,x)\,dy\\ &\quad - \int_0^t p(t-s, x-W_K(s))\,ds + R'_K(t,x).\end{aligned} \tag{2.77}$$

Equations (2.72) and (2.77) give the desired identities (2.64) and (2.66) with the explicit remainders $R_K(t,x)$ and $R'_K(t,x)$, as in (2.73) and (2.76). With this, it suffices to show that these remainders do satisfy the bounds (2.65) and (2.67). To this end, fixing arbitrary $T, n < \infty$ and $\gamma \in (0, \frac{1}{4})$, we let $C(k) < \infty$ denote a generic constant depending only on T, n, α, γ, D_*, $D_{\alpha,n}$, and the designated variable k.

We begin with a reduction. That is, in order to prove (2.65) and (2.67), we claim that it suffices to prove

$$\mathbf{P}\big(|R_K(t,x)| \le K^{-\gamma} t^{-\frac{3}{4}}, \forall t \le T, x \in [a, a+1]\big) \ge 1 - CK^{-n}, \tag{2.78}$$

$$\mathbf{P}\big(|R'_K(t,x)| \le K^{-\gamma}, \forall t \le T, x \in [a, a+1]\big) \ge 1 - CK^{-n}, \tag{2.79}$$

for all $a \in \mathbb{R}$. To see why such a reduction holds, we assume that (2.78) has been established, and take the union bound of (2.78) over $a \in \mathbb{Z} \cap [-K, K]$ to obtain

$$\mathbf{P}\big(|R_K(t,x)| \le K^{-\gamma} t^{-\frac{3}{4}}, \forall t \le T, |x| \le K\big) \ge 1 - CK^{-n+1}. \tag{2.80}$$

To cover the regime $|x| > K$ that is left out by (2.80), we use the fact that each $X_i^K(t)$ evolves as a Brownian motion with drift at most $\sqrt{K}$ (and absorption) to obtain

$$\mathbf{P}(\Omega_K) \ge 1 - CK^{-n}, \qquad \Omega_K := \left\{ |X_i^K(t)| \le \frac{1}{2}K, \forall t \le T, \forall i \right\}. \tag{2.81}$$

That is, with a sufficiently high probability, each particle $X_i^K(t)$ stays within $[-\frac{1}{2}K, \frac{1}{2}K]$ for all time. Use (2.72) to express $R_K(t,x) = \widetilde{U}_K(t,x) - f(t,x) -$

$\widetilde{G}_K(t,x)$, where $f(t,x) := \sum_{i=1}^{K} \int_0^{t\wedge\tau_i^K} \phi_i^K(s) p^{\mathrm{N}}(t-s, X_i^K(s), x)\,ds$. On the event Ω_K, the function $x \mapsto \widetilde{U}_K(t,x)$ remains constant on $\mathbb{R} \setminus (-K, K)$; and, for all $x > K$,

$$\begin{aligned}
&\bigl|f(t,\pm x) - f(t,\pm K)\bigr| \\
&\quad \le \sum_{i=1}^{K} \int_0^{t\wedge\tau_i^K} \phi_i^K(s) \bigl| p^{\mathrm{N}}\bigl(t-s, X_i^K(s), \pm x\bigr) \\
&\qquad - p^{\mathrm{N}}\bigl(t-s, X_i^K(s), \pm K\bigr)\bigr|\,ds \\
&\quad \le \int_0^t 4\left|p\left(t-s, \frac{K}{2}\right)\right| ds \le \frac{C}{K}.
\end{aligned}$$

From these, we conclude that, on Ω_K,

$$\begin{aligned}
\sup_{x\in\mathbb{R}} \bigl|R_K(t,x)\bigr| &\le \sup_{|x|\le K} \bigl|R_K(t,x)\bigr| + \left(\frac{C}{K} + \sup_{|x|\ge K} \bigl|\widetilde{G}_K(t,x)\bigr|\right) \\
&\le \sup_{|x|\le K} \bigl|R_K(t,x)\bigr| + \frac{C}{K}.
\end{aligned}$$

Combining this with (2.80) gives the desired bound (2.65). A similar argument shows that (2.79) implies (2.67).

Having shown that (2.78)–(2.79) imply the desired results, we now return to proving (2.78)–(2.79). This amounts to bounding each term on the right-hand side of the explicit expressions (2.73) and (2.76) of $R_K(t,x)$ and $R_K'(t,x)$. To this end, fixing $t \le T$, $a \in \mathbb{R}$ and $x \in [a, a+1]$, we establish bounds on the following terms in sequel:

(i) $|\widetilde{G}_K(t_K, x) - \widetilde{G}_K(t,x)|$;
(ii) $Q_K(t,x)$ and $Q_K'(t,x)$;
(iii) $E_K(t,x)$, $E_K'(0,t,x)$ and $E_K'(t,0,x)$; and
(iv) $N_K(t,x)$ and $M_K(t,x)$.

(i) By (2.46) for $m = 0$, we have that

$$\widetilde{G}_K(t,x) = \sqrt{K} \int_0^{\frac{1}{\sqrt{K}}} p^{\mathrm{N}}(s,y,x)\,dy. \tag{2.82}$$

Applying the bound (2.17) for $\alpha = \frac{1}{4}$ within (2.82), we obtain

$$\bigl|\widetilde{G}_K(t_K, x) - 2p(t,x)\bigr| \le C K^{-\frac{1}{4}} t^{-\frac{3}{4}}. \tag{2.83}$$

(ii) Applying (2.17) for $\alpha = \frac{1}{2}$ in (2.71) and in (2.75) yields

$$\bigl|Q_K(t,x)\bigr|, \bigl|Q_K'(t,x)\bigr| \le C K^{-\frac{1}{4}}. \tag{2.84}$$

(iii) For the Brownian distribution function $\Phi(t, y) = \mathbf{P}(B(t) \le y)$, it is standard to show that $t \mapsto |\Phi(t_K, y) - \Phi(t, y)|$ decreases in t, and that $|\Phi(\frac{1}{K}, y) - \Phi(0, y)| \le C \exp(-\sqrt{K}|y|)$. Further, fixing $\alpha \in (2\gamma, \frac{1}{2})$ and letting $I_{x,\alpha}$ be as in (2.54), we write

$$\begin{aligned}\exp(-\sqrt{K}|y-x|) &\le \mathbf{1}_{I_{x,\alpha}}(y) + \mathbf{1}_{\mathbb{R}\setminus I_{x,\alpha}}(y)\exp(-\sqrt{K}|y-x|)\\ &\le \mathbf{1}_{I_{x,\alpha}}(y) + \exp(-K^{\alpha-\frac{1}{2}}).\end{aligned}$$

From these bounds, we conclude

$$\begin{aligned}&\left|\Phi\left(\frac{1}{K}, y-x\right) - \Phi(0, y-x)\right| \\ &\qquad \le C\big(\mathbf{1}_{I_{x,\alpha}}(y) + \exp(-K^{\alpha-\frac{1}{2}})\big),\end{aligned} \tag{2.85a}$$

$$\begin{aligned}&\big|\Phi(t_K, y-x) - \Phi(t, y-x)\big| \\ &\qquad \le C\big(\mathbf{1}_{I_{x,\alpha}}(y) + \exp(-K^{\alpha-\frac{1}{2}})\big),\end{aligned} \tag{2.85b}$$

$$\begin{aligned}&\left|\Psi\left(\frac{1}{K}, y, x\right) - \Psi(0, y, x)\right| \\ &\qquad \le C\big(\mathbf{1}_{I_{x,\alpha}\cup I_{-x,\alpha}}(y) + \exp(-K^{\alpha-\frac{1}{2}})\big).\end{aligned} \tag{2.85c}$$

Recall the definition of $E(t, x)$ and $E'(u, t, x)$ from (2.70) and (2.74). Applying $\langle \nu_t^K, \cdot\rangle$ $\langle \nu_0^K, \cdot\rangle$ and $\langle \mu_t^K, \cdot\rangle$ on both sides of (2.85b)–(2.85c), respectively, we obtain

$$\begin{aligned}&\big|EY_K(t, 0, x)\big| \\ &\qquad \le C\langle \nu_t^K, \mathbf{1}_{I_{x,\alpha}}\rangle + C\exp(-K^{\alpha-\frac{1}{2}})\langle \nu_t^K, \mathbf{1}_{\mathbb{R}}\rangle,\end{aligned} \tag{2.86a}$$

$$\begin{aligned}&\big|E'_K(0, t, x)\big| \\ &\qquad \le C\langle \nu_0^K, \mathbf{1}_{I_{x,\alpha}}\rangle + C\exp(-K^{\alpha-\frac{1}{2}})\langle \nu_0^K, \mathbf{1}_{\mathbb{R}}\rangle,\end{aligned} \tag{2.86b}$$

$$\begin{aligned}&\big|E_K(t, x)\big| \\ &\qquad \le C\langle \mu_t^K, \mathbf{1}_{I_{x,\alpha}}\rangle + C\langle \mu_t^K, \mathbf{1}_{I_{-x,\alpha}}\rangle + C\exp(-K^{\alpha-\frac{1}{2}})\langle \mu_t^K, \mathbf{1}_{\mathbb{R}}\rangle.\end{aligned} \tag{2.86c}$$

On the right-hand side of (2.86a) sit two types of terms: the "concentrated terms" that concentrate on the small interval $\mathbf{1}_{I_{\pm x,\alpha}}$; and the "tail terms" with the factor $\exp(-K^{\alpha-\frac{1}{2}})$. For the tail terms, writing $\langle \nu_0^K, \mathbf{1}_{\mathbb{R}}\rangle = \langle \nu_t^K, \mathbf{1}_{\mathbb{R}}\rangle = \frac{1}{\sqrt{K}}\#\{Y_i^K(0)\}$ and $\langle \mu_t^K, \mathbf{1}_{\mathbb{R}}\rangle \le \frac{1}{\sqrt{K}}\#\{X_i^K(0)\}$, and using the bound (2.18) and $\#\{X_i^K(0)\} = K$, we bound the tail terms by $C\sqrt{K}\exp(-K^{\alpha-\frac{1}{2}})$, with probability $\ge 1 - CK^{-n}$. Further using $\sqrt{K}\exp(-K^{\alpha-\frac{1}{2}}) \le CK^{-\gamma}$, we have

$$\big|E'_K(t, 0, x)\big| \le C\langle \nu_t^K, \mathbf{1}_{I_{x,\alpha}}\rangle + CK^{-\gamma}, \tag{2.87a}$$

(2.87b) $$\big|E'_K(0,t,x)\big| \le C\langle \nu_0^K, \mathbf{1}_{I_{x,\alpha}}\rangle + CK^{-\gamma},$$

(2.87c) $$\big|E_K(t,x)\big| \le C\langle \mu_t^K, \mathbf{1}_{I_{x,\alpha}}\rangle + C\langle \mu_t^K, \mathbf{1}_{I_{-x,\alpha}}\rangle + CK^{-\gamma},$$

with probability $\ge 1 - CK^{-n}$. Next, to bound the concentrated terms, we consider the covering $\mathcal{X} := \{I_{y,\alpha} : |y| \le a+1\}$ of $[-a-1, a+1]$. With $x \in [a, a+1]$, we clearly have that $I_{\pm x,\alpha} \in \mathcal{X}$, so by Lemma 2.5 it follows that

$$\langle \nu_t^K, \mathbf{1}_{I_{x,\alpha}}\rangle, \langle \nu_0^K, \mathbf{1}_{I_{x,\alpha}}\rangle \le CK^{-\gamma}, \qquad \langle \nu_t^K, \mathbf{1}_{I_{\pm x,\alpha}}\rangle \le CK^{-\gamma} t^{-\frac{3}{4}},$$

with probability $1 - CK^{-n}$. Inserting this into (2.87) gives

(2.88a) $$\mathbf{P}\big(\big|E'_K(t,0,x)\big| \le CK^{-\gamma}, \forall t \le T, x \in [a, a+1]\big) \ge 1 - CK^{-n},$$

(2.88b) $$\mathbf{P}\big(\big|E'_K(0,t,x)\big| \le CK^{-\gamma}, \forall t \le T, x \in [a, a+1]\big) \ge 1 - CK^{-n},$$

(2.88c) $$\mathbf{P}\big(\big|E_K(t,x)\big| \le CK^{-\gamma} t^{-\frac{3}{4}}, \forall t \le T, x \in [a, a+1]\big) \ge 1 - CK^{-n}.$$

(iv) The strategy is to apply Lemma 2.7 for $F(t,x) := K^{1/4} N_K(t,x)$. With $N_K(t,x)$ defined as in (2.11), for such F we have $F(0,0) = 0$, so the condition (2.68) holds trivially. Turning to verifying the condition (2.69), we fix $t < t'$ and $x, x' \in \mathbb{R}$. With $N_K(t,x)$ defined as in (2.11), we telescope $F(t,x) - F(t',x')$ into $F_1 + F_2 - F_3$, where

$$F_1 := K^{-1/4} \sum_i \int_0^t f_1\big(s, Y_i^K(s)\big)\, dB_i^K(s),$$

$$F_2 := K^{-1/4} \sum_i \int_0^t f_2\big(s, Y_i^K(s)\big)\, dB_i^K(s),$$

$$F_3 := K^{-1/4} \sum_i \int_t^{t'} f_3\big(s, Y_i^K(s)\big)\, dB_i^K(s),$$

$f_1(s,y) := p(t_K - s, y - x) - p(t_K - s, y - x')$, $f_2(s,y) := p(t_K - s, y - x') - p(t'_K - s, y - x')$ and $f_3(s,y) := p(t'_K - s, y - x')$. Similar to the way we obtained (2.31), here by the BDG inequality we have

$$\|F_1\|_k^2 \le C(k) \int_0^t \big\|\langle \nu_s^K, f_1(s,\cdot)^2\rangle\big\|_{k/2}\, ds,$$

$$\|F_2\|_k^2 \le C(k) \int_0^t \big\|\langle \nu_s^K, f_2(s,\cdot)^2\rangle\big\|_{k/2}\, ds,$$

$$\|F_3\|_k^2 \le C(k) \int_t^{t'} \big\|\langle \nu_s^K, f_3(s,\cdot)^2\rangle\big\|_{k/2}\, ds,$$

for any fixed $k > 1$. On the right-hand side, the kernel functions f_1, f_2, f_3 appear in square (i.e., power of two). We use (2.16)–(2.17) to replace "one power" of them with $C(t-s)^{-\frac{3}{4}}|x - x'|^{\frac{1}{2}}$, $C(t-s)^{-\frac{3}{4}}|t - t'|^{\frac{1}{4}}$ and $C(t-s)^{-\frac{1}{2}}$, respectively, and then use (2.21) for $\alpha = \frac{3}{4}$ to bound $\|\langle \nu_s^K, f_j(s,\cdot)\rangle\|_{k/2}$, $j = 1, 2, 3$, whereby obtaining

$$\|F_1\|_k^2 \le C(k) \int_0^t |x - x'|^{\frac{1}{2}} \big((t-s)^{-\frac{7}{8}} + (t-s)^{-\frac{3}{4}} s^{-\frac{1}{8}}\big)\, ds \le C(k)|x - x'|^{\frac{1}{2}},$$

$$\|F_2\|_k^2 \le C(k) \int_0^t |t - t'|^{\frac{1}{4}} \big((t-s)^{-\frac{7}{8}} + (t-s)^{-\frac{3}{4}} s^{-\frac{1}{8}}\big)\, ds \le C(k)|t - t'|^{\frac{1}{4}},$$

$$\|F_3\|_k^2 \le C(k) \int_t^{t'} \big((t-s)^{-\frac{5}{8}} + (t-s)^{-\frac{1}{2}} s^{-\frac{1}{8}}\big)\, ds \le C(k)|t - t'|^{\frac{3}{8}}.$$

We have thus verified condition (2.69) for $(\alpha_1, \alpha_2) = (\frac{1}{8}, \frac{1}{4})$. Now apply Lemma 2.7 to obtain $\||N_K|_{L^\infty([0,T]\times[a,a+1])}\|_k \le C(k) K^{-\frac{1}{4}}$. From this and Markov's inequality, we conclude

$$\begin{aligned} &\mathbf{P}\big(|N_K(t,x)| \le K^{-\gamma} \\ &\qquad \forall t \le T, x \in [a, a+1]\big) \ge 1 - C(k) K^{-k(1-\gamma)} \ge 1 - C K^{-n}. \end{aligned} \tag{2.89}$$

The term $M_K(t,x)$ is bounded by similar procedures as in the preceding. The only difference is that the estimate (2.38), unlike (2.21), introduces a singularity of $M_K(t,x)$ as $t \to 0$, so we set $F(t,x) := t^{\frac{3}{4}} K^{1/4} M_K(t,x)$ [instead of $F(t,x) := K^{1/4} M_K(t,x)$]. The extra prefactor $t^{\frac{3}{4}}$ preserves the moment estimate (2.69) since $t^{\frac{3}{4}}$ is α-Hölder continuous for all $\alpha < \frac{3}{4}$. Consequently, following the preceding argument we obtain

$$\mathbf{P}\Big(\sup_{t \in [0,T], x \in [a,a+1]} \big(t^{\frac{3}{4}} |M_K(t,x)|\big) \le K^{-\gamma}\Big) \ge 1 - C K^{-n}. \tag{2.90}$$

Now, combining the bounds (2.83), (2.84), (2.88) and (2.89)–(2.90) from (i)–(iv), we conclude the desired results (2.65) and (2.67). □

3. The Stefan problem. In this section, we develop the necessary PDE tools. As stated in Remark 1.5, we take the integral identity and integral equations (1.9)–(1.10), instead of (1.14), as the definition of the Stefan problem. To motivate such a definition, we first prove the following:

LEMMA 3.1. *Let (u_2, z_2) be a classical solution to the following PDE, that is,*

$$\begin{aligned} &z_2 \in \mathcal{C}^1\big((0,\infty)\big) \cap \mathcal{C}\big([0,\infty)\big), \textit{ nondecreasing}, z(0) = 0, \\ &u_2 \in L^\infty(\overline{\mathcal{D}}) \cap L^1(\overline{\mathcal{D}}), \textit{ and has a} \\ &\qquad \mathcal{C}^2\textit{-extension onto a neighborhood of } \overline{\mathcal{D}}, \\ &\qquad \textit{where } \mathcal{D} := \{(t,x) : t > 0, x \ge z_2(t)\}, \end{aligned} \tag{3.1a}$$

$$\partial_t u_2 = \frac{1}{2}\partial_{xx} u_2 \qquad \forall 0 < t < T, x > z(t), \tag{3.1b}$$

$$u_2\bigl(t, z_2(t)\bigr) = 2 \qquad \forall t \geq 0, \tag{3.1c}$$

$$2\frac{d}{dt}z_2(t) + \frac{1}{2}\partial_x u_2\bigl(t, z_2(t)\bigr) = 0 \qquad \forall t > 0, \tag{3.1d}$$

Define the tail distribution function of u_2 *as* $\widetilde{U}_2(t,x) := \int_x^\infty u_2(t,y)\,dy$. *We have*

$$\begin{aligned}\widetilde{U}_2(t,x) = {}& \int_0^\infty p^N(t,y,x)\widetilde{U}_2(0,y)\,dy \\ & + \int_0^t p^N\bigl(t-s, z_2(s), x\bigr)\,ds \qquad \forall t, x \in \mathbb{R}_+,\end{aligned} \tag{3.2}$$

$$\begin{aligned}&\int_0^\infty p\bigl(t, z_2(t) - y\bigr)\bigl(\widetilde{U}_2(0,0) - \widetilde{U}_2(0,y)\bigr)\,dy \\ &\quad = \int_0^t p\bigl(t-s, z_2(t) - z_2(s)\bigr)\,ds.\end{aligned} \tag{3.3}$$

PROOF. Instead of the tail distribution function $\widetilde{U}_2(t,x)$, let us first consider the distribution function $U_2(t,x) := \int_{z_2(t)}^x u_2(t,y)\,dy$. We adopt the convention that $U_2(t,x)|_{x<z_2(t)} := 0$. By (3.1b), (3.1c)–(3.1d), $U_2(s,y)$ solves the heat equation in $\{(s,y) : s > 0, y > z(s)\}$. With this, for any fixed $t > 0$ and $x \in \mathbb{R}$, we integrate Green's identity

$$\frac{1}{2}\partial_y\bigl((\partial_y p)U_2 - p(\partial_y U_2)\bigr) + \partial_s(pU_2) = 0,$$

where

$$p = p(t-s, x-y), \qquad U_2 = U_2(s,y),$$

over $\{(s,y) : \varepsilon < s < t - \varepsilon, y > z(s) + \varepsilon\}$. Letting $\varepsilon \to 0$, and combining the result with $U_2(s, z_2(s)) = 0$ and (3.1c), we obtain

$$\begin{aligned}U_2(t,x) = {}& \int_0^\infty p(t, x-y)U_2(0,y)\,dy \\ & - \int_0^t p\bigl(t-s, x - z(s)\bigr)\,ds \qquad \forall t \in \mathbb{R}_+, x \in \mathbb{R}.\end{aligned} \tag{3.4}$$

Note that the preceding derivation of (3.4) applies to *all* $x \in \mathbb{R}$, including $x < z_2(t)$. Setting $x = z_2(t)$ in (3.4), on the left-hand side we have $U_2(t, z_2(0)) = 0$. Further using $U_2(0,y) = \widetilde{U}_2(0,0) - \widetilde{U}_2(0,y)$, we see that (3.3) follows.

We now turn to showing (3.2). A straightforward differentiation, following by using (3.1c)–(3.1d), gives

$$\begin{aligned}\partial_t U_2(t,\infty) &= \partial_t \int_{z_2(t)}^{\infty} u_2(t,x)\,dx \\ &= -z_2'(t)u_2\big(t,z(t)\big) + \int_{z(t)} \frac{1}{2}\partial_{xx}u_2(t,x)\,dx \\ &= -2z_2'(t) - \frac{1}{2}\partial_x u_2\big(t,z(t)\big) = 0,\end{aligned}$$

so in particular $U_2(0,\infty) = U_2(t,\infty)$. Consequently,

$$\widetilde{U}_2(t,x) = U_2(0,\infty) - U_2(t,x). \tag{3.5}$$

Further, as $U_2(t,x)|_{x\le 0} = 0$,

$$\begin{aligned}U_2(t,x) &= U_2(t,x) + U_2(t,-x) \\ &= \int_0^{\infty} p^{\mathrm{N}}(t,y,x)U_2(0,y)\,dy \\ &\quad - \int_0^t p^{\mathrm{N}}\big(t-s,z(s),x\big)\,ds \qquad \forall t,x\in\mathbb{R}_+.\end{aligned} \tag{3.6}$$

Inserting (3.6) into the last term in (3.5) yields

$$\begin{aligned}\widetilde{U}_2(t,x) = U_2(0,\infty) &- \int_0^{\infty} p^{\mathrm{N}}(t,y,x)U_2(0,y)\,dy \\ &+ \int_0^t p^{\mathrm{N}}\big(t-s,z(s),x\big)\,ds \qquad \forall t,x\in\mathbb{R}_+.\end{aligned}$$

Further using $U_2(0,\infty) = \int_0^{\infty} p^{\mathrm{N}}(t,y,x)U_2(0,\infty)\,dy$ to write

$$U_2(0,\infty) - \int_0^{\infty} p^{\mathrm{N}}(t,y,x)U_2(0,y)\,dy = \int_0^{\infty} p^{\mathrm{N}}(t,y,x)\widetilde{U}_2(0,y)\,dy,$$

we see that (3.2) follows. $\square$

We next turn to the well-posedness of (3.3). The existence of a solution to (3.3) will be established in Lemma 4.11, Section 4.2, as a by-product of establishing the hydrodynamic limit of certain Atlas models. Here, we focus the uniqueness and stability of (3.3). To this end, we consider $w \in \mathcal{C}([0,T])$ that satisfies

$$\begin{aligned}&\int_0^{\infty} p\big(t,w(t)-y\big)\Big(\widetilde{U}_\star\Big(\frac{1}{2},0\Big) - \widetilde{U}_\star\Big(\frac{1}{2},y\Big)\Big)\,dy \\ &\quad = f\big(t,w(t)\big) + \int_0^t p\big(t-s,w(t)-w(s)\big)\,ds,\end{aligned} \tag{3.7}$$

where $f \in L^\infty([0,T]\times\mathbb{R})$ is a generic perturbation. Define a seminorm

$$|w|'_{[0,T]} := \sup\{w(t) - w(t'), 0 \le t \le t' \le T\} \tag{3.8}$$

that measures how nondecreasing the given function is.

LEMMA 3.2. *Fixing $T < \infty$ and $f_1(t,x), f_2(t,x) \in L^\infty([0,T]\times\mathbb{R})$, we consider w_1 and w_2 satisfying (3.7) for $f = f_1$ and $f = f_2$, respectively. Let $L := \sup\{|w_1(t)|, |w_2(t)| : t \le T\} + 1$. There exists $C_1 = C_1(T,L) < \infty$ such that*

$$\sup_{0\le t\le T} \big(w_1(t) - w_2(t)\big) \le C_1 \sum_{i=1,2} \big(|w_i(0)| + |f_i|_{L^\infty([0,T]\times\mathbb{R})} + |w_i|'_{[0,T]}\big),$$

for all $f_1, f_2 \in L^\infty([0,T]\times\mathbb{R})$ satisfying $\sum_{i=1,2}(|w_i(0)| + |f_i|_{L^\infty([0,T]\times\mathbb{R})} + |w_i|'_{[0,T]}) \le \frac{1}{C_1}$.

Indeed, when $f_1 = f_2 = 0$, Lemma 3.2 yields the following.

COROLLARY 3.3. *The solution to (1.10) is unique.*

PROOF OF LEMMA 3.2. To simplify notation, let $\varepsilon := |f_1|_{L^\infty([0,T]\times\mathbb{R})} + |f_2|_{L^\infty([0,T]\times\mathbb{R})}$, $\varepsilon' := |w_1|'_{[0,T]} + |w_2|'_{[0,T]}$ and $\varepsilon'' := |w_1(0)| + |w_2(0)|$.
Let

$$\Lambda(t,z) := \int_0^\infty p(t, z-y)\Big(\widetilde{U}_\star\Big(\frac{1}{2}, 0\Big) - \widetilde{U}_\star\Big(\frac{1}{2}, y\Big)\Big)\,dy \tag{3.9}$$

$$= \int_{-\infty}^z p(t, y)\Big(\widetilde{U}_\star\Big(\frac{1}{2}, 0\Big) - \widetilde{U}_\star\Big(\frac{1}{2}, z-y\Big)\Big)\,dy \tag{3.10}$$

denote the expression on the left-hand side of (3.7). From the explicit expressions (1.11)–(1.12), we have that $\partial_x(-\widetilde{U}_\star(\frac{1}{2}, z-y)) = u_1(\frac{1}{2}, z-y) > 0$, $\forall y \le z$, so $\partial_z\Lambda(t,z) > 0$, $\forall z \ge 0$. Consequently, there exists $c_1 = c_1(T,L) > 0$ such that

$$\partial_z\Lambda(t,z) \ge c_1 > 0 \qquad \forall 0 \le z \le L, 0 \le t \le T. \tag{3.11}$$

Setting $C_1 := \frac{4}{c_1} \vee 1$ and $\delta := C_1(\varepsilon + \varepsilon' + \varepsilon'') \le 1$, we write $w_2^\delta(t) := w_2(t) + \delta$ to simplify notation, and consider the first time $t^* := \inf\{t \le T : w_1(t) \ge w_2^\delta(t)\}$ when w_1 hits w_2^δ. Indeed, since $C_1 \ge 1$, we have $w_1(0) \le w_2(0) + |w_2(0) - w_1(0)| < w_2(0) + \delta$, so in particular $t^* > 0$. Taking the difference of (3.7) for $(t, f) = (t^*, f_1)$ and for $(t, f) = (t^*, f_2)$, we obtain

$$\begin{aligned}\Lambda\big(t^*, w_1(t^*)\big) - \Lambda\big(t^*, w_2(t^*)\big) &= \Lambda\big(t^*, w_2^\delta(t^*)\big) - \Lambda\big(t^*, w_2(t^*)\big)\\ &\le \varepsilon + \int_0^{t^*} g^*(s)\,ds,\end{aligned} \tag{3.12}$$

where $g^*(s) := p(t^*-s, w_1(t^*)-w_1(s)) - p(t^*-s, w_2(t^*)-w_2(s))$. Next, using $w_1(s) \leq w_2(s)+\delta$, $\forall s \leq t^*$, we have

$$w_1(t^*) - w_1(s) = w_2(t^*) + \delta - w_1(s) \geq w_2(t^*) - w_2(s). \tag{3.13}$$

To bound the function $g^*(s)$, we consider the separate cases (i) $w_2(t^*) - w_2(s) \geq 0$; and (ii) $w_2(t^*) - w_2(s) < 0$. For case (i), by (3.13) we have $|w_1(t^*) - w_1(s))| \geq |w_2(t^*) - w_2(s)|$, so $g^*(s) \leq 0$. For case (ii), using $0 > w_2(t^*) - w_2(s) \geq -\varepsilon'$ we have $g^*(s) \leq p(t^*-s, 0) - p(t^*-s, \varepsilon')$. Combining these bounds with the readily verified identity

$$\int_0^t p(t-s, x)\, ds = 2tp(t,x) - 2|x|\widetilde{\Phi}(t, |x|), \tag{3.14}$$

we obtain

$$\begin{aligned}\int_0^{t^*} g_*(s)\, ds &\leq \int_0^{t^*} \bigl(p(t^*-s, 0) - p(t^*-s, \varepsilon')\bigr)\, ds \\ &\leq 2t_* p(t^*, 0) - 2t_* p(t^*, \varepsilon') + 2|\varepsilon'| \\ &= \sqrt{\frac{2t^*}{\pi}}\biggl(1 - \exp\biggl(-\frac{\varepsilon'^2}{2t^*}\biggr)\biggr) + 2\varepsilon' < 4\varepsilon',\end{aligned} \tag{3.15}$$

where we used $(1-e^{-\xi}) \leq 2\sqrt{\xi}$, $\forall \xi \in \mathbb{R}_+$, in the last inequality. Now, if $t^* \leq T$, combining (3.15) with (3.12) and (3.11) yields $\delta c_1 < \varepsilon + 4\varepsilon'$, leading to a contradiction. Consequently, we must have $t^* > T$. □

We next establish a property of $(\widetilde{U}_\star(t,x), z_\star(t))$, that will be used toward the proof of Theorems 1.2 and 1.6.

LEMMA 3.4. *For any* $(\widetilde{U}_\star(t,x), z_\star(t); t \geq \frac{1}{2})$ *satisfying* (1.9)–(1.10), *we have*

$$\widetilde{U}_\star(t, z_\star(t)) = \frac{4}{\sqrt{\pi}} \qquad \forall t \geq \frac{1}{2}. \tag{3.16}$$

REMARK 3.5. As $\widetilde{U}_\star(t,x)$ represents the hydrodynamic limit of the scaled tail distribution function $\widetilde{U}_K(t,x) := \frac{1}{\sqrt{K}}\#\{X_i^K(t) > x\}$, equation (3.16) is a statement of *conservation of particles* within the moving boundary phase, in the hydrodynamic limit.

PROOF. Fixing such $(\widetilde{U}_\star(t,x), z_\star(t))$, we *define*

$$\begin{aligned}U_\star\Bigl(t+\frac{1}{2}, x\Bigr) := &\int_0^\infty p(t, x-y)\Bigl(\widetilde{U}_\star\Bigl(\frac{1}{2}, 0\Bigr) - \widetilde{U}_\star\Bigl(\frac{1}{2}, y\Bigr)\Bigr)\, dy \\ &- \int_0^t p\Bigl(t-s, x - z_\star\Bigl(s+\frac{1}{2}\Bigr)\Bigr)\, ds.\end{aligned}$$

From this expression, it is straightforward to verify that, for any fixed $T < \infty$, $U_\star(\cdot + \frac{1}{2}, \cdot) \in \mathcal{C}([0,T] \times \mathbb{R}) \cap L^\infty([0,T] \times \mathbb{R})$ solves the heat equation on $\{(t,x) : t > 0, x < z_\star(s)\}$. Further, $U_\star(\frac{1}{2}, x)|_{x \le 0} = 0$ and, by (1.10),

$$U_\star\Big(t + \frac{1}{2}, z_\star\Big(t + \frac{1}{2}\Big)\Big) = 0. \tag{3.17}$$

From these properties of $U_\star(t + \frac{1}{2}, x)$, by the uniqueness of the heat equation on the domain $\{(t,x) : t \in \mathbb{R}, x < z_\star(t)\}$, we conclude that $U_\star(t + \frac{1}{2}, x)|_{x \le z_\star(t+\frac{1}{2})} = 0$. Therefore,

$$\begin{aligned}
U_\star\Big(\frac{1}{2} + t, x\Big) &= U_\star\Big(\frac{1}{2} + t, x\Big) + U_\star\Big(\frac{1}{2} + t, -x\Big) \\
&= \int_0^\infty p^{\mathrm{N}}(t,y,x)\Big(\widetilde{U}_\star\Big(\frac{1}{2}, 0\Big) - \widetilde{U}_\star\Big(\frac{1}{2}, y\Big)\Big)\,dy \\
&\quad - \int_0^t p^{\mathrm{N}}\Big(t - s, z_\star\Big(s + \frac{1}{2}\Big), x\Big)\,ds \\
&= \widetilde{U}_\star\Big(\frac{1}{2}, 0\Big) - \int_0^\infty p^{\mathrm{N}}(t,y,x)\widetilde{U}_\star\Big(\frac{1}{2}, y\Big)\,dy \\
&\quad - \int_0^t p^{\mathrm{N}}\Big(t - s, z_\star\Big(s + \frac{1}{2}\Big), x\Big)\,ds \qquad \forall x \in \mathbb{R}_+.
\end{aligned} \tag{3.18}$$

Next, set $t = \frac{1}{2}$ in (1.9), and write $2p(\frac{1}{2}, y) = p^{\mathrm{N}}(t, 0, y)$ to obtain

$$\widetilde{U}_\star\Big(\frac{1}{2}, y\Big) = p^{\mathrm{N}}\Big(\frac{1}{2}, 0, y\Big) + \int_0^{\frac{1}{2}} p^{\mathrm{N}}\Big(\frac{1}{2} - s, z_\star(s), y\Big)\,ds. \tag{3.19}$$

Inserting this expression (3.19) of $\widetilde{U}_\star(\frac{1}{2}, y)$ into (3.18), followed by using the semigroup property $\int_0^\infty p^{\mathrm{N}}(t,y,x)p^{\mathrm{N}}(\frac{1}{2}, z, y)\,dy = p^{\mathrm{N}}(t + \frac{1}{2}, z, y)$, we obtain

$$\begin{aligned}
U_\star\Big(t + \frac{1}{2}, x\Big) &= \widetilde{U}_\star\Big(\frac{1}{2}, 0\Big) - p^{\mathrm{N}}\Big(t + \frac{1}{2}, 0, x\Big) \\
&\quad - \int_0^{t+\frac{1}{2}} p^{\mathrm{N}}\Big(t - s, z_\star\Big(s + \frac{1}{2}\Big), x\Big)\,ds \\
&= \widetilde{U}_\star\Big(\frac{1}{2}, 0\Big) - \widetilde{U}_\star\Big(t + \frac{1}{2}, x\Big) \qquad \forall x \in \mathbb{R}_+.
\end{aligned}$$

Setting $x = z_\star(t + \frac{1}{2})$ and using (3.17) on the left-hand side, we conclude the desired identity (3.16). $\square$

As will be needed toward the proof of Theorems 1.2 and 1.6, we next show that $z_\star(t)$ grows quadratically near $t = \frac{1}{2}$.

LEMMA 3.6. *For any solution $z_\star(\cdot+\frac{1}{2})$ to the integral equation* (1.10), *we have*

$$\lim_{t\downarrow 0}\left\{t^{-2}z_\star\left(t+\frac{1}{2}\right)\right\}=\frac{2}{\sqrt{\pi}}. \tag{3.20}$$

REMARK 3.7. For sufficiently smooth solutions to the PDE (1.14), one can easily calculate $\frac{d^2}{dt^2}z_\star(\frac{1}{2})=-\frac{1}{8}\partial_{xxx}u_1(\frac{1}{2},0)=\frac{2}{\sqrt{\pi}}$ by differentiating (1.14e) and (1.14b). Here, as we take the integral equation (1.10) as the definition of the Stefan problem, we prove Lemma 3.6 by a different, indirect method, which does not assume the smoothness of of $z_\star$.

PROOF OF LEMMA 3.6. We begin by deriving a useful identity. Write $\int_0^t p(t-s,x)\,ds=-\int_0^t\int_{-\infty}^x\partial_{yy}\Phi(t-s,y)\,ds$, use $-\partial_{yy}\Phi(t-s,y)=2\partial_s\Phi(t-s,y)$, swap the double integrals, and integrate over $s\in[0,t]$. With $\Phi(0,y)=\mathbf{1}_{[0,\infty)}(y)$, we obtain

$$\begin{aligned}\int_0^t p(t-s,x)\,ds&=2\int_{-\infty}^x\big(\Phi(t,y)-\mathbf{1}_{[0,\infty)}(y)\big)\,ds\\&=2\int_{-\infty}^{-|x|}\Phi(t,y)\,ds.\end{aligned}\tag{3.21}$$

We now begin the proof of (3.20). Let $\Lambda(t,x)$ be as in (3.9). Recall from (1.11) that $\partial_y\widetilde{U}_\star(\frac{1}{2},y)=-u_1(\frac{1}{2},y)$, for $u_1(\frac{1}{2},y)$ defined in (1.12). Integrating by parts followed by a change of variable $y\mapsto\frac{y}{\sqrt{t}}$ yields

$$\begin{aligned}\Lambda(t,x)&=-\int_0^\infty\partial_y\left(\widetilde{U}_\star\left(\frac{1}{2},0\right)-\widetilde{U}_\star\left(\frac{1}{2},y\right)\right)\Phi(t,x-y)\,dy\\&=\sqrt{t}\int_0^\infty u_1\left(\frac{1}{2},\sqrt{t}y\right)\Phi\left(1,\frac{x}{\sqrt{t}}-y\right)dy.\end{aligned}\tag{3.22}$$

From the explicit expression (1.12) of $u_1(\frac{1}{2},y)$, we see that $u_1(\frac{1}{2},\cdot)\in\mathcal{C}^\infty(\mathbb{R}_+)\cap L^\infty(\mathbb{R}_+)$, and that $u_1(\frac{1}{2},0)=2$, $\partial_yu_1(\frac{1}{2},0)=\partial_{yy}u_1(\frac{1}{2},0)=0$, and $-\partial_{yyy}u_\star(0)=\frac{16}{\sqrt{\pi}}=:a_3$. Using these properties to Taylor-expand $u_1(\frac{1}{2},\sqrt{t}y)$ in (3.22) yields

$$\Lambda(t,x)=t^{\frac{1}{2}}\Lambda_0\left(\frac{x}{\sqrt{t}}\right)-t^2\Lambda_3\left(\frac{x}{\sqrt{t}}\right)+t^{\frac{5}{2}}\Lambda_4(t,x), \tag{3.23}$$

where $\Lambda_4(t,x)$ is a *bounded* remainder function in the sense that $\lim_{t\downarrow 0}\sup_{x\in\mathbb{R}}|\Lambda_4(t,x)|<\infty$, and

$$\Lambda_0(x):=2\int_0^\infty\Phi(1,x-y)\,dy=2\int_{-\infty}^x\Phi(1,y)\,dy, \tag{3.24}$$

$$\Lambda_3(x):=\frac{a_3}{6}\int_0^\infty y^3\Phi(1,x-y)\,dy. \tag{3.25}$$

Insert the expression (3.23) into (1.10), we obtain

$$\begin{aligned} &t^{\frac{1}{2}}\Lambda_0\Big(\frac{w(t)}{\sqrt{t}}\Big) - t^2\Lambda_3\Big(\frac{w(t)}{\sqrt{t}}\Big) + t^{\frac{1}{2}}\Lambda_4\big(t, w(t)\big) \\ &\quad = \int_0^t p\big(t-s, w(t)-w(s)\big)\,ds. \end{aligned} \tag{3.26}$$

The strategy of the proof is to extract upper and lower bounds on $\frac{w(t)}{\sqrt{t}}$ from (3.26). We begin with the upper bound. On the right-hand side of (3.26), using $p(t-s, w(t)-w(s)) \le p(t-s, 0)$, followed by applying the identity (3.21), we have that

$$t^{\frac{1}{2}}\Lambda_0\Big(\frac{w(t)}{\sqrt{t}}\Big) - t^2\Lambda_3\Big(\frac{w(t)}{\sqrt{t}}\Big) + t^{\frac{5}{2}}\Lambda_4(t,x) \le t^{\frac{1}{2}}\Lambda_0(0). \tag{3.27}$$

Dividing (3.27) by $t^{\frac{1}{2}}$ and letting $t \downarrow 0$, we conclude that $\lim_{t\downarrow 0}\Lambda_0(\frac{w(t)}{\sqrt{t}}) = \Lambda_0(0)$. As $x \mapsto \Lambda_0(x)$ is strictly increasing, we must have $\lim_{t\downarrow 0}\frac{w(t)}{\sqrt{t}} = 0$. Now, dividing both sides of (3.27) by t^2, and letting $t \downarrow 0$ using $\lim_{t\downarrow 0}\frac{w(t)}{\sqrt{t}} = 0$, we further deduce that

$$\lim_{t\downarrow 0} t^{-\frac{3}{2}}\Big(\Lambda_0\Big(\frac{w(t)}{\sqrt{t}}\Big) - \Lambda_0(0)\Big) - \Lambda_3(0) \le 0. \tag{3.28}$$

From the explicit expression (3.24) of $\Lambda_0(x)$, we have that

$$\frac{d}{dx}\Lambda_0(0) = 1. \tag{3.29}$$

Using (3.29) to Taylor-expand the expression $\Lambda_0(\frac{w(t)}{\sqrt{t}})$ in (3.28) to the first order, we thus conclude the desired upper bound $\limsup_{t\downarrow 0}\frac{w(t)}{t^2} \le \Lambda_3(0) = \frac{a_3}{8} = \frac{2}{\sqrt{\pi}}$.

Having established the desired upper bound on $\frac{w(t)}{t^2}$, we now turn to the lower bound. Let $b := \liminf_{t\downarrow 0}\frac{w(t)}{t^2}$. Since $0 \le b \le \frac{2}{\sqrt{\pi}} < \infty$, there exists $t_n \downarrow 0$ such that

$$\Big|\frac{w(t_n)}{t_n^2} - b\Big| < \frac{1}{n}, \qquad \frac{w(s)}{s^2} > b - \frac{1}{n} \qquad \forall s \in (0, t_n]. \tag{3.30}$$

As $t \mapsto w(t)$ is nondecreasing, by (3.30) we have

$$\begin{aligned} \big|w(t_n) - w(s)\big| &= w(t_n) - s(s) \\ &\le \Big(bt_n^2 + \frac{t_n^2}{n}\Big) - \Big(bs^2 - \frac{s^2}{n}\Big) \\ &\le b\big(t_n^2 - s^2\big) + \frac{2t_n^2}{n} \qquad \forall s \le t_n. \end{aligned}$$

Taking the square of the preceding inequality further yields

$$\begin{aligned} &\big|w(t_n) - w(s)\big|^2 \\ &\leq \left(\frac{2t_n^2}{n}\right)^2 + \frac{4bt_n^2(t_n^2 - s^2)}{n} + b^2\big(t_n^2 - s^2\big)^2 \qquad \forall s \leq t_n. \end{aligned} \tag{3.31}$$

Use inequality (3.31) to write

$$\begin{aligned} &p\big(t_n - s, w(t_n) - w(s)\big) \\ &\quad \geq p\left(t_n - s, \frac{2t_n^2}{n}\right)\exp\left(-\frac{1}{2(t_n - s)}\left(\frac{4bt_n^2(t_n^2 - s^2)}{n} + b^2\big(t_n^2 - s^2\big)^2\right)\right) \\ &\quad = p\left(t_n - s, \frac{2t_n^2}{n}\right)\exp\left(-\frac{2bt_n^2(t_n + s)}{n}\right)\exp\left(-\frac{b^2}{2}(t_n - s)(t_n + s)^2\right) \\ &\quad \geq p\left(t_n - s, \frac{2t_n^2}{n}\right)\exp\left(-\frac{2bt_n^2(2t_n)}{n}\right)\exp\left(-\frac{b^2}{2}(t_n - s)(2t_n)^2\right). \end{aligned}$$

Within the last expression, using $e^{-\xi} \geq 1 - \xi$ for $\xi = \frac{b^2}{2}(t_n - s)(2t_n)^2$, and using $|p(t_n - s, \frac{2t_n^2}{n})\exp(-\frac{2bt_n^2(2t_n)}{n})| \leq \frac{1}{\sqrt{t_n - s}}$, we obtain

$$\begin{aligned} &p\big(t_n - s, w(t_n) - w(s)\big) \\ &\quad \geq p\left(t_n - s, \frac{2t_n^2}{n}\right)\exp\left(-\frac{2bt_n^2(2t_n)}{n}\right) - \frac{b^2}{2}(t_n - s)^{\frac{1}{2}}(2t_n)^2. \end{aligned} \tag{3.32}$$

Now, integrate (3.32) over $s \in [0, t_n]$, using identity (3.21) to obtain

$$\begin{aligned} &\int_0^{t_n} p\big(t_n - s, w(t_n) - w(s)\big)\,ds \\ &\quad \geq e^{-\frac{2bt_n^2(2t_n)}{n}} \int_0^{t_n} p\left(t_n - s, \frac{2t_n^2}{n}\right)ds - 4b^2(t_n)^{\frac{5}{2}} \\ &\quad = e^{-\frac{4bt_n^3}{n}} \sqrt{t_n}\Lambda_0\left(-\frac{2t_n^2}{n\sqrt{t_n}}\right) - 4b^2(t_n)^{\frac{5}{2}} \\ &\quad \geq \sqrt{t_n}\Lambda_0\left(-\frac{2t_n^2}{n\sqrt{t_n}}\right) - C(t_n)^{\frac{5}{2}}, \end{aligned} \tag{3.33}$$

for some constant $C < \infty$. Now, set $t = t_n$ in (3.27) and combine the result with (3.33). After dividing both sides of the result by t_n^2 and letting $n \to \infty$, we arrive at

$$\lim_{n\to\infty}(t_n)^{-\frac{3}{2}}\left(\Lambda_0\left(\frac{w(t_n)}{\sqrt{t_n}}\right) - \Lambda_0\left(-\frac{2}{n}(t_n)^{\frac{3}{2}}\right)\right) - \Lambda_3(0) \geq 0. \tag{3.34}$$

Using (3.29) to Taylor-expand the expressions $\Lambda_0(\frac{w(t_n)}{\sqrt{t_n}})$ and $\Lambda_0(-\frac{2}{n}(t_n)^{\frac{3}{2}})$, with $\liminf_{n\to\infty}\frac{w(t_n)}{t_n^2}=b$ [by (3.30)], we conclude the desired lower bound $b\geq \Lambda_3(0)=\frac{2}{\sqrt{\pi}}$. $\square$

4. Proof of Theorem 1.6. Equipped with the tools developed previously, in this section prove Theorem 1.6. To this end, throughout this section we specialize $(\phi_i^K(t): t\geq 0)$ to the push-the-laggard strategy (1.2). Recalling that τ_i denote the absorption of the ith particle $X_i(t)$, we let

$$\tau_{\mathrm{ext}} := \max_i \tau_i, \qquad \tau_{\mathrm{ext}}^K := K^{-1}\tau_{\mathrm{ext}} \tag{4.1}$$

denote the extinction times (unscaled and scaled). Under the push-the-laggard strategy, Proposition 2.6(a) gives

$$\widetilde{U}_K(t,x) = \widetilde{G}_K(t,x) + \int_0^{t\wedge\tau_{\mathrm{ext}}^K} p^{\mathrm{N}}\big(t-s, Z_K(s), x\big)\,ds + R_K(t,x). \tag{4.2}$$

We first establish a lower bound on the extinction time.

LEMMA 4.1. *For any fixed $T, n<\infty$, there exists $C=C(T,n)<\infty$ such that*

$$\mathbf{P}\big(\tau_{ext}^K > T\big) \geq 1 - CK^{-n}. \tag{4.3}$$

PROOF. Consider the modified process $(X_i^{\mathrm{ab}}(t); t\geq 0)_{1\leq i\leq K}$ consisting of K independent Brownian motions starting at $x=1$ and absorbed once they reach $x=0$, and let $\tau'_{\mathrm{ext}} := \inf\{t : X_i(t)=0, \forall i\}$ denote the corresponding extinction time. Under the natural coupling of $(X_i^{\mathrm{ab}}(t))_i$ and $(X_i(t))_i$ (by letting them sharing the underlying Brownian motions), we clearly have $\tau_{\mathrm{ext}}\geq\tau'_{\mathrm{ext}}$. For the latter, it is straightforward to verify that

$$\mathbf{P}\big(\tau'_{\mathrm{ext}}\leq KT\big) = \Big(\mathbf{P}\Big(\inf_{t\leq T}\big(B(Kt)+1\big)\leq 0\Big)\Big)^K \leq \exp\Big(-\frac{1}{C(T)}K^{1/2}\Big),$$

where $B(\cdot)$ denotes a standard Brownian motion. From this, the desired result follows. $\square$

By Lemma 4.1, toward the end of proving Theorem 1.6, without loss of generality we remove the localization $\cdot\wedge\tau_{\mathrm{ext}}^K$ in (4.2).

Next, using the expression (2.82) of $\widetilde{G}_K(t,x)$, from the heat kernel estimate (2.16) we have

$$\big|\widetilde{G}_K(t,x) - 2p(t,x)\big| \leq C(\alpha)K^{-\frac{\alpha}{2}}t^{-\frac{1+\alpha}{2}} \qquad \forall\alpha\in(0,1), \tag{4.4}$$

$$\int_0^t \left|p^{\mathrm{N}}(t-s,x,z) - p^{\mathrm{N}}(t-s,x,z')\right| ds \le C(\alpha')\left|z-z'\right|^{\alpha'} t^{\frac{1-\alpha'}{2}} \qquad \forall \alpha' \in (0,1). \tag{4.5}$$

For any fixed $\gamma \in (0, \frac{1}{4})$ and $\alpha' \in (0, 1)$, taking the difference of (2.1) and (4.2), followed by using the estimates (4.4)–(4.5) and (2.65), we obtain

$$\begin{aligned}
&\left|\widetilde{U}_K(t,x) - \widetilde{U}_\star(t,x)\right| \\
&\quad \le \left|\widetilde{G}_K(t,x) - 2p(t,x)\right| \\
&\qquad + \int_0^t \left|p^{\mathrm{N}}(t-s,x,Z_K(t)) - p^{\mathrm{N}}(t-s,x,z')\right| ds + \left|R_K(t,x)\right| \\
&\quad \le C(\gamma)t^{-\frac{3}{4}}K^{-\gamma} + C(\gamma,\alpha')\sup_{s\le T}\left|Z_K(s) - z(s)\right|^{\alpha'} \qquad \forall x \in \mathbb{R}, t \le T,
\end{aligned}$$

with probability $\ge 1 - C(n,T)K^{-n}$. From this, we see that the hydrodynamic limit (1.15) of $\widetilde{U}_K(t,x)$ follows immediately from the hydrodynamic limit (1.16) of Z_K. Focusing on proving (1.16) hereafter, in the following we settle (1.16) in the absorption phase and the moving boundary phase separately. For technical reasons, instead of using $t_\star = \frac{1}{2}$ as the separation of these two phases, in the following we use $\frac{1}{2} + \frac{1}{7}K^{-2\gamma}$ for the separation of the two phases, where $\gamma \in (0, \frac{1}{96})$ is fixed. More precisely, the desired hydrodynamic result (1.16) follows immediately from the following two propositions [by setting $\beta = \gamma$ in Part(a)].

PROPOSITION 4.2. *For any fixed $\gamma < \gamma_1 \in (0, \frac{1}{96})$ and $n < \infty$, there exists $C = C(\gamma, \gamma_1, n) < \infty$ such that*:

(a) *for all $\beta \le 4\gamma_1$ and $K < \infty$,*

$$\mathbf{P}\left(\left|Z_K(t) - z_\star(t)\right| \le CK^{-\beta}, \forall t \in \left[0, \frac{1}{2} + \frac{1}{7}K^{-2\beta}\right]\right) \ge 1 - CK^{-n}; \tag{4.6}$$

(b) *for all $K < \infty$,*

$$\mathbf{P}\left(\left|Z_K(t) - z_\star(t)\right| \le CK^{-\gamma}, \forall t \in \left[\frac{1}{2} + \frac{1}{7}K^{-2\gamma}, T\right]\right) \ge 1 - CK^{-n}. \tag{4.7}$$

We settle Proposition 4.2(a)–(b) in Sections 4.1–4.2 in the following, respectively. To this end, we fix $\gamma < \gamma_1 \in (0, \frac{1}{96})$, $n < \infty$ and $T < \infty$, and, to simplify notation, use $C < \infty$ to denote a generic constant that depends only on γ, γ_1, n, T.

4.1. *Proof of Proposition* 4.2(a). Fix $\beta \le 4\gamma_1$. We begin with a reduction. Since $z_\star(t)|_{t\le\frac{1}{2}} = 0$, by Lemma 3.6, we have $\sup_{t\le\frac{1}{2}+\frac{1}{7}K^{-2\beta}} |z_\star(t)| \le CK^{-4\beta} \le CK^{-\beta}$. From this, we see that is suffices to prove

$$\mathbf{P}\left(Z_K(t) \le K^{-\beta}, \forall t \le \frac{1}{2} + \frac{1}{7}K^{-2\beta}\right) \ge 1 - CK^{-n}. \tag{4.8}$$

To the end of showing (4.8), we recall the following classical result from [8].

LEMMA 4.3 ([8], Chapter X.5, Example (c)). *Let* $(B(t); t \geq 0)$ *be a standard Brownian motion* (*starting from* 0), *and let* $0 < a < b < \infty$. *Defining* $\rho(t, a, b) := \mathbf{P}(0 < B(s) + a < b, \forall s \leq t)$, *we have*

$$\rho(t, a, b) = \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} \sin\left(\frac{(2n+1)\pi a}{b}\right) \exp\left(-\frac{(2n+1)^2\pi^2}{2b^2} t\right). \tag{4.9}$$

With $\beta \leq 4\gamma_1 < \frac{1}{24}$, we have $4\beta < \frac{1}{2} - 8\beta$. Fixing $\alpha \in (4\beta, \frac{1}{2} - 8\beta) \subset (0, \frac{1}{2})$, we begin with a short-time estimate.

LEMMA 4.4. *There exists* $C < \infty$ *such that*

$$\mathbf{P}(Z_K(t) \leq K^{-\alpha}, \forall t \leq K^{-2\alpha}) \geq 1 - CK^{-n}. \tag{4.10}$$

PROOF. We consider first the modified process $(\widehat{X}_i^{\mathrm{ab}}(t); t \geq 0)_{i=1}^K$, which consists of K independent Brownian motions starting at $x = 1$, and absorbed at $x = 0$ and $x = \frac{1}{2}K^{\frac{1}{2}-\alpha}$. Let $\widehat{X}_i^{K,\mathrm{ab}}(t) := \frac{1}{\sqrt{K}}\widehat{X}_i^{\mathrm{ab}}(Kt)$ denote the scaled process, and consider

$$\widehat{N}^{\mathrm{ab}} := \#\left\{i : 0 < \widehat{X}_i^{K,\mathrm{ab}}(t) < \frac{1}{2}K^{-\alpha}, \forall t \leq K^{-2\alpha}\right\}, \tag{4.11}$$

the number of surviving $\widehat{X}^{K,\mathrm{ab}}$-particles of up to time $K^{-2\alpha}$. Let

$$\begin{aligned}\rho_K^* &:= \rho\left(K^{-2\alpha}, K^{-\frac{1}{2}}, \frac{1}{2}K^{-\alpha}\right)\\ &:= \mathbf{P}\left(0 < \frac{1}{\sqrt{K}}(B(Kt) + 1) < \frac{1}{2}K^{-\alpha}, \forall t \leq K^{-2\alpha}\right).\end{aligned}$$

From the definition (4.11), we see that $\widehat{N}^{\mathrm{ab}}$ is the sum of i.i.d. Bernoulli(ρ_K^*) random variables. Hence, by the Chernov bound we have

$$\mathbf{P}\left(\widehat{N}^{\mathrm{ab}} \geq \frac{1}{2}K\rho_K^*\right) \geq 1 - \exp\left(-\frac{1}{8}K\rho_K^*\right). \tag{4.12}$$

Specialize (4.9) at $(t, a, b) = (K^{-2\alpha}, K^{-\frac{1}{2}}, \frac{1}{2}K^{-\alpha})$ to obtain

$$\rho_K^* = \sum_{n=0}^{\infty} \rho'_{K,n} \exp(-2(2n+1)^2\pi^2),$$

where

$$\rho'_{K,n} := \frac{4}{(2n+1)\pi} \sin(2(2n+1)\pi K^{\alpha-\frac{1}{2}}).$$

With $\alpha < \frac{1}{2}$, we have $\lim_{K\to\infty}(K^{\frac{1}{2}-\alpha}\rho'_{K,n}) = 8$ and $|\rho'_{K,n}| \le 8K^{\alpha-\frac{1}{2}}$, and it is straightforward to show that

$$\lim_{K\to\infty}(K^{\frac{1}{2}-\alpha}\rho^*_K) = 8\sum_{n=0}^{\infty}\exp(-2(2n+1)^2\pi^2) > 0.$$

Consequently, $\rho^*_K \ge \frac{1}{C}K^{\alpha-\frac{1}{2}}$. Inserting this into (4.12), we arrive at

$$\mathbf{P}\Big(\widehat{N}^{\mathrm{ab}} \ge \frac{1}{C}K^{\alpha+\frac{1}{2}}\Big) \ge 1 - \exp\Big(-\frac{1}{C}K^{\alpha+\frac{1}{2}}\Big) \ge 1 - CK^{-n}.$$

Next, we consider the process $(X^{\mathrm{ab}}_i(t); t \ge 0)_{i=1}^K$, consisting of K independent Brownian motions starting at $x = 1$ and absorbed only at $x = 0$, coupled to $(\widehat{X}^{\mathrm{ab}}_i(t))_i$ by the natural coupling that each ith particle share the same underlying driving Brownian motion. Let $X^{\mathrm{ab},K}_i(t) := \frac{1}{\sqrt{K}}X^{\mathrm{ab}}_i(Kt)$ denote the scaled process, let $\Gamma := \{X^{\mathrm{ab},K}_i(K^{-2\alpha}) : 0 < X^{\mathrm{ab},K}_i(t) < \frac{1}{2}K^{-\alpha}, \forall t \le K^{-2\alpha}\}$ denote the set of all $X^{\mathrm{ab},K}$-particles that stay within $(0, \frac{1}{2}K^{-\alpha})$ for all $t \le K^{-2\alpha}$ and let $N^{\mathrm{ab}} := \#\Gamma$. We clearly have $N^{\mathrm{ab}} \ge \widehat{N}^{\mathrm{ab}}$ and, therefore,

$$\mathbf{P}\Big(N^{\mathrm{ab}}(K^{-2\alpha}) \ge \frac{1}{C}K^{\alpha+\frac{1}{2}}\Big) \ge 1 - CK^{-n}. \tag{4.13}$$

Now, couple $(X^{\mathrm{ab},K}(t))$ and $(X^K(t))$ by the aforementioned natural coupling. On the event $\{N^{\mathrm{ab}} \ge \frac{1}{C}K^{\alpha+\frac{1}{2}}\}$, to move all X^K-particles in Γ to the level $x = K^{-\alpha}$ requires at least a drift of $N^{\mathrm{ab}}(\frac{1}{2}K^{-\alpha}) \ge \frac{1}{C}K^{\frac{1}{2}}$, while the total amount of (scaled) drift at disposal is $K^{-2\alpha+\frac{1}{2}}$. This is less than $\frac{1}{C}K^{\frac{1}{2}}$ for all large enough K. Consequently, the desired result (4.10) follows from (4.13). $\square$

Equipped with the short-time estimate (4.10), we now return to showing (4.8). Consider the threshold function

$$z^*(t) = K^{-\alpha}\mathbf{1}_{\{t\le K^{-2\alpha}\}} + (\sqrt{t}K^{-\beta})\mathbf{1}_{\{t>K^{-2\alpha}\}}, \tag{4.14}$$

and the corresponding hitting time $\tau := \inf\{t \in \mathbb{R}_+ : Z_K(t) \ge z^*(t)\}$. It suffices to show $\mathbf{P}(\tau > \frac{1}{2} + \frac{1}{7}K^{-2\beta}) \ge 1 - CK^{-n}$. To this end, by Lemma 4.4, without loss of generality we assume $\tau \in (K^{-2\alpha}, 1)$. As the trajectory of Z_K is continuous except when it hits 0, we have $Z_K(\tau) \ge z^*(\tau)$. Hence at time τ, no particle exists between 0 and $z^*(\tau)$, or equivalently $\widetilde{U}_K(\tau, z^*(\tau)) = \widetilde{U}_K(\tau, 0)$. With this, taking the difference of (4.2) at $x = z^*(\tau)$ and at $x = 0$, and multiplying the result by $\sqrt{\frac{\pi\tau}{2}}$, we obtain

$$h_1 = h_2 + \sqrt{\frac{\pi\tau}{2}}\big(R_K(\tau, z^*(\tau)) - R_K(\tau, 0)\big),$$

where

$$h_1 := \sqrt{\frac{\pi\tau}{2}}\big(\widetilde{G}_K(\tau,0) - \widetilde{G}_K\big(\tau, z^*(\tau)\big)\big),$$

$$h_2 := \sqrt{\frac{\pi\tau}{2}} \int_0^\tau f_2\big(s, Z_K(s), z^*(\tau)\big)\,ds$$

and

$$f_2(s,z,z') := p^{\mathrm{N}}(\tau-s,z,z') - p^{\mathrm{N}}(\tau-s,z,0). \tag{4.15}$$

Further using (2.65), for fixed $\delta \in (0, \frac{1-2\alpha}{4} - 4\beta)$, to control the remainder term $(R_K(\tau,z^*) - R_K(\tau,0))$, with $\tau \geq K^{-2\alpha}$, we have

$$h_1 \leq h_2 + CK^{\frac{2\alpha-1}{4}+\delta}, \tag{4.16}$$

with probability $1 - CK^{-n}$. Given the inequality (4.16), the strategy of the proof is to extract the bound $\tau \geq \frac{1}{2} + \frac{1}{7}K^{-2\beta}$ from (4.16). To this end, we next derive a lower bound on h_1 and an upper bound on h_2.

With $\widetilde{G}_K(t,x)$ defined as in (2.8), we have

$$h_1 = h_1\big(K^{-\beta}\big),$$

where

$$h_1(a) = \sqrt{K\tau}\int_0^{\frac{1}{\sqrt{K\tau}}} \left(e^{-\frac{y^2}{2}} - \frac{1}{2}e^{-\frac{(y+a)^2}{2}} - \frac{1}{2}e^{-\frac{(y-a)^2}{2}}\right) dy.$$

Taylor-expanding $h_1(a)$ to the fifth order gives $h_1(a) \geq a^2 h_{12} + a^4 h_{14} - Ca^6$, where $h_{12} := \sqrt{K\tau}\int_0^{\frac{1}{\sqrt{K\tau}}} e^{-y^2/2}(\frac{1}{2} - \frac{1}{2}y^2)\,dy$ and $h_{14} := \sqrt{K\tau}\int_0^{\frac{1}{\sqrt{K\tau}}} e^{-y^2/2}(-\frac{1}{8} + \frac{1}{4}y^2 - \frac{1}{24}y^4)\,dy$. Further Taylor-expanding h_{12} and h_{14} in $\frac{1}{\sqrt{K\tau}}$ yields $h_{12} \geq \frac{1}{2} - \frac{C}{K\tau}$ and $h_{14} \geq -\frac{1}{8} - \frac{C}{K\tau}$ and, therefore,

$$\begin{aligned} h_1 &\geq \frac{1}{2}a^2 - \frac{1}{8}a^4 - Ca^2\left(\frac{1}{K\tau} + a^4\right) \\ &\geq \frac{1}{2}a^2 - \frac{1}{8}a^4 - Ca^2\big(K^{2\alpha-1} + a^4\big) \qquad \text{for } a = K^{-\beta}. \end{aligned} \tag{4.17}$$

Turning to estimating h_2, we first observe that the function $f_2(s,z,z')$ as in (4.15) increases in z, $\forall z \leq z'$, as is readily verified by taking derivative as follows:

$$\begin{aligned} &\sqrt{2\pi(\tau-s)^3}\partial_z f_2(s,z,z') \\ &\quad = z'\big(e^{-(z-z')^2/2} - e^{-(z+z')^2/2}\big) \\ &\qquad - z\big(e^{-(z-z')^2/2} + e^{-(z+z')^2/2}\big) + 2ze^{-z^2/2} \\ &\quad \geq z\big(e^{-(z-z')^2/2} - e^{-(z+z')^2/2}\big) \\ &\qquad - z\big(e^{-(z-z')^2/2} + e^{-(z+z')^2/2}\big) + 2ze^{-z^2/2} \geq 0. \end{aligned}$$

Now, since $t \mapsto z^*(t)$ is increasing for all $t \geq K^{-2\alpha}$, to obtain an upper bound on h_2 we replace $Z_K(s)$ with $z^*(\tau)$ for $s \geq K^{-2\alpha}$. Further, with $\int_0^{K^{-2\alpha}} p^{\mathrm{N}}(\tau - s, z, z')\,ds \leq CK^{-\alpha}$, we obtain $h_2 \leq CK^{-\alpha} + \sqrt{\frac{\pi\tau}{2}} \int_0^\tau f_2(s, z^*(\tau), z^*(\tau))\,ds$. With $z^*(\tau) = K^{-\beta}\sqrt{\tau}$, the last integral is evaluated explicitly by using (3.14), yielding

$$h_2 \leq \tau h_2\bigl(K^{-\beta}\bigr) + CK^{-\alpha}$$

where

$$h_2(a) = 1 + e^{-2a^2} - 2e^{-a^2/2} + 2a \int_a^{2a} e^{-y^2/2}\,dy.$$

Taylor-expanding $h_2(a)$ to the fifth order, we further obtain

$$h_2 \leq \tau\left(a^2 - \frac{7}{12}a^4 + Ca^6\right) + CK^{-\alpha} \qquad \text{for } a = K^{-\beta}. \tag{4.18}$$

Now, combining (4.16)–(4.18), we arrive at

$$\tau \geq \frac{\frac{1}{2} - \frac{1}{8}a^2 - C(K^{2\alpha-1} + a^4 + a^{-2}K^{-\alpha} + a^{-2}K^{\frac{2\alpha-1}{4}+\delta})}{1 - \frac{7}{12}a^2 + Ca^4} \tag{4.19}$$

$$\text{for } a = K^{-\beta}.$$

With α and δ chosen as in the preceding, it is now straightforward to check that, for $a = K^{-\beta}$,

$$\begin{aligned}
&\frac{\frac{1}{2} - \frac{1}{8}a^2 - C(K^{2\alpha-1} + a^4 + a^{-2}K^{-\alpha} + a^{-2}K^{\frac{2\alpha-1}{4}+\delta})}{1 - \frac{7}{12}a^2 + Ca^4} \\
&\quad = \frac{\frac{1}{2} - \frac{1}{8}a^2}{1 - \frac{7}{12}a^2} + (\text{higher order terms}) \\
&\quad = \frac{1}{2} + \frac{1}{6}K^{-2\gamma} + (\text{higher order terms}).
\end{aligned}$$

From this, the we conclude the desired result: $\tau > \frac{1}{2} + \frac{1}{7}K^{-2\gamma}$, with probability $\geq 1 - CK^{-n}$.

4.2. *Proof of Proposition* 4.2(b). To simplify notation, we let $\sigma_K := \frac{1}{2} + \frac{1}{7}K^{-2\gamma}$. Define the scaled distribution function of surviving X-particles as

$$U_K(t, x) := \frac{1}{\sqrt{K}}\#\{0 < X_i^K(t) \leq x\} = \langle \mu_t^K, \mathbf{1}_{(0,x]}\rangle, \tag{4.20}$$

and, to simplify notation, we let

$$U_{\star\star}(x) := \widetilde{U}_\star\left(\frac{1}{2}, 0\right) - \widetilde{U}_\star\left(\frac{1}{2}, x\right) = \int_0^x u_1\left(\frac{1}{2}, y\right)dy, \tag{4.21}$$

where $u_1(t,y)$ is defined in (1.12). Recall that $\gamma < \gamma_1 \in (0, \frac{1}{96})$ are fixed. Fix furthering $\gamma_3 < \gamma_2 \in (\gamma, \gamma_1)$, we begin with an estimate on $U_K(t,x)$.

LEMMA 4.5. *There exists $C < \infty$ such that*

$$\mathbf{P}\left(\left|U_K\left(\frac{1}{2}, x\right) - U_{\star\star}(x)\right| \le CK^{-4\gamma_2}, \forall x \in \mathbb{R}\right) \ge 1 - CK^{-n}, \tag{4.22}$$

$$\mathbf{P}\big(\big|U_K(\sigma_K, x) - U_{\star\star}(x)\big| \le K^{-\gamma}, \forall x \in \mathbb{R}\big) \ge 1 - CK^{-n}. \tag{4.23}$$

PROOF. With $U_K(t,x) = \widetilde{U}_K(t,0) - \widetilde{U}_K(t,x)$ and $U_{\star\star}(x)$ defined in (4.21), we have that $|U_K(t,x) - U_{\star\star}(x)| \le 2\sup_{y\in\mathbb{R}} |\widetilde{U}_K(t,y) - \widetilde{U}_\star(\frac{1}{2}, y)|$. To bound the right-hand side, we take the difference of the integral identities (4.2) and (2.1) to obtain

$$\begin{aligned}
&\left|\widetilde{U}_K(t,x) - \widetilde{U}_\star\left(\frac{1}{2}, x\right)\right| \\
&\quad \le \big|\widetilde{G}_K(t,x) - 2p(t,x)\big| + \left|2p(t,x) - 2p\left(\frac{1}{2}, x\right)\right| && (4.24)\\
&\qquad + \int_0^t \big|p^{\mathrm{N}}\big(t-s, Z_K(s), x\big) - p^{\mathrm{N}}\big(t-s, z_\star(s), x\big)\big|\, ds && (4.25)\\
&\qquad + \int_{\frac{1}{2}}^t p^{\mathrm{N}}\big(t-s, z_\star(s), x\big)\, ds \\
&\qquad + \big|R_K(t,x)\big|. && (4.26)
\end{aligned}$$

We next bound each of the the terms in (4.24)–(4.26):

- Using (4.4) for $\alpha = 2\gamma_2$ yields $|\widetilde{G}_K(t,x) - 2p(t,x)| \le CK^{-4\gamma_2}$.
- Using (2.17) for $\alpha = 1$, gives $|2p(t,x) - 2p(\frac{1}{2}, x)| \le C|\frac{1}{2} - t|$, $\forall t \ge \frac{1}{2}$.
- Using (4.5) for $\alpha' = \frac{\gamma_2}{\gamma_1}$ and (4.6) for $\beta = 4\gamma_1$, we have $\int_0^t |p^{\mathrm{N}}(t-s, Z_K(s), x) - p^{\mathrm{N}}(t-s, z_\star(s), x)|\, ds \le C \sup_{s\le t} |Z_K(s) - z_\star(s)|^{\alpha'} \le CK^{-4\gamma_2}$, with probability $\ge 1 - CK^{-n}$.
- Using $p^{\mathrm{N}}(t-s, z_\star(s), s) \le \frac{2}{\sqrt{2\pi(t-s)}}$, we obtain $\int_{\frac{1}{2}}^t p^{\mathrm{N}}(t-s, z_\star(s), x)\, ds \le \sqrt{\frac{2}{\pi}|t - \frac{1}{2}|}$.
- Using (2.65), we have $|R_K(t,x)| \le CK^{-4\gamma_2}$, $\forall t \in [\frac{1}{2}, \sigma_K]$, $x \in \mathbb{R}$, with probability $\ge 1 - CK^{-n}$.

Combining these bounds yields

$$\begin{aligned}
&\left|\widetilde{U}_K(t,x) - \widetilde{U}_\star\left(\frac{1}{2}, x\right)\right| \\
&\quad \le CK^{-4\gamma_2} + C\left|t - \frac{1}{2}\right| + \sqrt{\frac{2}{\pi}\left|t - \frac{1}{2}\right|} \qquad \forall t \in \left[\frac{1}{2}, \sigma_K\right],
\end{aligned} \tag{4.27}$$

with probability $\geq 1 - CK^{-n}$. Substituting in $t = \frac{1}{2}$ in (4.27) yields (4.22). Similarly, substituting in $t = \sigma_K$ in (4.27), with $|\sigma_K - \frac{1}{2}| = \frac{K^{-2\gamma}}{7}$, we have, with probability $\geq 1 - CK^{-n}$,

$$\left|\widetilde{U}_K(\sigma_K, x) - \widetilde{U}_\star\left(\frac{1}{2}, x\right)\right| \leq CK^{-4\gamma_2} + CK^{-2\gamma} + \sqrt{\frac{2}{7\pi}}K^{-\gamma} < K^{-\gamma},$$

for all K large enough. This concludes (4.23). $\square$

Recall the definition of Atlas models from the beginning of Section 2. Our strategy of proving Proposition 4.2(b) is to *reduce* the problem of the particle system $(X(s); s \geq \sigma_K)$ to a problem of certain Atlas models $(\overline{Y}(t) : t \geq 0)$ and $(\underline{Y}(t) : t \geq 0)$, constructed as follows. To construct such Atlas models, recalling the expression of $u_1(\frac{1}{2}, x)$ from (1.12), we define

$$\overline{u}(x) := \begin{cases} u_1\left(\frac{1}{2}, x\right) & \text{when } x \geq K^{-\gamma}, \\ 0 & \text{when } x < K^{-\gamma}, \end{cases} \tag{4.28}$$

$$\underline{u}(x) := \begin{cases} u_1\left(\frac{1}{2}, x\right) & \text{when } x > 0, \\ u_1\left(\frac{1}{2}, 0\right) = 2 & \text{when } -K^{-4\gamma_3} \leq x \leq 0, \\ 0 & \text{when } x < -K^{-4\gamma_3}. \end{cases} \tag{4.29}$$

Adopting the notation $\mathrm{PPP}(f(x))$ for a Poisson point process on $\mathbb{R}$ with density $f(x)$, for each $K < \infty$ we let $(\overline{Y}(t; K) : t \geq 0)$ and $(\underline{Y}(t; K) : t \geq 0)$ be Atlas models starting from the following initial conditions:

$$(\overline{Y}_i(0; K))_i \sim \mathrm{PPP}\left(\overline{u}\left(\frac{x}{\sqrt{K}}\right)\right), \qquad (\underline{Y}_i(0; K))_i \sim \mathrm{PPP}\left(\underline{u}\left(\frac{x}{\sqrt{K}}\right)\right), \tag{4.30}$$

and let $\overline{W}(t; K) := \min_i \overline{Y}_i(t; K)$ and $\underline{W}(t; K) := \min_i \underline{Y}_i(t; K)$ denote the corresponding laggards.

REMARK 4.6. The notation $\overline{Y}_i(t; K)$, etc., are intended to highlight the dependence on K of the processes, as is manifest from (4.30). To simplify notation, however, hereafter we omit the dependence, and write $\overline{Y}_i(t; K) = \overline{Y}_i(t)$, etc., unless otherwise noted.

We let $\overline{Y}_i^K(t) := \frac{1}{\sqrt{K}}\overline{Y}_i(Kt)$, denote the scaled process, and similarly for $\underline{Y}_i^K(t)$, $\overline{W}_K(t)$ and $\underline{W}_K(t)$. Under this notation, equation (4.30) translates into

$$(\overline{Y}_i^K(0))_i \sim \mathrm{PPP}(\overline{u}(x)), \qquad (\underline{Y}_i^K(0))_i \sim \mathrm{PPP}(\underline{u}(x)). \tag{4.31}$$

We let $\overline{V}_K(t,x) := \frac{1}{\sqrt{K}}\#\{\overline{Y}_i^K(t) \le x\}$ and $\underline{V}_K(t,x) := \frac{1}{\sqrt{K}}\#\{\underline{Y}_i^K(t) \le x\}$ denote the corresponding scaled distribution functions.

Having introduced the Atlas models $\overline{Y}$ and $\underline{Y}$, we next establish couplings that *relate* these models to the relevant particle system X. Recall the definition of the extinction time τ_{ext}^K from (4.1). We let

$$\tau_{\mathrm{abs}}^K := \inf\{t > \sigma_K : Z_K(t) = 0\} \tag{4.32}$$

denote the first absorption time (scaled by K^{-1}) after σ_K.

LEMMA 4.7. *There exists a coupling of* $(X^K(s+\frac{1}{2}); s \ge 0)$ *and* $(\underline{Y}^K(s); s \ge 0)$ *under which*

$$\mathbf{P}\left(\underline{W}_K(s) \le Z_K\left(\frac{1}{2}+s\right), \forall s + \frac{1}{2} < \tau_{ext}^K\right) \ge 1 - CK^{-n}. \tag{4.33}$$

Similarly, there exists a coupling of $(X^K(s+\sigma_K); s \ge 0)$ *and* $(\overline{Y}^K(s); s \ge 0)$ *under which*

$$\mathbf{P}(\overline{W}_K(s) \ge Z_K(s+\sigma_K), \forall s + \sigma_K < \tau_{ext}^K \wedge \tau_{abs}^K) \ge 1 - CK^{-n}. \tag{4.34}$$

The proof requires a coupling result from [21].

LEMMA 4.8 ([21], Corollary 3.9). *Let* $(Y_i(s); s \ge 0)_{i=1}^m$ *and* $(Y_i'(s); s \ge 0)_{i=1}^{m'}$ *be Atlas models, and let* $W(s)$ *and* $W'(s)$ *denote the corresponding laggards. If* $Y'(0)$ *dominates* $Y(0)$ *componentwisely in the sense that*

$$m' \le m, \qquad Y_i'(0) \ge Y_i(0), \qquad i = 1, \dots, m', \tag{4.35}$$

then there exists a coupling of Y *and* Y' *(for* $s > 0$*) such that the dominance continues to hold for* $s > 0$*, that is,* $Y_i'(s) \ge Y_i(s)$, $i = 1, \dots, m'$*. In particular,* $W'(s) \ge W(s)$.

PROOF OF LEMMA 4.7. As will be more convenient for the notation for this proof, we work with *unscaled* processes $X(s+\frac{1}{2}K)$, $X(s+\sigma_K K)$ and $Y(s)$, and construct the coupling accordingly.

We consider first $\underline{Y}$ and prove (4.33). At $s = 0$, order the particles as $(\underline{W}(0) = \underline{Y}_1(0) \le \underline{Y}_2(0) \le \cdots)$, and $(Z(\frac{1}{2}K) = X_1(\frac{1}{2}K) \le X_2(\frac{1}{2}K) \le \cdots)$. We claim that, regardless of the coupling, the following holds with probability $1 - CK^{-n}$:

$$\begin{aligned} &\#\{\underline{Y}_i(0)\} \ge \#\left\{X_i\left(\frac{1}{2}K\right)\right\} \quad \text{and} \\ &\underline{Y}_i(0) \le X_i\left(\frac{1}{2}K\right) \qquad \forall 1 \le i \le \#\left\{X_j\left(\frac{1}{2}K\right)\right\}. \end{aligned} \tag{4.36}$$

Recalling from (4.20) that $U_K(t,x)$ denotes the *scaled* distribution function of $X(t)$, with $U_K(t,x)|_{x<0}=0$, we see that (4.36) is equivalent to the following:

$$\mathbf{P}\Big(\underline{V}_K(0,x) \geq U_K\Big(\frac{1}{2},x\Big), \forall x \in \mathbb{R}_+\Big) \geq 1 - CK^{-n}. \tag{4.37}$$

To see why (4.37) holds, with $(\underline{Y}_i^K(0))_i$ distributed in (4.31), we note that $x \mapsto \sqrt{K}\,\underline{V}_K(0,x)$, $x \in [-K^{-4\gamma_3}, \infty)$ is an inhomogeneous Poisson process with density $\sqrt{K}\underline{u}(x)$. From this, it is standard (using Doob's maximal inequality and the BDG inequality) to show that

$$\Big\|\sup_{x\in\mathbb{R}} |\underline{V}_K(0,x) - \int_{-K^{-4\gamma_3}}^{x} \underline{u}(y)\,dy|\Big\|_m \leq C(m)K^{-\frac{1}{4}} \qquad \forall m \geq 2. \tag{4.38}$$

Further, with $\underline{u}$ defined in (4.29), we have

$$\int_{-K^{-4\gamma_3}}^{x} \underline{u}(y)\,dy = U_{\star\star}(x) + 2K^{-4\gamma_3} \qquad \forall x \geq 0.$$

Inserting this into (4.38), followed by using Markov's inequality $\mathbf{P}(|\xi| > K^{-\frac{1}{8}}) \leq K^{-\frac{m}{8}}\mathbf{E}(|\xi|^m)$ for $m = 8n$, we arrive at

$$\mathbf{P}\big(\big|\underline{V}_K(0,x) - U_{\star\star}(x) - 2K^{-4\gamma_3}\big| \leq K^{-\frac{1}{8}}, \forall x \in \mathbb{R}_+\big) \geq 1 - CK^{-n}. \tag{4.39}$$

Combining (4.39) and (4.22) yields

$$\underline{V}_K(0,x) - U_K\Big(\frac{1}{2},x\Big) \geq -K^{-\frac{1}{8}} - CK^{-4\gamma_2} + 2K^{-4\gamma_3} \qquad \forall x \in \mathbb{R}_+, \tag{4.40}$$

with probability $\geq 1 - CK^{-n}$. With $\gamma_3 < \gamma_2 < \frac{1}{96}$, the right-hand side of (4.40) is positive for all K large enough, so (4.37) holds.

Assuming the event (4.36) holds, we proceed to construct the coupling for $s > 0$. Let $\tau_1 := \inf\{t \geq \frac{1}{2}K : Z(t) = 0\}$ be the first absorption time after time $\frac{1}{2}K$. For $s \in [0, \tau_1 - \frac{1}{2}K)$, both processes $\underline{Y}(s)$ and $X(s + \frac{1}{2}K)$ evolve as Atlas models. Hence, by Lemma 4.8 for $(Y(s), Y'(s)) = (\underline{Y}(s), X(s + \frac{1}{2}K))$, we have a coupling such that

$$\underline{Y}_i(s) \leq X_i\Big(s + \frac{1}{2}K\Big) \qquad \forall 1 \leq i \leq \#\Big\{X_j\Big(\frac{1}{2}K\Big)\Big\}, s \in \Big[0, \tau_1 - \frac{1}{2}K\Big).$$

At time $t = \tau_1$, the system X *loses* a particle, so by reorder $(\underline{Y}_i(\tau_1 - \frac{1}{2}K))_i$ and $(X_i(\tau_1))_i$, we retain the type of dominance as in (4.36). Based on this, we iterate the prescribed procedure to the second absorption $\tau_2 := \inf\{s > \tau_1 : Z(s) = 0\}$. As absorption occurs at most K times, the iteration procedure yields the desired coupling until the extinction time τ_{ext}. We have thus constructed a coupling of $(\underline{Y}(s); s \geq 0)$ and $(X(s + \frac{1}{2}K) : s \geq 0)$ under which (4.33) holds.

We now turn to $\overline{Y}$ and construct the analogous coupling of $(\underline{Y}(s); s \geq 0)$ and $(X(s + \sigma_K K) : s \geq 0)$. Similar to (4.38), for $\overline{V}_K(0,x)$ we have that

$$\begin{aligned} &\mathbf{P}\biggl(\biggl|\overline{V}_K(0,x) - \int_{K^{-\gamma}}^{x} u_1\biggl(\frac{1}{2}, y\biggr)\,dy\biggr| \leq K^{-\frac{1}{8}}, \forall x \geq K^{-\gamma}\biggr) \\ &\quad \geq 1 - CK^{-n}. \end{aligned} \tag{4.41}$$

As seen from the expression (1.12), $u_1(\frac{1}{2}, 0) = 2$ and $x \mapsto u_1(\frac{1}{2}, x)$ is smooth with bounded derivatives, so in particular,

$$\biggl|\int_{K^{-\gamma}}^{x} u_1\biggl(\frac{1}{2}, y\biggr)\,dy - \bigl(U_{\star\star}(x) - 2K^{-\gamma}\bigr)\biggr| \leq CK^{-2\gamma} \qquad \forall x \geq K^{-\gamma}.$$

Inserting this estimate into (4.41), and combining the result with (4.23), we obtain that, with probability $\geq 1 - CK^{-n}$,

$$\begin{aligned} \overline{V}_K(0,x) &\leq U_K(\sigma_K, x) - 2K^{-\gamma} + K^{-\gamma} + CK^{-2\gamma} \\ &\leq U_K(\sigma_K, x) \qquad \forall x \geq K^{-\gamma}, \end{aligned}$$

for all K large enough. This together with $\overline{V}_K(0,x)|_{x<K^{-\gamma}} = 0$ yields the following dominance condition:

$$\begin{aligned} &\#\{\overline{Y}_i(0)\} \leq \#\{X_i(\sigma_K K)\} \quad \text{and} \\ &\overline{Y}_i(0) \geq X_i(\sigma_K K) \qquad \forall 1 \leq i \leq \#\{\overline{Y}_j(0)\}, \end{aligned} \tag{4.42}$$

with probability $\geq 1 - CK^{-n}$. Based on this, we construct the coupling for $\overline{Y}$ and X similar to the proceeding. Unlike in the proceeding, however, when an absorption occurs, dominance properties of the type (4.42) may be destroyed. Hence here we obtain the coupling with the desired property only up to the first absorption time, as in (4.34). $\square$

We see from Lemma 4.7 that $\overline{W}_K$ and $\underline{W}_K$ serve as suitable upper and lower bounds for Z_K. With this, we now turn our attention to the Atlas models $\overline{Y}$ and $\underline{Y}$, and aim at establishing the hydrodynamic limits of $\overline{W}_K$ and $\underline{W}_K$. To this end, recalling from (3.8) the definition of $|\cdot|'_{[0,T]}$ and that $T < \infty$ is fixed, we begin by establishing the following estimates on $|\overline{W}_K|'_{[0,T]}$ and $|\underline{W}_K|'_{[0,T]}$.

LEMMA 4.9. *There exists $C < \infty$ such that*

$$\mathbf{P}\bigl(|\overline{W}_K|'_{[0,T]} \leq CK^{-\frac{1}{8}}\bigr) \geq 1 - CK^{-n}, \tag{4.43}$$

$$\mathbf{P}\bigl(|\underline{W}_K|'_{[0,T]} \leq CK^{-\frac{1}{8}}\bigr) \geq 1 - CK^{-n}, \tag{4.44}$$

PROOF. The proofs of (4.43)–(4.44) are similar, and we work out only the former here.

At any given time $s \in \mathbb{R}_+$, let us *order* the $\overline{Y}$-particles as $W(s) = \overline{Y}_1(s) \le \overline{Y}_2(s) \le \cdots \le \overline{Y}_N(s)$, where $N := \#\{\overline{Y}_j(0)\}$, and let $\overline{G}_i(s) := \overline{Y}_{i+1}(s) - \overline{Y}_i(s)$ denote the corresponding gap process. We adopt the convention that $\overline{G}_i(s) := \infty$ if $i+1 > N$, so that $\overline{G}(s) := (\overline{G}_i(s))_{i=1}^{\infty}$ is $[0,\infty]^{\infty}$-valued.

We begin with a stochastic comparison of the gap process $\overline{G}(s)$. More precisely, given any $[0,\infty]^{\infty}$-valued random vectors ξ and ζ, we say ξ stochastically dominate ζ, denoted $\xi \succeq \zeta$, if there exists a coupling of ξ and ζ under which $\xi_i \ge \zeta_i$, $i = 1, 2, \ldots$. Since $(\overline{Y}_i(0))$ is distributed as in (4.31), with $\overline{u}(x) \le 2$, $\forall x \in \mathbb{R}$, we have that

$$\overline{G}(0) \succeq \bigotimes_{i=1}^{\infty} \operatorname{Exp}(2). \tag{4.45}$$

By [20], Theorem 4.7, for any Atlas model satisfying the dominance property (4.45), the dominance will continue to hold for $s > 0$, that is, $\overline{G}(s) \succeq \bigotimes_{i=1}^{\infty} \operatorname{Exp}(2)$. (Theorem 4.7 of [20] does not state $\overline{G}(s) \succeq \bigotimes_{i=1}^{\infty} \operatorname{Exp}(2)$ explicitly, but the statement appears in the first line of the proof, wherein $\pi = \bigotimes_{i=1}^{\infty} \operatorname{Exp}(2)$; cf., [20] and [18], Example 1.)

Having established the stochastic comparison of $\overline{G}(s)$, we now return to estimating $|\overline{W}_K|'_{[0,T]}$. The seminorm $|\cdot|'_{[0,T]}$, defined in (3.8), measures how nondecreasing the given function is. To the end of bounding $|\overline{W}_K|'_{[0,T]}$, we fix $s_* \in [0, T]$, and begin by bounding the quantity

$$\sup_{s \in [s_*, T]} \big(\overline{W}_K(s_*) - \overline{W}_K(s)\big). \tag{4.46}$$

We consider an *infinite Atlas model* $(Y_i^*(s); t \ge 0)_{i=1}^{\infty}$, which is defined analogously to (2.2) via the following stochastic differential equations:

$$\begin{aligned} &dY_i^*(s) = \mathbf{1}_{\{Y_i^*(s) = W^*(s)\}}\, dt + dB_i(s), \qquad i = 1, 2, \ldots, \\ &W^*(s) := \min\nolimits_{i=1}^{\infty}\{Y_i^*(s)\}, \end{aligned} \tag{4.47}$$

with the following initial condition:

$$\begin{aligned} &Y_1^*(0) := \underline{Y}_1(Ks_*), \qquad \text{and, independently} \\ &\big(Y_{i+1}^*(0) - Y_i^*(0)\big)_{i=1}^{\infty} \sim \bigotimes_{i=1}^{\infty} \operatorname{Exp}(2). \end{aligned} \tag{4.48}$$

General well-posedness conditions for (4.47) are studied in [14, 22]. In particular, the distribution (4.48) is an admissible initial condition, and is in fact a stationary distribution of the gaps [18]. Under such stationary gap distribution, the laggard $W^*(s)$ remains very close to a constant under diffusive scaling. More precisely, letting $W_K^*(s) := \frac{1}{\sqrt{K}} W^*(sK)$, by [7], Proposition 2.3, Remark 2.4, we have

$$\mathbf{P}\Big(\sup_{s \in [s_*, T]} \big|W_K^*(s) - W_K^*(0)\big| \le K^{-\eta}\Big) \ge 1 - C(\eta) K^{-n-2}, \tag{4.49}$$

for any fixed $\eta \in (0, \frac{1}{4})$. In view of the bound (4.49), the idea of bounding the quantity (4.46) is to couple $(Y^*(s); s \geq 0)$ and $(\overline{Y}(s + s_*); s \geq 0)$. As we showed previously $G(Ks_*) \succeq \bigotimes_{i=1}^{\infty} \mathrm{Exp}(2)$. With $(Y_i^*(0))_i$ distributed in (4.48), we couple $(\overline{Y}_i(s_*))_i$ and $(Y_i^*(0))_i$ in such a way that

$$Y_i^*(0) \leq \overline{Y}_i(Ks_*), \qquad i = 1, 2, \dots, \#\{\overline{Y}_i(Ks_*)\}. \tag{4.50}$$

Equation (4.50) gives a generalization of the dominance condition (4.35) to the case where $m = \infty$. For such a generalization, we have the analogous coupling result from [21], Corollary 3.9, Remark 9, which gives a coupling of $(Y^*(s); s \geq 0)$ and $(\overline{Y}(s + s_*K); s \geq 0)$ such that

$$W^*(s) \leq \overline{W}(s + s_*K) \qquad \forall s \in \mathbb{R}_+. \tag{4.51}$$

Combining (4.49) for $\eta = \frac{1}{8}$ and (4.51), together with $W^*(0) = \overline{W}(s_*K)$ [by (4.48)], we obtain

$$\mathbf{P}\Big(\sup_{s \in [s_*, T]} \big(\overline{W}_K(s_*) - \overline{W}_K(s)\big) \leq K^{-\frac{1}{8}}\Big) \geq 1 - CK^{-n-2}. \tag{4.52}$$

Having established the bound (4.52) for fixed $s_* \in [0, T]$, we now take the union bound of (4.52) over $s_* = s_\ell := K^{-2}T\ell$, $1 \leq \ell \leq K^2$, to obtain

$$\mathbf{P}\Big(\sup_{s \in [s', T]} \big(\overline{W}_K(s') - \overline{W}_K(s)\big) \leq K^{-\frac{1}{8}}, \forall s' = s_1, s_2, \dots\Big) \geq 1 - CK^{-n}. \tag{4.53}$$

To pass from the "discrete time" $s' = s_1, s_2, \dots$ to $s' \in [0, T]$, adopting the same procedure we used for obtaining (2.61), we obtain the following continuity estimate:

$$\mathbf{P}\Big(\sup_{s \in [s_\ell, s_{\ell+1}]} \big|\overline{W}_K(s) - \overline{W}_K(s_\ell)\big| \leq K^{-\frac{1}{8}}, 1 \leq \ell \leq K^2\Big) \geq 1 - CK^{-n}. \tag{4.54}$$

Combining (4.53)–(4.54) yields

$$\mathbf{P}\Big(\sup_{s' < s \in [0, T]} \big(\overline{W}_K(s') - \overline{W}_K(s)\big) \leq 2K^{-\frac{1}{8}}\Big) \geq 1 - CK^{-n}.$$

This concludes the desired result (4.43). $\square$

We next establish upper bonds on $|\overline{W}_K|$ and $|\underline{W}_K|$.

LEMMA 4.10. *There exists $C < \infty$ and a constant $L = L(T) < \infty$ such that*

$$\mathbf{P}\big(\big|\overline{W}_K(t)\big| \leq L, \forall t \leq T\big) \geq 1 - CK^{-n}, \tag{4.55}$$

$$\mathbf{P}\big(\big|\underline{W}_K(t)\big| \leq L, \forall t \leq T\big) \geq 1 - CK^{-n}. \tag{4.56}$$

PROOF. We first establish (4.55). The first step is to derive an integral equation for $\overline{W}_K$. Recalling that $\overline{V}_K(t,x)$ denote the scaled distribution function of $\overline{Y}$, we apply Lemma 2.1(b) for $Y = \overline{Y}$ to obtain the following integral identity:

$$\begin{aligned}\overline{V}_K(t,x) = &\int_0^\infty p(t,x-y)\overline{V}_K(0,y)\,dy \\ &- \int_0^t p\big(t-s, x - W_K(s)\big)\,ds + R'_K(t,x).\end{aligned} \tag{4.57}$$

Note that the conditions (2.18)–(2.19) hold for $\overline{Y}(0)$, which is distributed as in (4.31). Using the approximating (4.39), we have

$$\begin{aligned}&\left|\int_0^\infty p(t,x-y)\overline{V}_K(0,y)\,dy - \int_0^t p(t,x-y)U_{\star\star}(y)\,dy\right| \\ &\quad\le CK^{-4\gamma_2},\end{aligned} \tag{4.58}$$

with probability $\ge 1 - CK^{-n}$. Using (4.58) and (2.65) in (4.57), we rewrite the integral identity as

$$\begin{aligned}\overline{V}_K(t,x) = &\int_0^\infty p(t,x-y)U_{\star\star}(y)\,dy \\ &- \int_0^t p\big(t-s, x - W_K(s)\big)\,ds + \overline{F}'_K(t,x),\end{aligned} \tag{4.59}$$

for some $\overline{F}'_K(t,x)$ such that

$$\mathbf{P}\big(|\overline{F}'_K|_{L^\infty([0,T]\times\mathbb{R})} \le CK^{-4\gamma_2}\big) \ge 1 - CK^{-n}. \tag{4.60}$$

Further, with $(\overline{Y}_i^K(0))$ distributed as in (4.31), it is standard to verify that

$$\mathbf{P}\big(|\overline{W}_K(0)| \le CK^{-4\gamma_3}\big) \ge 1 - CK^{-n}. \tag{4.61}$$

By definition, $\overline{V}_K(t, \overline{W}_K(t)) = \frac{1}{\sqrt{K}}\#\{\overline{Y}_i^K(t) \in (-\infty, \overline{W}_K(t)]\} = \frac{1}{\sqrt{K}}$, so setting $x = \overline{W}_K(t)$ in (4.59) we obtain the following integral equations:

$$\begin{aligned}&\int_0^\infty p\big(t, \overline{W}_K(t) - y\big)U_{\star\star}(y)\,dy \\ &\quad= \int_0^t p\big(t-s, \overline{W}_K(t) - \overline{W}_K(s)\big)\,ds + \overline{F}_K\big(t, \overline{W}_K(t)\big),\end{aligned} \tag{4.62}$$

where $\overline{F}_K(t,x) := \frac{1}{\sqrt{K}} - \overline{F}'_K(t,x)$, which, by (4.60), satisfies

$$\mathbf{P}\big(|\overline{F}_K|_{L^\infty([0,T]\times\mathbb{R})} \le CK^{-4\gamma_3}\big) \ge 1 - CK^{-n}. \tag{4.63}$$

Having derived the integral equation (4.62) for $\overline{W}_K$, we proceed to showing (4.55) based on (4.62). To this, we define $w^*(t) := \overline{W}_K(0) + at$, for some $a \in \mathbb{R}_+$

to be specified later, and consider the first hitting time $\tau := \inf\{t : \overline{W}_K(t) \geq w^*(t)\}$. As $w \mapsto \int_0^\infty p(\tau, w-y)U_{\star\star}(y)\,dy$ is nondecreasing, by (4.61) we have

$$\begin{aligned}\int_0^\infty & p(\tau, \overline{W}_K(0) + a\tau - y)U_{\star\star}(y)\,dy \\ &\geq \int_0^\infty p(\tau, 1 + a\tau - y)U_{\star\star}(y)\,dy \\ &:= f_1(\tau),\end{aligned} \tag{4.64}$$

with probability $\geq 1 - CK^{-n}$. Using $\overline{W}_K(\tau) - \overline{W}_K(s) \geq a(\tau - s)$, $\forall s \leq \tau$, we obtain

$$\int_0^\tau p(\tau - s, \overline{W}_K(\tau) - \overline{W}_K(s))\,ds \leq \int_0^\infty p(s, as)\,ds := f_2(a). \tag{4.65}$$

For the functions f_1 and f_2, we clearly have $\inf_{t \leq T} f_1(t) := f_* > 0$ and $\lim_{a\to\infty} f_2(a) = 0$. With this, we now fix some large enough a with $f_2(a) < \frac{1}{2}f_*$, and insert the bounds (4.63)–(4.65) into (4.62) to obtain

$$\mathbf{P}\bigg(\bigg\{f_* \leq \frac{1}{2}f_* + K^{-4\gamma_3}\bigg\} \cap \{\tau \geq T\}\bigg) \geq 1 - CK^{-n}.$$

Since $f_* > 0$, the event $\{f_* \leq \frac{1}{2}f_* + K^{-4\gamma_3}\}$ is empty for all large enough K, so

$$\mathbf{P}(\overline{W}_K(t) \leq \overline{W}_K(0) + aT, \forall t \leq T) \geq 1 - CK^{-n}.$$

This together with (4.61) gives the upper bound $\mathbf{P}(\overline{W}_K(t) \leq L, \forall t \leq T) \geq 1 - CK^{-n}$ for $L := 1 + aT$. A lower bound $\mathbf{P}(\overline{W}_K(t) \geq -L, \forall t \leq T) \geq 1 - CK^{-n}$ follows directly from (4.52) for $s_* = 0$. From these, we conclude the desired result (4.55).

Similar to (4.62), for $\underline{V}_K(t,x)$ we have

$$\begin{aligned}\int_0^\infty & p(t, \underline{W}_K(t) - y)U_{\star\star}(y)\,dy \\ &= \int_0^t p(t - s, \underline{W}_K(t) - \underline{W}_K(s))\,ds + \underline{F}_K(t, \underline{W}_K(t)),\end{aligned} \tag{4.66}$$

for some $\underline{F}_K(t,x)$ satisfying

$$\mathbf{P}(|\underline{F}_K|_{L^\infty([0,T]\times\mathbb{R})} \leq CK^{-\gamma}) \geq 1 - CK^{-n}. \tag{4.67}$$

From this, the same argument in the proceeding gives the desired bound (4.56) for $L = 1 + aT$. □

We now establish the hydrodynamic limit of $\overline{W}_K$ and $\underline{W}_K$.

LEMMA 4.11. *There exists* $z_\star(\cdot + \frac{1}{2}) \in \mathcal{C}(\mathbb{R}_+)$ *that solves* (1.10) (*which is unique by Corollary* 3.3). *Furthermore, for some* $C < \infty$, *we have*

$$\mathbf{P}\Big(\Big|\underline{W}_K(s) - z_\star\Big(\frac{1}{2} + s\Big)\Big| \le CK^{-4\gamma_3}, \forall s \in [0, T]\Big) \ge 1 - CK^{-n}, \tag{4.68}$$

$$\mathbf{P}\Big(\Big|\overline{W}_K(s) - z_\star\Big(s + \frac{1}{2}\Big)\Big| \le CK^{-\gamma}, \forall s \in [0, T]\Big) \ge 1 - CK^{-n}. \tag{4.69}$$

PROOF. The strategy of the proof is to utilize the fact that $\overline{W}_K$ and $\underline{W}_K$ satisfy the integral equations (4.62) and (4.66), respectively, and apply the stability estimate Lemma 3.2 to show the convergence of $\overline{W}_K$ and $\underline{W}_K$. Given the estimates (4.63) and (4.67), (4.43)–(4.44) and (4.55)–(4.56), the proof of (4.68) and (4.69) are similar, and we present only the former.

Such a $z_\star$ will be constructed as the unique limit point of $\overline{W}_K$. We begin by showing the convergence of $\overline{W}_K$. To this end, we fix $K_1 < K_2$, and consider the processes $\overline{W}_{K_1}$ and $\overline{W}_{K_2}$. Since they satisfy the integral equation (4.62), together with the estimates (4.61), (4.63), (4.43) and (4.55), we apply Lemma 3.2 for $(w_1, w_2) = (\overline{W}_{K_1}, \overline{W}_{K_2})$ to obtain

$$\mathbf{P}\big(\big|\overline{W}_{K_1}(s) - \overline{W}_{K_2}(s)\big| \le CK_1^{-4\gamma_3}, \forall s \in [0, T]\big) \ge 1 - C{K_1}^{-n}. \tag{4.70}$$

We now consider the subsequence $\{\overline{W}_{K_m}\}_{m=1}^\infty$, for $K_m := 2^m$. Setting $(K_1, K_2) = (2^m, 2^{m+j})$ in (4.70), and taking union bound of the result over $j \in \mathbb{N}$, we obtain

$$\mathbf{P}\Big(\sup_{t\in[0,T]} \big|\overline{W}_{K_m}(t) - \overline{W}_{K_{m'}}(t)\big| \le CK_m^{-4\gamma_3}, \forall m' > m\Big) \ge 1 - CK_m^{-n}. \tag{4.71}$$

From this, we conclude that $\{\overline{W}_{K_m}\}_m$ is almost surely Cauchy in $\mathcal{C}([0, T])$, and hence converges to a possibly random limit $W \in \mathcal{C}([0, T])$. Now, letting $K \to \infty$ in (4.62), with (4.63), we see that W must solve (1.10). Further, by (4.43) and (4.61), $t \mapsto W(t)$ is nondecreasing with $W(0) = 0$. Since, by Corollary 3.3, the solution to (1.10) is unique, $W(t) =: z_\star(t + \frac{1}{2})$ must in fact be deterministic. Now, letting $m' \to \infty$ in (4.71) yields

$$\mathbf{P}\Big(\sup_{s\in[0,T]}\Big|\overline{W}_{K_m}(s) - z_\star\Big(s + \frac{1}{2}\Big)\Big| \le CK_m^{-4\gamma_3}\Big) \ge 1 - CK_m^{-n}. \tag{4.72}$$

Combining (4.70) and (4.72), we conclude the desired result (4.68). □

Having established the hydrodynamic limit of the laggards $\overline{W}_K$ and $\underline{W}_K$ of the Atlas models $\overline{Y}$ and $\underline{Y}$, we now return to proving Proposition 4.2(b), that is, proving the hydrodynamic limit (4.7) of $Z_K(t)$ for $t \in [\sigma_K, T]$. We recall from Lemma 4.7 that we have a coupling of Z_K and $\underline{W}_K$ under which $\underline{W}_K(t - \frac{1}{2}) \le$

$Z_K(t)$, $\forall t \in [\frac{1}{2}, \tau^K_{\mathrm{ext}})$, with probability $\geq 1 - CK^{-n}$. By using the lower bound (4.3) on the scaled extinction time τ^K_{ext}, we have that

$$\mathbf{P}\Big(\underline{W}_K\Big(t-\frac{1}{2}\Big) \leq Z_K(t), \forall t \in \Big[\frac{1}{2}, T\Big]\Big) \geq 1 - CK^{-n}.$$

Combining this with (4.68) yields

$$\mathbf{P}\Big(Z_K(t) \geq z_\star(t) - CK^{-4\gamma_3}, \forall t \in \Big[\frac{1}{2}, T\Big]\Big) \geq 1 - CK^{-n}. \tag{4.73}$$

Equation (4.73) gives the desired lower bound on Z_K. Further, it provides a lower bound on the absorption time τ^K_{abs} [as defined in (4.32)]. To see this, we use (4.73) to write

$$\mathbf{P}\Big(\inf_{t\in[\sigma_K,T]} Z_K(t) \geq \inf_{t\in[\sigma_K,T]} z_\star(t) - CK^{-4\gamma_3}\Big) \geq 1 - CK^{-n}. \tag{4.74}$$

With $\sigma_K = \frac{1}{2} + \frac{1}{7}K^{-2\gamma}$ and $t \mapsto z_\star(t)$ being nondecreasing, the quadratic growth (3.20) of $z_\star(t)$ near $t = \frac{1}{2}$ gives

$$\inf_{t\in[\sigma_K,T]} z_\star(t) = z_\star(\sigma_K) \geq \frac{1}{C}K^{-4\gamma}. \tag{4.75}$$

Combining (4.75) with (4.74), followed by using $\gamma < \gamma_3$, we obtain

$$\mathbf{P}\Big(\inf_{t\in[\sigma_K,T]} Z_K(t) > 0\Big) = \mathbf{P}\big(\tau^K_{\mathrm{abs}} > T\big) \geq 1 - CK^{-n}. \tag{4.76}$$

Using the bounds (4.76) and (4.3) on τ^K_{abs} and τ^K_{ext} within the coupling (4.34), we have that $Z_K(t) \leq \overline{W}_K(t)$, $\forall t \in [\sigma_K, T]$, with probability $\geq 1 - CK^{-n}$. From this and (4.69), we conclude

$$\mathbf{P}\big(Z_K(t) \leq z_\star(t) + CK^{-\gamma}, \forall t \in [\sigma_K, T]\big) \geq 1 - CK^{-n}. \tag{4.77}$$

As $4\gamma_3 > \gamma$, the bounds (4.73) and (4.77) conclude the desired hydrodynamic limit (4.7) of $Z_K(t)$.

5. Proof of Theorem 1.2. We first settle part (a). To this end, we fix an arbitrary strategy $\phi(t) = (\phi_i(t))^K_{i=1}$, fix $\gamma \in (0, \frac{1}{4})$ and $n < \infty$, and use $C = C(\gamma, n) < \infty$ to denote a generic constant that depends only on γ, n, and *not* on the strategy in particular. Our goal is to establish an upper on $\widetilde{U}_K(\infty) := \lim_{t\to\infty} \widetilde{U}_K(t, Z_K(0))$, the total number of ever-surviving particles, scaled by $\frac{1}{\sqrt{K}}$. To this end, with $\widetilde{U}_K(\infty) \leq \widetilde{U}_K(\frac{1}{2}, 0)$, we set $t = \frac{1}{2}$ in (2.64) to obtain

$$\begin{aligned}\widetilde{U}_K(\infty) \leq \widetilde{G}_K\Big(\frac{1}{2}, 0\Big) &+ \sum_{i=1}^{K} \int_0^{\frac{1}{2}\wedge\tau_i^K} \phi_i^K(s)p^{\mathrm{N}}\Big(\frac{1}{2} - s, X_i^K(s), 0\Big)\,ds \\ &+ R_K\Big(\frac{1}{2}, x\Big).\end{aligned} \tag{5.1}$$

On the right-hand side of (5.1), we:

– use (2.83) to approximate $\widetilde{G}_K(\frac{1}{2}, 0)$ with $2p(t, x)$;
– use $p^{\mathrm{N}}(\frac{1}{2} - s, X_i^K(s), 0) \leq 2p(\frac{1}{2} - s, 0)$ and $\sum_{i=1}^K \phi_i^K(s) \leq 1$ to bound the integral term;
– use (2.65) to bound the remainder term $R_K(\frac{1}{2}, x)$.

We then obtain

$$\widetilde{U}_K(\infty) \leq 2p\left(\frac{1}{2}, 0\right) + \int_0^{\frac{1}{2}} 2p\left(\frac{1}{2} - s, 0\right) ds + CK^{-\gamma}, \tag{5.2}$$

with probability $\geq 1 - CK^{-n}$. Comparing the right-hand side of (5.2) with the right-hand side of (1.7), followed by using $\widetilde{U}_\star(\frac{1}{2}, 0) = \frac{4}{\sqrt{\pi}}$ [from (3.16)] we obtain

$$\widetilde{U}_K(\infty) \leq \widetilde{U}_\star\left(\frac{1}{2}, 0\right) + CK^{-\gamma} = \frac{4}{\sqrt{\pi}} + CK^{-\gamma},$$

with probability $\geq 1 - CK^{-n}$. This concludes the desired result (1.4) of part (a).

We now turn to the proof of part (b). Fix $\gamma \in (0, \frac{1}{96})$ and $n < \infty$, and specialize $\phi(t)$ to the push-the-laggard strategy hereafter. Using Theorem 1.6 for $T = 1$, with $\widetilde{U}_K(t) := \widetilde{U}_K(t, Z_K(t))$, we have that $\sup_{t \in [\frac{1}{2}, 1]} |\widetilde{U}_K(t) - \widetilde{U}_\star(t, z_\star(t))| \leq CK^{-\gamma}$, with probability $\geq 1 - CK^{-n}$. Combining this with (3.16) yields

$$\mathbf{P}\left(\left|\widetilde{U}_K(t) - \frac{4}{\sqrt{\pi}}\right| \leq CK^{-\gamma}, \forall t \in \left[\frac{1}{2}, 1\right]\right) \geq 1 - CK^{-n}. \tag{5.3}$$

Having established (5.3), we next establish that

$$\mathbf{P}\left(\inf_{t \in [1, \infty)} \underline{W}_K\left(t - \frac{1}{2}\right) > 0\right) \geq 1 - CK^{-n}. \tag{5.4}$$

We claim that (5.4) is the desired property in order to complete the proof. To see this, recall from Lemma 4.7 that we have a coupling under which (4.33) holds, and by Lemma 4.1, we assume without lost of generality that $\tau_{\mathrm{ext}}^K > 1$. Under this setup, the event in (5.4) implies $Z_K(t) \geq \underline{W}_K(t - \frac{1}{2}) > 0$, $\forall t \in [1, \tau_{\mathrm{ext}}^K)$ which then forces $\tau_{\mathrm{ext}} = \infty$. That is, the statement (5.4) implies $\mathbf{P}(Z_K(t) > 0, \forall t > 1) \geq 1 - CK^{-n}$, and hence $\mathbf{P}(\widetilde{U}_K(t) = \widetilde{U}_K(1), \forall t \geq 1) \geq 1 - CK^{-n}$. This together with (5.3) concludes (1.5).

Returning to the proof of (5.4), we recall from Remark 4.6 that the Atlas model $\underline{Y}(t)$ as well as its laggard $\underline{W}(t)$ actually depend on K, which we have omitted up until this point for the sake of notation. Here, we restore such a dependence and write $\underline{Y}(t; K)$ and $\underline{W}(t; K)$, etc. Recall that the initial condition of the Atlas model $(\underline{Y}_i(0; K))_i$ is sampled from the Poisson point process $\mathrm{PPP}(\underline{u}(\frac{x}{\sqrt{K}}))$ in (4.30). From the definition (4.29) of $\underline{u}$ and the explicit formula (1.12) of $u_1(\frac{1}{2}, x)$, it is straightforward to verify that the density function $x \mapsto \underline{u}(\frac{x}{\sqrt{K}})$ is nonincreasing on its support $[-K^{\frac{1}{2} - 4\gamma_3}, \infty)$. Consequently, fixing $K_1 < K_2$, we have

$$\underline{u}\left(\frac{1}{\sqrt{K_1}}\left(x + K_1{}^{\frac{1}{2} - 4\gamma_3}\right)\right) \leq \underline{u}\left(\frac{1}{\sqrt{K_2}}\left(x + K_2{}^{\frac{1}{2} - 4\gamma_3}\right)\right) \qquad \forall x \in \mathbb{R}.$$

With this, it is standard to construct a coupling of $\underline{Y}(0; K_1)$ and $\underline{Y}(0; K_2)$ under which

$$\#\{\underline{Y}_i(0; K_1)\} \le \#\{\underline{Y}_i(0; K_2)\},$$
$$\underline{Y}_i(0; K_2) + K_2^{\frac{1}{2}-\gamma_3} \le \underline{Y}_i(0; K_1) + K_1^{\frac{1}{2}-\gamma_3} \qquad \forall i = 1, \dots, \#\{\underline{Y}_i(0; K_1)\}.$$

By Lemma 4.8, such a dominance coupling at $s = 0$ is leveraged into a dominance coupling for all $s > 0$, yielding

$$\begin{aligned} \underline{W}(s; K_1) &\ge \underline{W}(s; K_2) + K_2^{\frac{1}{2}-4\gamma_3} - K_1^{\frac{1}{2}-4\gamma_3} \\ &\ge \underline{W}(s; K_2) \qquad \forall s \ge 0. \end{aligned} \tag{5.5}$$

Now, fix $K < \infty$ and consider the geometric subsequence $L_m := K2^m$. We use the union bound to write

$$\mathbf{P}\Big(\inf_{s \in [\frac{1}{2}K, \infty)} \underline{W}(s; K) \le 0\Big) \le \sum_{m=1}^{\infty} \mathbf{P}\Big(\inf_{s \in [L_{m-1}, L_m]} \underline{W}(s; K) \le 0\Big).$$

Within each mth term in the last expression, use the coupling (5.5) for $(K_1, K_2) = (L_{m-1}, L_m)$ to obtain

$$\mathbf{P}\Big(\inf_{s \in [L_{m-1}, L_m]} \underline{W}(s; K) \le 0\Big) \le \mathbf{P}\Big(\inf_{s \in [L_{m-1}, L_m]} \underline{W}(s; L_m) \le 0\Big). \tag{5.6}$$

Next, set $T = 1$ and $K = L_m$ in (4.68) and rewrite the resulting equation in in the prescaled form as

$$\begin{aligned} &\mathbf{P}\bigg(\bigg|\underline{W}(s; L_m) - \sqrt{L_m} z_\star\Big(\frac{s}{L_m} + \frac{1}{2}\Big)\bigg| \le C L_m^{\frac{1}{2}-4\gamma_3} \forall s \in [0, L_m]\bigg) \\ &\quad \ge 1 - C L_m^{-n}. \end{aligned} \tag{5.7}$$

Further, by Lemma 3.6 and the fact that $t \mapsto z_\star(t)$ is nondecreasing, we have that

$$\inf_{s \in [L_{m-1}, L_m]} z_\star\Big(\frac{s}{L_m} + \frac{1}{2}\Big) = \inf_{t \in [\frac{1}{2}, 1]} z_\star(t) = z_\star\Big(\frac{1}{2}\Big) > 0. \tag{5.8}$$

Combining (5.7)–(5.8) yields $\mathbf{P}(\inf_{s \in [L_{m-1}, L_m]} \underline{W}(s, L_m) \le 0) \le C L_m^{-n}$. Inserting this bound into (5.6), and summing the result over m, we arrive at

$$\mathbf{P}\Big(\inf_{s \in [\frac{1}{2}K, \infty)} \underline{W}(s; K) \le 0\Big) \le C \sum_{m=1}^{\infty} L_m^{-n} = C K^{-n}.$$

This concludes (5.4), and hence completes the proof of part (b).

Acknowledgments. We thank David Aldous for suggesting this problem for research. WT thanks Jim Pitman for helpful discussion throughout this work, and Craig Evans for pointing out the relation between (1.14) and parabolic variational inequalities. Li-Cheng Tsai thanks Amir Dembo for enlightening discussion at the early stage of this work.

We thank the anonymous reviewers for their careful reading of the manuscript.

REFERENCES

[1] ALDOUS, D. (2002). Unpublished. Available at http://www.stat.berkeley.edu/~aldous/Research/OP/river.pdf.

[2] BANNER, A. D., FERNHOLZ, R. and KARATZAS, I. (2005). Atlas models of equity markets. *Ann. Appl. Probab.* **15** 2296–2330. MR2187296

[3] CABEZAS, M., DEMBO, A., SARANTSEV, A. and SIDORAVICIUS, V. Brownian particles of rank-dependent drifts: Out of equilibrium behavior. Preprint. Available at arXiv:1708.01918.

[4] CHATTERJEE, S. and PAL, S. (2010). A phase transition behavior for Brownian motions interacting through their ranks. *Probab. Theory Related Fields* **147** 123–159. MR2594349

[5] CHATTERJEE, S. and PAL, S. (2011). A combinatorial analysis of interacting diffusions. *J. Theoret. Probab.* **24** 939–968. MR2851239

[6] DEMBO, A., SHKOLNIKOV, M., VARADHAN, S. R. S. and ZEITOUNI, O. (2016). Large deviations for diffusions interacting through their ranks. *Comm. Pure Appl. Math.* **69** 1259–1313. MR3503022

[7] DEMBO, A. and TSAI, L.-C. (2017). Equilibrium fluctuation of the Atlas model. *Ann. Probab.* **45** 4529–4560. DOI:10.1214/16-AOP1171.

[8] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications. Vol. II*, 2nd ed. Wiley, New York. MR0270403

[9] FERNHOLZ, E. R. (2002). *Stochastic Portfolio Theory. Stochastic Modelling and Applied Probability. Applications of Mathematics (New York)* **48**. Springer, New York. MR1894767

[10] FRIEDMAN, A. (1982). *Variational Principles and Free-Boundary Problems*. Wiley, New York. MR0679313

[11] HAN, W. (2013). Available at http://hanweijian.com/research/2013-research-projects/random-particle-motion/.

[12] HERNÁNDEZ, F., JARA, M. and VALENTIM, F. J. (2017). Equilibrium fluctuations for a discrete Atlas model. *Stochastic Process. Appl.* **127** 783–802. MR3605711

[13] ICHIBA, T. and KARATZAS, I. (2010). On collisions of Brownian particles. *Ann. Appl. Probab.* **20** 951–977. MR2680554

[14] ICHIBA, T., KARATZAS, I. and SHKOLNIKOV, M. (2013). Strong solutions of stochastic equations with rank-based coefficients. *Probab. Theory Related Fields* **156** 229–248. MR3055258

[15] ICHIBA, T., PAPATHANAKOS, V., BANNER, A., KARATZAS, I. and FERNHOLZ, R. (2011). Hybrid atlas models. *Ann. Appl. Probab.* **21** 609–644. MR2807968

[16] KUNITA, H. (1997). *Stochastic Flows and Stochastic Differential Equations. Cambridge Studies in Advanced Mathematics* **24**. Cambridge Univ. Press, Cambridge. Reprint of the 1990 original. MR1472487

[17] MCKEAN, H. P. and SHEPP, L. A. (2005). The advantage of capitalism vs. socialism depends on the criterion. *J. Math. Sci.* **139** 6589–6594. DOI:10.1007/s10958-006-0374-5.

[18] PAL, S. and PITMAN, J. (2008). One-dimensional Brownian particle systems with rank-dependent drifts. *Ann. Appl. Probab.* **18** 2179–2207. MR2473654

[19] PAL, S. and SHKOLNIKOV, M. (2014). Concentration of measure for Brownian particle systems interacting through their ranks. *Ann. Appl. Probab.* **24** 1482–1508. MR3211002

[20] SARANTSEV, A. (2017). Infinite systems of competing Brownian particles. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** 22792–2315. DOI:10.1214/16-AIHP791.

[21] SARANTSEV, A. (2018). Comparison techniques for competing Brownian particles. *J. Theoret. Probab.* To appear. Available at arXiv:1305.1653.

[22] SHKOLNIKOV, M. (2011). Competing particle systems evolving by interacting Lévy processes. *Ann. Appl. Probab.* **21** 1911–1932. MR2884054

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