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INTRODUCTION TO MEMORIAL ISSUE FOR DONALD BURKHOLDER (1927–2013)

Don Burkholder had a long and distinguished career in mathematics. He is recognized worldwide for his deep and lasting contributions to martingale theory and its applications to other areas of mathematics. As a person, colleague and friend, he was kind and generous, serving as mentor and role model for many young people entering the field. Burkholder published many of his foundational papers in IMS journals, particularly *The Annals of Mathematical Statistics* and *The Annals of Probability*. His many contributions to probability and analysis have been reviewed in three recent publications: (1) "Don Burkholder's work on Banach spaces," (2) "Donald Burkholder's work in martingales and analysis" and (3) "The foundational inequalities of D. L. Burkholder and some of their ramifications." The first two articles, one written by Gilles Pisier and the other by Rodrigo Bañuelos and Burgess Davis, respectively, appeared in "*Selected Works of Donald L. Burkholder*," published by Springer in 2010. The third article, written by Bañuelos, appeared in the special volume *Don Burkholder: A collection of articles in his honor*, of the *Illinois Journal of Mathematics*, 2011. The reader is referred to these articles for an in-depth look at Burkholder's work, its applications and connections to different fields in mathematics where his work has had, and continues to have, an impact. Here, we briefly touch upon a few of his papers.

In 1966, Burkholder published his celebrated "Martingale Transforms" paper in *The Annals of Mathematical Statistics*. In the 1930s, Marcinkiewicz and Paley (in separate papers) proved an inequality for the Haar system of functions in the unit interval, which is equivalent to the boundedness of dyadic martingale transforms with the predictable sequence taking values in $\{1, -1\}$. Burkholder's paper extended this result to the general setting of martingales. This paper was inspired by the seminal work of Donald Austin, published in the same volume of *The Annals of Statistics*, which showed the finiteness of the square function of L^1 bounded martingales. The groundbreaking paper, "Extrapolation and Interpolation of Quasi-Linear Operators on Martingales," written with Richard Gundy, was published in *Acta Mathematica* in 1970. The paper introduced the "good λ method," now important in many areas of mathematics for comparing norms of operators, and used it to prove the famous Burkholder–Gundy martingale inequalities which compare the norms of the square and maximal functions for all continuous path martingales and many other regular martingales. In a subsequent paper Burkholder, together with Gundy and Martin Silverstein, used these inequalities to solve a longstanding open problem of Hardy and Littlewood concerning conjugate functions. This revolutionary paper inspired many harmonic analysts to learn probability and many probabilists to learn harmonic analysis, greatly enriching both fields.

For the next several years, Burkholder continued to work on applications of martingale theory and produced many other influential papers. His 1977 publication in *Advances in Mathematics*, for example, examines the exit times of Brownian motion from cones and other domains in Euclidean spaces and applies this to the behavior of harmonic functions. This paper influenced many subsequent papers on this and related subjects.

In 1984 Burkholder returned to boundedness of martingale transforms. In his *Annals of Probability* paper, "Boundary value problems and sharp inequalities for martingale transforms," he introduced the now called "Burkholder method." This was a new approach that re-proved his 1966 boundedness of martingale transforms results with optimal constants and provided extensions to the setting of Banach spaces. This allowed Burkholder and others to answer many questions about the geometry of Banach spaces for which both martingale transforms and Calderón–Zygmund singular integral operators (most notably the Hilbert transform) acting on functions taking values in those Banach spaces, are bounded on L^p, $1 < p < \infty$. The question of characterizing the Banach spaces for which this property holds arose from work of Bochner and Taylor in the 1930s. The ideas introduced by Burkholder in this paper were far ahead of their time and it took some 20 years for others to fully understand and explore them. The ideas and techniques in the 1984 article have been extensively used in recent years on problems in probability and harmonic analysis where the tools of L^2 theory are either not available or lead to estimates which are not optimal. The wider implications, in particular its application to sharp L^p bounds for singular integral operators and ramifications in quasiconformal mappings, problems from the calculus of variations dealing with rank-one convex and quasiconvex functions, problems on optimal control and the theory of Bellman functions, are all topics of current interest. One should note that these fields, on the surface, are far removed from martingale theory.

Burkholder served as president of the IMS from 1975 to 1976 and was editor of *The Annals of Mathematical Statistics* from 1964 to 1967. He served on multiple editorial boards and scientific advisory committees for both the IMS and other scientific societies. He supervised 19 Ph.D. students and several postdocs. He was elected member of the National Academy of Sciences in 1992. He was a fellow of the IMS, the American Academy of Arts and Sciences, the Society for Industrial and Applied Mathematics, the American Association for the Advancement of Science and an inaugural fellow of the American Mathematical Society. In 1970 he was an invited speaker at the ICM held in Nice, France.

Don Burkholder grew up on a farm in Nebraska during the years of the Great Depression and the Dust Bowl, which perhaps had something to do with his ferocious work ethic. After he retired from his distinguished professorship position at the University of Illinois he was asked if he was now taking it easy, to which he replied "I am, I no longer work on Sunday afternoons." His character also showed in how kind and helpful he was to others and specially to young mathematicians. He was always encouraging and never intimidating.

Burkholder attended Earlham College where he met Jean, his wife and the mother of their three children. He received his Ph.D. in mathematical statistics from the University of North Carolina at Chapel Hill in 1955 where he worked under the mathematical statistician Wassily Hoeffding, one of the founders of non-parametric statistics. His thesis, “On a certain class of stochastic approximation processes,” published in *The Annals of Mathematical Statistics*, extended work of Julius Blum, Kai Lai Chung, Joseph Hodes, and Erich Leo Lehmann on “asymptotic normality.” Immediately after graduating he took a job at the University of Illinois at Urbana–Champaign where the legendary Joseph L. Doob, working on martingales and applications of probability to potential theory and complex analysis, was a professor. Almost certainly influenced by Doob, Burkholder soon turned his attention to these subjects.

During his entire career, Burkholder wrote only six joint research papers. His closest collaborator, and the only mathematician with whom he wrote more than one paper, was Richard Gundy. Their work had a deep and lasting impact in probability and its applications. Burkholder’s later work in the eighties on martingales taking values in Banach spaces led him to new techniques to prove sharp martingale inequalities. The latter have had applications in areas which on the surface are far removed from martingales. The wide use of Burkholder’s ideas, results and techniques, shows the universality of his mathematics, although he seemingly “only” touched martingales.

Rodrigo Bañuelos and Burgess Davis
West Lafayette, Indiana 47906
USA
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