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A Skellam GARCH model

Ghadah A. Alomani, Abdulhamid A. Alzaid and Maha A. Omair

King Saud University

Abstract. This paper considers the modeling of nonstationary integer valued time series with conditional heteroskedasticity using Skellam distribution. Two approaches of estimation of the model's parameters are treated and discussed. The obtained results are verified through some numerical simulation. In addition, the proposed model is applied to real time series.

1 Introduction

Various models have been proposed for stationary discrete time series. Al-Osh and Alzaid (1987) and McKenzie (1986) introduced an integer-valued autoregressive-moving average (INARMA) models. The nonstationary integer-valued time series are frequently encountered in real-life problems. Such problems contains valuable data that is required to be modeled using times series One way to address this non-stationary is to take the difference which results in integer valued time series that takes positive and negative values. Kim and Park (2008) introduced an integer-valued autoregressive process of order p with signed binomial thinning operator (INARS(p)). Karlis and Andersson (2009) defined the ZINAR process, as an extension of the INAR model using the signed binomial thinning operator and studied the case where the innovation has Skellam distribution. Alzaid and Omair (2014) defined a Poisson difference (Skellam) integer valued autoregressive model of order one (PDINAR(1)). Many time series exhibit time varying volatility. Engle (1982) proposed the autoregressive conditional heteroskedastic model (ARCH) that allows the conditional variance to change over time as a function of past errors by which makes the unconditional variance constant. Bollerslev (1986) introduced a more general class of ARCH process, the generalized autoregressive conditional heteroskedastic (GARCH) having more flexible lag structure. Ferland, Latour and Oraichi (2006) have introduced an integer-valued analogue of the classical GARCH model using Poisson deviates instead of the normal deviates abbreviated by (INGARCH). The INGARCH model has been studied by many researchers. Zhu and Li (2009) have studied the moment and Bayesian estimation of parameters in the INGARCH$(1, 1)$ model. Zhu, Li and Wang (2010) have introduced a mixture integer-valued ARCH model. Heinen (2003) has defined autore-

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gressive conditional Poisson model (ACP(1, 1)). Ghahramani and Thavaneswaran (2009) have extended Heinen's results to higher order ACP(p, q). Zhu (2011) has derived the negative binomial INGARCH model for overdispersion and extreme observations. In many time series, there are two important phenomena that occur simultaneously. These phenomena are the nonstationarity and the time varying variance. In literature, there are few tries that have been used to overcome those phenomena. Here we introduce an integer-valued GARCH model based on Skellam distribution to tackle those phenomena. To get more insights and understand our model, some real-life data applications have been used.

This paper is organized as follows. The Skellam distribution is defined in Section 2. In Section 3, the Skellam GARCH model is introduced. We study and investigate the estimation of the model using two approaches in Section 4. A numerical simulation is carried out in Section 5. In Section 6, real life application is used to validate our proposed model and to compare out obtained results with existed results in literature.

2 Skellam distribution

For any pair of variables (X, Y) where $X = W_1 + W_3$ and $Y = W_2 + W_3$ such that $W_1 \sim \text{Poisson}(\theta_1)$ independent of $W_2 \sim \text{Poisson}(\theta_2)$ and W_3 follows any distribution, the probability mass function of $Z = X - Y$ is given by:

$$P(Z = z) = e^{-\theta_1-\theta_2}\left(\frac{\theta_1}{\theta_2}\right)^{\frac{z}{2}} I_z(2\sqrt{\theta_1\theta_2}), \qquad z = \dots, -1, 0, 1, \dots, \tag{2.1}$$

where

$$I_z(x) = \left(\frac{x}{2}\right)^z \sum_{k=0}^{\infty} \frac{(x^2/4)^k}{k!(z+k)!}$$

and Z is said to have Skellam (Poisson difference) distribution, denoted by Skellam(θ_1, θ_2) with mean $\mu = \theta_1 - \theta_2$ and variance $\sigma^2 = \theta_1 + \theta_2$. One can easily see that $\theta_1 = \frac{\sigma^2+\mu}{2}$ and $\theta_2 = \frac{\sigma^2-\mu}{2}$.

Equivalently, the probability mass function of the Skellam distribution can be rewritten in terms of the mean and the variance as follows:

$$P(X = x) = e^{-\sigma^2}\left(\frac{\sigma^2 + \mu}{\sigma^2 - \mu}\right)^{\frac{x}{2}} I_x\left(\sqrt{\sigma^4 - \mu}\right), \qquad x = \dots, -1, 0, 1, \dots. \tag{2.2}$$

3 Skellam GARCH(1, 1) model

Let $\{Y_t\}$ be an integer valued time series and let F_t be defined as σ-field generated by observations up to and including time t. We assume that conditional on the past observations, $\{Y_t|F_{t-1}\}$ are independent and follow symmetric Skellam $(\frac{\sigma^2_{t|t-1}}{2}, \frac{\sigma^2_{t|t-1}}{2})$ with the conditional variance satisfying,

$$\sigma^2_{t|t-1} = \omega + \alpha y^2_{t-1} + \beta\sigma^2_{t-1|t-2}, \qquad t \geq 2, \tag{3.1}$$

as in GARCH(1, 1) model (Bollerslev (1986)).

In (3.1), the parameters ω, α and β satisfy the following constraints: $\omega > 0$, $0 < \alpha < 1$, $0 < \beta < 1$ and $\alpha + \beta < 1$ which are a necessary and sufficient for stationarity of the process (3.1). We refer to the above model as Skellam GARCH(1, 1). The simplest case of the model with $\beta = 0$ will be denoted by Skellam ARCH(1).

Therefore, for Skellam GARCH(1, 1), we have:

$$\begin{aligned} P(Y_t = y_t|F_{t-1}) &= e^{-\sigma^2_{t|t-1}} I_{y_t}(\sigma^2_{t|t-1}), \qquad y_t = \dots, -1, 0, 1, \dots, \\ E(Y_t|F_{t-1}) &= 0 \end{aligned} \tag{3.2}$$

and

$$\mathrm{Var}(Y_t|F_{t-1}) = \sigma^2_{t|t-1}.$$

Unlike, the ACP(1, 1) of Heinen (2003), we have

$$\mathrm{Var}(Y_t|F_{t-1}) = \sigma^2_{t|t-1} > E(Y_t|F_{t-1}).$$

4 Parameters estimation

In this section, we use the conditional maximum likelihood method and conditional least squares method to estimate the parameters as following.

4.1 Conditional maximum likelihood method

For any $t \in (1, \dots, n)$ and $\Theta = (\omega, \alpha, \beta)$, the conditional likelihood function is given by:

$$\mathrm{CL}(\Theta, y) = \prod_{t=2}^{n} [e^{-\sigma^2_{t|t-1}} I_{y_t}(\sigma^2_{t|t-1})].$$

Here, the conditional variances are computed recursively from formula (3.1), using the initial value $\sigma^2_{1|0} = \frac{\omega}{1-\alpha-\beta}$ to insure stationarity. There is no closed-form solution for maximum likelihood estimators of ω, α and β, but they can be computed numerically by maximizing the conditional log likelihood function.

The conditional log likelihood equations are:

$$\begin{aligned}
\frac{\partial \ln \mathrm{CL}(\Theta, y)}{\partial \omega} &= \sum_{t=2}^{n}\left(-1+\frac{y_t}{\sigma^2_{t|t-1}}+\frac{I_{y_t+1}(\sigma^2_{t|t-1})}{I_{y_t}(\sigma^2_{t|t-1})}\right) \\
&\quad \times \left(\frac{1-\alpha-\beta+\alpha\beta^{t-1}}{(1-\beta)(1-\alpha-\beta)}\right) \\
&= 0,
\end{aligned} \tag{4.1a}$$

$$\begin{aligned}
\frac{\partial \ln \mathrm{CL}(\Theta, y)}{\partial \alpha} &= \sum_{t=2}^{n}\left(-1+\frac{y_t}{\sigma^2_{t|t-1}}+\frac{I_{y_t+1}(\sigma^2_{t|t-1})}{I_{y_t}(\sigma^2_{t|t-1})}\right) \\
&\quad \times \left(\sum_{k=1}^{t-1}\beta^{k-1}y^2_{t-k}+\frac{\omega\beta^{t-1}}{(1-\alpha-\beta)^2}\right) \\
&= 0,
\end{aligned} \tag{4.1b}$$

$$\begin{aligned}
\frac{\partial \ln \mathrm{CL}(\Theta, y)}{\partial \beta} &= \sum_{t=2}^{n}\left(-1+\frac{y_t}{\sigma^2_{t|t-1}}+\frac{I_{y_t+1}(\sigma^2_{t|t-1})}{I_{y_t}(\sigma^2_{t|t-1})}\right) \\
&\quad \times \left(\sum_{k=1}^{t-1}\beta^{k-1}\sigma^2_{t-k|t-(k+1)}+\frac{\omega\beta^{t-1}}{(1-\alpha-\beta)^2}\right) \\
&= 0.
\end{aligned} \tag{4.1c}$$

Proof of the conditional log likelihood equations are obtained in the Appendix.

4.2 Conditional least squares method

In this section, we will use the conditional least square method with the conditional variance prediction error. Brännäs and Quoreshi (2010) have used the conditional variance prediction error as a second step to obtain feasible generalized least square estimator for long-lag integer valued moving average model. The same method has been utilized in Alzaid and Omair (2014) to estimate the parameter of Poisson difference integer-valued autoregressive model of order one. Now, the conditional variance prediction error is defined by:

$$e_{2t} = \left(Z_t - E(Z_t|Z_{t-1})\right)^2 - V(Z_t|Z_{t-1}).$$

To estimate the parameters ω, α and β, we have to minimize the following formula:

$$\begin{aligned} S_{2\mathrm{CLS}} &= \sum_{t=2}^{n} e_{2t}^2 = \sum_{t=2}^{n} (y_t^2 - \sigma_{t|t-1}^2)^2 \\ &= \sum_{t=2}^{n} \left(y_t^2 - \left(\omega \left(\frac{1-\beta^{t-1}}{1-\beta} + \frac{\beta^{t-1}}{1-\alpha-\beta} \right) + \alpha \sum_{k=1}^{t-1} \beta^{t-k-1} y_k^2 \right) \right)^2 \end{aligned}$$

with respect to ω, α and β. Differentiating the above formula with respect to those parameters yield:

$$\begin{aligned} \frac{\partial S_{2\mathrm{CLS}}}{\partial \omega} &= \sum_{t=2}^{n} \left(y_t^2 - \left(\omega \left(\frac{1-\beta^{t-1}}{1-\beta} + \frac{\beta^{t-1}}{1-\alpha-\beta} \right) + \alpha \sum_{k=1}^{t-1} \beta^{t-k-1} y_k^2 \right) \right) \\ &\quad \times \left(\frac{1-\alpha-\beta+\alpha\beta^{t-1}}{(1-\beta)(1-\alpha-\beta)} \right) \\ &= 0, \end{aligned} \tag{4.2a}$$

$$\begin{aligned} \frac{\partial S_{2\mathrm{CLS}}}{\partial \alpha} &= \sum_{t=2}^{n} \left(y_t^2 - \left(\omega \left(\frac{1-\beta^{t-1}}{1-\beta} + \frac{\beta^{t-1}}{1-\alpha-\beta} \right) + \alpha \sum_{k=1}^{t-1} \beta^{t-k-1} y_k^2 \right) \right) \\ &\quad \times \left(\frac{\omega\beta^{t-1}}{(1-\alpha-\beta)^2} + \sum_{k=1}^{t-1} \beta^{t-k-1} y_k^2 \right) \\ &= 0, \end{aligned} \tag{4.2b}$$

$$\begin{aligned} \frac{\partial S_{2\mathrm{CLS}}}{\partial \beta} &= \sum_{t=2}^{n} \left(y_t^2 - \left(\omega \left(\frac{1-\beta^{t-1}}{1-\beta} + \frac{\beta^{t-1}}{1-\alpha-\beta} \right) + \alpha \sum_{k=1}^{t-1} \beta^{t-k-1} y_k^2 \right) \right) \\ &\quad \times \left(\omega \left(\frac{(1-t)(\beta^{t-2}-\beta^{t-1}) + 1 - \beta^{t-1}}{(1-\beta)^2} \right. \right. \\ &\quad \left. + \frac{(t-1)(1-\alpha-\beta)\beta^{t-2} + \beta^{t-1}}{(1-\alpha-\beta)^2} \right) \\ &\quad \left. + \alpha \sum_{k=1}^{t-1} (t-k-1) \beta^{t-k-2} y_k^2 \right) \\ &= 0. \end{aligned} \tag{4.2c}$$

Since it is hard to get explicit form for the solution of the system (4.2a)–(4.2c), numerical solutions are carried out to estimate its parameters.

5 Numerical simulation

To provide an idea about the relative merits of each of the methods of estimation discussed in the previous section, a Monte Carlo study is conducted. The selected parameters are: $(\omega, \alpha, \beta) = (0.5, 0.4, 0.1)$, $(0.5, 0.5, 0.2)$, $(0.5, 0.6, 0.2)$, $(0.7, 0.5, 0.2)$, $(0.7, 0.6, 0.2)$, $(3, 0.2, 0.2)$ and $(3, 0.5, 0.2)$. Samples of size 100, 200 and 500 with 1000 replications for each choice of parameters are adopted. The bias and the relative mean square error (RMSE) are calculated according to the following formulas:

$$\mathrm{BIAS}(\hat{\alpha}) = \frac{1}{r}\sum_{i=1}^{r}(\hat{\alpha}_i - \alpha), \tag{5.1}$$

$$\mathrm{RMSE}(\hat{\alpha}) = \frac{1}{\alpha}\left[\frac{1}{r}\sum_{i=1}^{r}(\hat{\alpha}_i - \alpha)^2\right]^{\frac{1}{2}}. \tag{5.2}$$

Algorithm steps for GARCH(1, 1):

Step1: begin with the values of ω, α and β.
Step2: Take the initial value of $\sigma^2_{1|0}$ as $\sigma^2_{1|0} = \frac{\omega}{1-\alpha-\beta}$.
Step3: Generate the observation $X_{11} \sim \text{Poisson}(\frac{\sigma^2_{1|0}}{2})$.
Step4: Generate the observation $X_{21} \sim \text{Poisson}(\frac{\sigma^2_{1|0}}{2})$.
Step5: Calculate the difference $Y_1 = X_{11} - X_{21}$.
Step6: For $t = 2, \dots, n$:

(i) Calculate $\sigma^2_{t|t-1} = \omega + \alpha y^2_{t-1} + \beta\sigma^2_{t-1|t-2}$.
(ii) Generate the observation $X_{1t} \sim \text{Poisson}(\frac{\sigma^2_{t|t-1}}{2})$.
(iii) Generate the observation $X_{2t} \sim \text{Poisson}(\frac{\sigma^2_{t|t-1}}{2})$.
(iv) Calculate the difference $Y_t = X_{1t} - X_{2t}$.

Mathematica 8 software is used to solve the systems of equations (4.1a)–(4.1c) and (4.2a)–(4.2c) to get our parameters estimates. Wolfram Language in Mathematica program has full coverage of all standard Bessel-related functions. The notation BesselI$[n, z]$ calculates the modified Bessel function of the first kind $I_n(z)$. For more information see https://reference.wolfram.com/language/ref/BesselJ.html. Tables 1 and 2 present some of the obtained results. It can be seen that using both methods of estimation, the estimates seem to converge to the true parameter values as the sample size increases.The bias results for Skellam GARCH$(1, 1)$ using CLS method are extremely higher than the bias redusing CML method for the parameters ω and α. The bias of the simulation results reflect the inverse relation between ω and $\alpha + \beta$, where the biases in ω and $\alpha + \beta$

Table 1 *Bias results for Skellam* GARCH(1, 1)

				BIAS(ω)		BIAS(α)		BIAS(β)	
ω	α	β	n	CML	CLS	CML	CLS	CML	CLS
0.5	0.4	0.1	100	−0.0196	0.0575	−0.0353	−0.1317	0.0548	0.0465
			200	−0.0074	0.0486	−0.0183	−0.1066	0.0217	0.0387
			500	−0.0037	0.0416	−0.0080	−0.0763	0.0081	0.0207
0.5	0.5	0.2	100	0.0376	0.2219	−0.0348	−0.2025	−0.0004	0.0073
			200	0.0233	0.2132	−0.0110	−0.1633	0.0083	−0.0020
			500	0.0115	0.1654	−0.0039	−0.1333	−0.0063	0.0110
0.5	0.6	0.2	100	0.0573	0.4005	−0.0507	−0.2674	−0.0072	0.0059
			200	0.0339	0.4042	−0.0168	−0.2297	−0.0122	0.0126
			500	0.0174	0.3377	−0.0062	−0.2027	−0.0087	0.0346
0.7	0.5	0.2	100	0.0608	0.3080	−0.0431	−0.1991	−0.0012	0.0004
			200	0.0473	0.2938	−0.0123	−0.1565	−0.0150	−0.0111
			500	0.0206	0.2261	−0.0072	−0.1282	−0.0075	0.0094
0.7	0.6	0.2	100	0.0876	0.5545	−0.0610	−0.2678	−0.0067	0.0105
			200	0.0549	0.5643	−0.0195	−0.2280	−0.0146	0.0146
			500	0.0219	0.4911	−0.0067	−0.2044	−0.0081	0.0364
3	0.2	0.2	100	−0.0536	0.0883	−0.0097	−0.0507	0.0189	0.0136
			200	−0.0209	0.0981	−0.0064	−0.0289	0.0087	−0.0012
			500	0.0082	0.0584	−0.0052	−0.0187	0.0000	0.0015
3	0.5	0.2	100	0.2527	1.3883	−0.0342	−0.2025	−0.0113	−0.0010
			200	0.1925	1.2470	−0.0131	−0.1610	−0.0174	−0.0021
			500	0.0914	1.0212	−0.0068	−0.1297	−0.0096	0.0052

have different signs. The maximum likelihood estimates are better than the conditional least squares estimates in terms of relative mean square error and in terms of bias, so it is recommended to use the CML for estimation of the parameters.

6 Application

In this section, we present an application of the Skellam GARCH model. The application consists of number of drug crime along 144 months from 1990 until 2001 in one Precinct of Pittsburgh city (www.forecastingprinciples.com/). The time series plot of the data is illustrated in Figure 1. The autocorrelation function (ACF) and partial autocorrelation function (PACF) for drug crime are given in Figures 2 and 3, respectively.

In order to get more information about the data, Table 3 displays some descriptive statistics for the drug crime along with the lag one difference.

Table 2 *RMSE results for Skellam* GARCH(1, 1)

				RMSE(ω)		RMSE(α)		RMSE(β)	
ω	α	β	n	CML	CLS	CML	CLS	CML	CLS
0.5	0.4	0.1	100	0.3811	0.4525	0.4841	0.5050	2.1071	2.1541
			200	0.2814	0.3863	0.3482	0.4265	1.6093	2.0000
			500	0.1855	0.2871	0.2264	0.3410	1.0770	1.4967
0.5	0.5	0.2	100	0.4779	0.8567	0.4162	0.5099	0.9823	1.1587
			200	0.3341	0.7451	0.3053	0.4271	0.7433	1.0392
			500	0.2126	0.5692	0.1929	0.3521	0.4873	0.8732
0.5	0.6	0.2	100	0.4980	1.4687	0.3532	0.5185	0.8874	1.1325
			200	0.3406	1.6993	0.2635	0.4522	0.6671	1.0464
			500	0.2117	1.1515	0.1691	0.3958	0.4093	0.9474
0.7	0.5	0.2	100	0.4729	0.8414	0.4118	0.5056	0.9811	1.1651
			200	0.3540	0.7127	0.3124	0.4200	0.7566	1.0308
			500	0.2190	0.5568	0.1929	0.3504	0.4924	0.8902
0.7	0.6	0.2	100	0.5334	1.3758	0.3674	0.5148	0.9179	1.1511
			200	0.3617	2.0027	0.2672	0.4491	0.6633	1.0724
			500	0.2129	1.4191	0.1700	0.3993	0.4062	0.9618
3	0.2	0.2	100	0.4069	0.4241	0.7228	0.6595	1.2390	1.3134
			200	0.3743	0.3749	0.5315	0.5500	1.1790	1.2196
			500	0.2960	0.3171	0.3464	0.4062	0.9618	1.0536
3	0.5	0.2	100	0.4771	0.8435	0.4167	0.5060	0.9734	1.1705
			200	0.3561	0.6994	0.3046	0.4224	0.7297	1.0223
			500	0.2265	0.5564	0.1918	0.3481	0.4822	0.8775

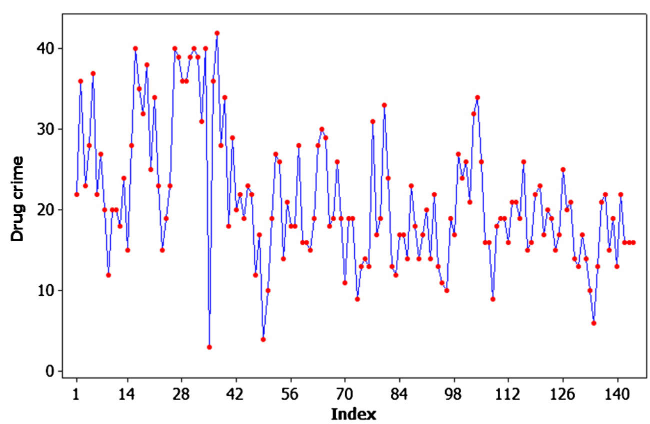

Figure 1 *Time series of drug crime.*

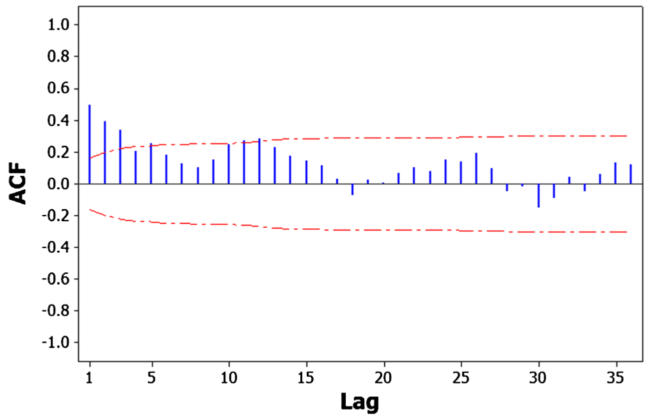

Figure 2 *ACF of drug crime.*

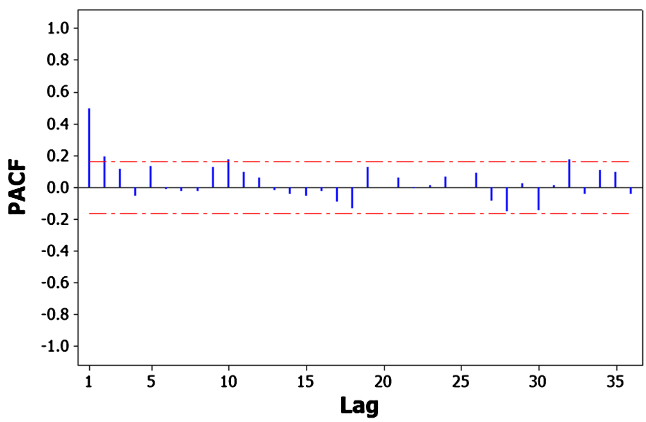

Figure 3 *PACF of drug crime.*

Table 3 *Descriptive statistics of the drug crime data with their lag one difference*

Variable	No. of observation	Mean	Variance	Minimum	Median	Maximum
Drug crime	144	21.396	68.423	3	19	42
Drug crime lag one difference	143	−0.042	68.984	−37	0	33

After differencing, a discrete model that allow for negative integers should be used. The time series are illustrated in Figure 4. The autocorrelation function and the partial autocorrelation function for the square of the difference series are shown in Figures 5 and 6, respectively.

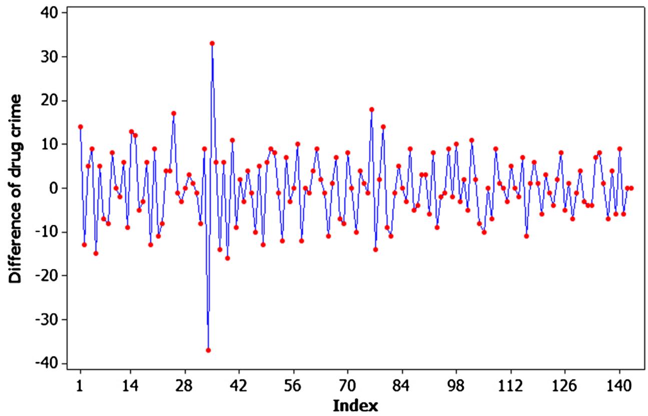

Figure 4 *Time series of difference of drug crime.*

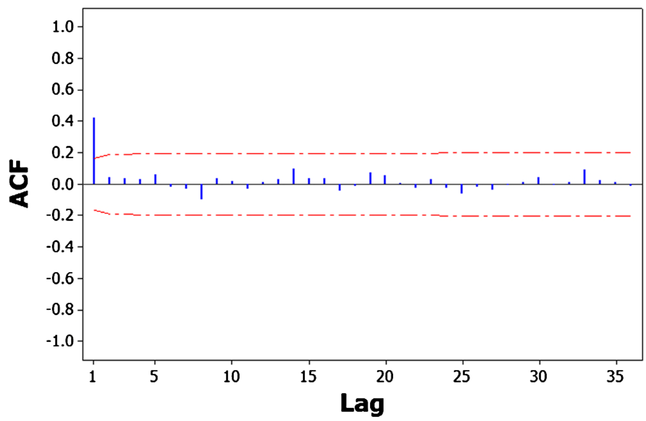

Figure 5 *ACF for square of difference of drug crime.*

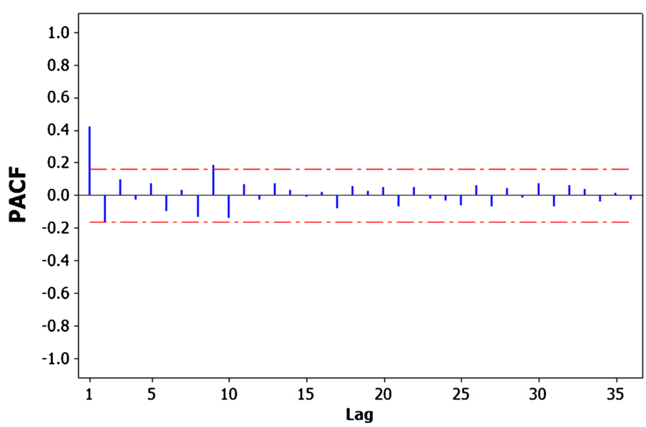

Figure 6 *PACF for square of difference of drug crime.*

Table 4 *Parameters estimates of the models with their standard errors, Akaike Information criterion (AIC) and Bayesian information criterion (BIC)*

	Model	ω	α	β	AIC	BIC
ARCH(1)	negative binomial ($r = 2$)	6.0512 (0.9170)	0.2156 (0.0500)	–	1080.684	1086.624
	Poisson	11.4154 (1.6186)	0.4654 (0.0760)	–	1042.926	1084.866
	Skellam	47.183 (8.2490)	0.2712 (0.1325)	–	991.302	997.228
GARCH(1, 1)	negative binomial ($r = 2$)	2.2051 (1.4504)	0.1822 (0.0488)	0.4299 (0.1761)	1080.836	1089.745
	Poisson	4.2151 (2.6084)	0.3682 (0.0792)	0.4357 (0.1542)	1025.958	1034.867
	Skellam (CML)	33.4414 (13.2303)	0.2659 (0.1338)	0.2173 (0.2243)	992.168	1001.057
	Skellam (CLS)	38.5149 (11.8284)	0.4217 (0.1846)	1.0974×10^{-9} (0.1801)	–	–

It is clear that the differenced data is stationary in the mean and exhibits conditional heteroskedasticity. We fit the differenced data to Skellam GARCH and Skellam ARCH models. We also fit the original data to the Poisson GARCH, Poisson ARCH, negative binomial GARCH and negative binomial ARCH models. The parameters estimates of the models with their standard errors, Akaike Information criterion (AIC) and Bayesian information criterion (BIC) values are shown in Table 4. The values of AIC and BIC show that the Skellam ARCH and GARCH fit better than the other corresponding models.

The fitted conditional variance of Skellam ARCH(1) and GARCH(1, 1) models with the squares of the differenced drug crime data are plotted in Figures 7 and 8, respectively. These two figures show that the ARCH and GARCH models capture the variation on squared differences of the drug crime data.

7 Conclusions

In this article, we have discussed the analysis of integer time series model with time varying variance and nonstationarity in the mean and a Skellam GARCH model was proposed. For estimating the parameters of the model, the conditional maximum likelihood and conditional least squares methods have been developed. The numerical simulation has confirmed that the CML estimates have better performance. A real application has shown the efficacy of the model.

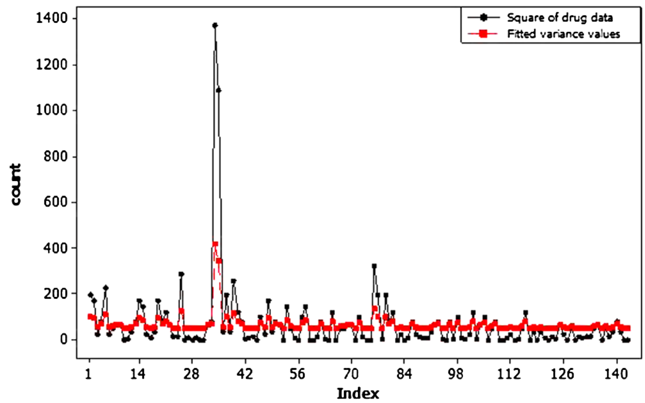

Figure 7 *Square of the differenced drug crime data and fitted variance value of Skellam* ARCH(1) *model.*

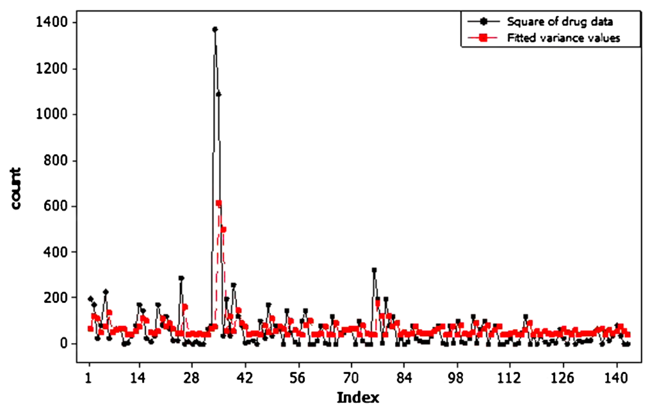

Figure 8 *Square of the differenced drug crime data and fitted variance value of Skellam* GARCH(1, 1) *model.*

Appendix. Parameter's estimation by using the conditional likelihood method

The conditional log likelihood equation are:

$$\frac{\partial \ln \mathrm{CL}(\Theta, y)}{\partial \omega} = -\sum_{t=2}^{n} \frac{\partial \sigma_{t|t-1}^2}{\partial \omega} + \sum_{t=2}^{n} \left(\frac{y_t}{\sigma_{t|t-1}^2} + \frac{I_{y_t+1}(\sigma_{t|t-1}^2)}{I_{y_t}(\sigma_{t|t-1}^2)} \right) \times \frac{\partial \sigma_{t|t-1}^2}{\partial \omega},$$

where

$$\frac{\partial \sigma_{t|t-1}^2}{\partial \omega} = 1 + \beta + \beta^2 + \cdots + \beta^{t-2} + \beta^{t-1} \frac{\partial \sigma_{1|0}^2}{\partial \omega}$$

$$= \sum_{k=0}^{t-2} \beta^k + \frac{\beta^{t-1}}{1-\alpha-\beta}$$
$$= \frac{1-\alpha-\beta+\alpha\beta^{t-1}}{(1-\beta)(1-\alpha-\beta)}.$$

Therefore

$$\frac{\partial \ln \mathrm{CL}(\Theta, y)}{\partial \omega} = \sum_{t=2}^{n}\left(-1 + \frac{y_t}{\sigma^2_{t|t-1}} + \frac{I_{y_t+1}(\sigma^2_{t|t-1})}{I_{y_t}(\sigma^2_{t|t-1})}\right)\left(\frac{1-\alpha-\beta+\alpha\beta^{t-1}}{(1-\beta)(1-\alpha-\beta)}\right),$$
$$\frac{\partial \ln \mathrm{CL}(\Theta, y)}{\partial \alpha} = -\sum_{t=2}^{n} \frac{\partial \sigma^2_{t|t-1}}{\partial \alpha} + \sum_{t=2}^{n}\left(\frac{y_t}{\sigma^2_{t|t-1}} + \frac{I_{y_t+1}(\sigma^2_{t|t-1})}{I_{y_t}(\sigma^2_{t|t-1})}\right) \times \frac{\partial \sigma^2_{t|t-1}}{\partial \alpha},$$

where

$$\frac{\partial \sigma^2_{t|t-1}}{\partial \alpha} = y^2_{t-1} + \beta y^2_{t-2} + \beta^2 y^2_{t-3} + \cdots + \beta^{t-2} y^2_1 + \beta^{t-1}\frac{\partial \sigma^2_{1|0}}{\partial \alpha}$$
$$= y^2_{t-1} + \beta y^2_{t-2} + \beta^2 y^2_{t-3} + \cdots + \beta^{t-2} y^2_1 + \frac{\omega\beta^{t-1}}{(1-\alpha-\beta)^2}$$
$$= \sum_{k=1}^{t-1} \beta^{k-1} y^2_{t-k} + \frac{\omega\beta^{t-1}}{(1-\alpha-\beta)^2}.$$

Therefore

$$\frac{\partial \ln \mathrm{CL}(\Theta, y)}{\partial \alpha} = \sum_{t=2}^{n}\left(-1 + \frac{y_t}{\sigma^2_{t|t-1}} + \frac{I_{y_t+1}(\sigma^2_{t|t-1})}{I_{y_t}(\sigma^2_{t|t-1})}\right) \times \left(\sum_{k=1}^{t-1} \beta^{k-1} y^2_{t-k} + \frac{\omega\beta^{t-1}}{(1-\alpha-\beta)^2}\right),$$
$$\frac{\partial \ln \mathrm{CL}(\Theta, y)}{\partial \beta} = -\sum_{t=2}^{n} \frac{\partial \sigma^2_{t|t-1}}{\partial \beta} + \sum_{t=2}^{n}\left(\frac{y_t}{\sigma^2_{t|t-1}} + \frac{I_{y_t+1}(\sigma^2_{t|t-1})}{I_{y_t}(\sigma^2_{t|t-1})}\right) \times \frac{\partial \sigma^2_{t|t-1}}{\partial \beta},$$

where

$$\frac{\partial \sigma^2_{t|t-1}}{\partial \beta} = \sigma^2_{t-1|t-2} + \beta\sigma^2_{t-2|t-3} + \beta^2\sigma^2_{t-3|t-4} + \cdots + \beta^{t-2}\sigma^2_{1|0} + \beta^{t-1}\frac{\partial \sigma^2_{1|0}}{\partial \beta}$$
$$= \sigma^2_{t-1|t-2} + \beta\sigma^2_{t-2|t-3} + \beta^2\sigma^2_{t-3|t-4} + \cdots + \beta^{t-2}\sigma^2_{1|0} + \frac{\omega\beta^{t-1}}{(1-\alpha-\beta)^2}$$
$$= \sum_{k=1}^{t-1} \beta^{k-1}\sigma^2_{t-k|t-(k+1)} + \frac{\omega\beta^{t-1}}{(1-\alpha-\beta)^2}.$$

Therefore,

$$\frac{\partial \ln \mathrm{CL}(\Theta, y)}{\partial \beta} = \sum_{t=2}^{n}\left(-1 + \frac{y_t}{\sigma^2_{t|t-1}} + \frac{I_{y_t+1}(\sigma^2_{t|t-1})}{I_{y_t}(\sigma^2_{t|t-1})}\right) \times \left(\sum_{k=1}^{t-1} \beta^{k-1}\sigma^2_{t-k|t-(k+1)} + \frac{\omega\beta^{t-1}}{(1-\alpha-\beta)^2}\right).$$

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Department of Statistics
and Operations Research
King Saud University
Jabal Azeb, Riyadh
Saudi Arabia
E-mail: ghadah_24@hotmail.com
alzaid@ksa.edu.sa
maomair@ksu.edu.sa