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A CODE ARITHMETIC APPROACH FOR QUATERNARY CODE DESIGNS AND ITS APPLICATION TO (1/64)TH-FRACTIONS[1]

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The study of good nonregular fractional factorial designs has received significant attention over the last two decades. Recent research indicates that designs constructed from quaternary codes (QC) are very promising in this regard. The present paper aims at exploring the fundamental structure and developing a theory to characterize the wordlengths and aliasing indexes for a general $(1/4)^p$th-fraction QC design. Then the theory is applied to $(1/64)$th-fraction QC designs. Examples are given, indicating that there exist some QC designs that have better design properties, and are thus more cost-efficient, than the regular fractional factorial designs of the same size. In addition, a result about the periodic structure of $(1/64)$th-fraction QC designs regarding resolution is stated.

1. Introduction. In many scientific researches and investigations, the interest lies in the study of effects of many factors simultaneously. One may choose a full factorial design which is able to estimate all possible level combinations of factors, but it usually involves many unnecessary trials. To be more cost-efficient, a fractional factorial design is suggested. A good choice of fractional factorial design allows us to study many factors with relatively small run size but enables us to estimate a large number of effects.

Designs that can be constructed through defining relations among factors are called regular designs, and all other designs that do not possess this kind of defining relation are called nonregular designs. Wu and Hamada (2000) and Mukerjee and Wu (2006) provide detailed discussions on optimality criteria such as resolution and minimum aberration for choosing fractional factorial designs. Nonregular designs have received particular attention in the past ten to twenty years. The notions of resolution and aberration have been generalized with statistical justifications to these designs; see Deng and Tang (1999) and Tang and Deng (1999). It is well recognized that although nonregular designs have a complex aliasing structure, they can outperform their regular counterparts with regard to resolution or projectivity, and this is a major motivating force for the current surge of interest in

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these designs. A comprehensive review on the development of nonregular designs is referred to Xu, Phoa and Wong (2009).

A recent major development in nonregular two-level designs has been the use of quaternary codes for their simple construction, and the resulting two-level designs are generally called QC designs. Xu and Wong (2007) pioneered research on QC designs and reported theoretical as well as computational results. Phoa and Xu (2009) investigated the properties of quarter-fraction QC designs. In addition to giving theoretical results on the aliasing structure of such designs, they constructed optimal quarter-fraction QC designs under several criteria. Zhang et al. (2011) introduced a trigonometric representation for the study of QC designs and successfully derived the properties of $(1/8)$th- and $(1/16)$th-fractions QC designs. The optimal $(1/8)$th- and $(1/16)$th-fractions QC designs under maximum resolution criterion were reported in Phoa, Mukerjee and Xu (2012).

The present paper aims at exploring the fundamental structure and developing the underlying theorems of a general QC design. In Section 2 we recall some concepts about the design construction method via quaternary codes. Then we introduce some new notation that is related to wordlengths and aliasing indexes of words. This new notation provides clear and simple presentations for theorems and examples in the later sections. Section 3 contains some rules and corollaries about the structure of QC designs. One can derive the wordlengths and aliasing indexes of a word in a general QC design using these rules. In addition, two theorems are stated about the structure of the k-equation and their necessary and sufficient conditions. These theorems are applied in Section 4, leading to a theorem about the properties of $(1/64)$th-fraction QC designs. An example demonstrates the use of the theorem to derive the generalized resolutions and generalized wordlength patterns of QC designs. Based on the properties of the derived classes of QC designs, the structure periodicity of $(1/64)$th-fraction QC designs with high resolution is suggested. The proofs of these theorems are given in the last section.

2. Definitions and notation. We recall some concepts in Phoa and Xu (2009) here. A quaternary code takes on values from $Z_4 = \{0, 1, 2, 3\}$. Let G by an $n \times m$ generator matrix over Z_4. All possible linear combinations of the rows in G over Z_4 form a quaternary linear code, denoted by C. Then each Z_4 entry of C is transformed into two binary codes in its binary image $D = \phi(C)$ via the Gray map, which is defined as follows:

$$\phi : 0 \to (1, 1), \quad 1 \to (1, -1), \quad 2 \to (-1, -1), \quad 3 \to (-1, 1).$$

Note that D is a binary $2^{2n} \times 2m$ matrix or a two-level design with 2^{2n} runs and $2m$ factors.

In general, for highly-fractionated QC designs, we consider an $n \times (n + p)$ generator matrix $G = (V, I_n)$, where $V = (\vec{v}_1, \ldots, \vec{v}_p)$ is a matrix over Z_4 that consists of p vectors of lengths n and I_n is an $n \times n$ identity matrix. It leads to a two-level design D with 2^{2n} runs and $2n + 2p$ factors, that is,

$D = (d_1, \dots, d_{2p}, d_{2p+1}, \dots, d_{2p+2n})$. It is easy to verify that the identity matrix I_n generates a full $2^{2n} \times 2n$ design. Therefore, the properties of D depend on the matrix V only.

For $s = \{c_1, c_2, \dots, c_k\}$, a subset of $k \le 2n + 2p$ columns of D, define $j_k(s; D) = \sum_{i=1}^{2^{2n}} c_{s1} \cdots c_{sk}$, where c_{ij} is the ith entry of c_j. The $j_k(s; D)$ values are called the J-characteristics of design D [Deng and Tang (1999), Tang (2001)]. It is evident that $|j_k(s; D)| \le 2^{2n}$. Following Cheng, Li and Ye (2004), we define the aliasing index as $\rho_k(s) = \rho_k(s; D) = |j_k(s; D)|/2^{2n}$, which measures the amount of aliasing among columns in s. It is obvious that $0 \le \rho_k(s) \le 1$. When $\rho_k(s) = 1$, the columns in s are fully aliased with each other and form a complete word of length k. It is equivalent to the defining relations in regular designs. When $0 < \rho_k(s) < 1$, the columns in s are partially aliased with each other and form a partial word of length k with aliasing index $\rho_k(s)$. When $\rho_k(s) = 0$, the columns in s are orthogonal and do not form a word.

Throughout this paper, for $\vec{i}$ to be a quaternary row vector, let $f_{\vec{i}}$ be the number of times that $\vec{i}$ appears in the rows of V. Define $\vec{w} = (w_1, \dots, w_p)$ to be a word type that describes the structure of a word. All w_i are quaternary with the following meanings. For $i = 1, \dots, p$, if $w_i = 0$, none of the $(2i - 1)$th and $(2i)$th in D are included in the word; if $w_i = 2$, both the $(2i - 1)$th and $(2i)$th in D are included in the word; if w_i is odd, either the $(2i - 1)$th or $(2i)$th in D is included in the word. If there are q odd entries in $\vec{w}$, where $q < p$, there are 2^q different column choices. Therefore, we denote $w_i = 1$ or 3 for different i to represent different column choices. For example, in (1/16)th-fraction QC designs, there are four possible forms of words, namely, $(1, 1)$, $(1, 3)$, $(3, 1)$ and $(3, 3)$, representing the cases that select one column from the first two columns of D and select another column from the next two columns of D.

Let $k_{\vec{w}}$ be the wordlength equation, or simply called k-equation, of the word described by $\vec{w}$. In addition, denote $C(p)$ by a $4^p \times p$ matrix consisting of all possible combinations of quaternary entries. With reference to the matrix V, $k_{\vec{w}}$ can be written as the linear combination of $f_{\vec{i}}$, where $\vec{i}$ represents the ith row of $C(p)$, that is, $k_{\vec{w}} = \sum_{\vec{i} \in C(p)} c_{\vec{i}} f_{\vec{i}}$ for $c_{\vec{i}} = 0, 1, 2$. Furthermore, if there exists two k-equations $k_{\vec{w}_1}$ and $k_{\vec{w}_2}$ with the corresponding coefficient vectors $c_{\vec{i}}$ and $c'_{\vec{i}}$ in their summations, then we define a code arithmetic (CA) operator $\oplus$ in the following way:

$$k_{\vec{w}_1} \oplus k_{\vec{w}_2} = \left(\sum_{\vec{i} \in C(p)} c_{\vec{i}} f_{\vec{i}} \right) \oplus \left(\sum_{\vec{i} \in C(p)} c'_{\vec{i}} f_{\vec{i}} \right) = \sum_{\vec{i} \in C(p)} L_w(c_{\vec{i}} + c'_{\vec{i}}) f_{\vec{i}},$$

where $L_w(x)$ represents the Lee weight of x and the Lee weights of $0, 1, 2, 3 \in Z_4$ are $0, 1, 2, 1$, respectively. Notice that the wordlength of a word is not equal to the value of k-equations directly, but it is equal to that plus a constant showing the number of columns among the first $2p$ columns of D (generated from V) that are included in the word.

The above definitions and concepts are demonstrated in the following example.

EXAMPLE 1. Consider a general $(1/16)$th-fraction QC design D (i.e., $p = 2$) generated by a generator matrix $G = (V, I_n)$, where $V = (u, v)$ for convenience. There are 16 possible combinations of quaternary entries for $\vec{i} = (i_1, i_2)$ for $i_1, i_2 \in \{0, 1, 2, 3\}$. Given a word formed by a specific group of columns $\vec{w}$, its k-equations $k_{\vec{w}}$ can always be written as linear combinations of these 16 combinations of $\vec{i}$. For example,

$$\begin{aligned}
k_{10} = {} & 0(f_{00} + f_{01} + f_{02} + f_{03}) \\
& + 1(f_{10} + f_{11} + f_{12} + f_{13} + f_{30} + f_{31} + f_{32} + f_{33}) \\
& + 2(f_{20} + f_{21} + f_{22} + f_{23}) = l_1, \\
k_{02} = {} & 0(f_{00} + f_{02} + f_{10} + f_{12} + f_{20} + f_{22} + f_{30} + f_{32}) \\
& + 2(f_{01} + f_{03} + f_{11} + f_{13} + f_{21} + f_{23} + f_{31} + f_{33}) = l_6,
\end{aligned}$$

where l_1 and l_6 are defined in Zhang et al. (2011). If we perform a CA operation on these two k-equations,

$$\begin{aligned}
k_{10} \oplus k_{02} = {} & 0(f_{00} + f_{02} + f_{21} + f_{23}) \\
& + 1(f_{10} + f_{11} + f_{12} + f_{13} + f_{30} + f_{31} + f_{32} + f_{33}) \\
& + 2(f_{01} + f_{03} + f_{20} + f_{22}).
\end{aligned}$$

In the resulting k-equation, the coefficient of f_{11} and f_{21} come from $L_w(1+2) = 1$ and $L_w(2+2) = 0$, respectively.

For a simpler notation, we may write a set of k-equations into a matrix form $K = CF$, where K and F are the k-equations and frequency vectors, C is the wordlength equation coefficient matrix or simply called k-matrix. For $(1/4)$th-fractions, $F = (f_0, f_1, f_2, f_3)^T$, $K = (k_1, k_2)^T$ and the equations of k_1 and k_2 in Phoa and Xu (2009) are rewritten as

$$C = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 0 & 2 \end{pmatrix}.$$

For $(1/16)$th-fractions, $F = (f_{00}, f_{01}, f_{02}, f_{03}, f_{10}, f_{11}, f_{12}, f_{13}, f_{20}, f_{21}, f_{22}, f_{23}, f_{30}, f_{31}, f_{32}, f_{33})^T$, $K = (k_{01}, k_{10}, k_{02}, k_{11}, k_{13}, k_{20}, k_{12}, k_{21}, k_{22})^T$ and the equations of $l_1, \dots, k_{10}$ in Zhang et al. (2011) are rewritten as

$$C = \begin{pmatrix} 0 & 1 & 2 & 1 & 0 & 1 & 2 & 1 & 0 & 1 & 2 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 \\ 0 & 1 & 2 & 1 & 1 & 2 & 1 & 0 & 2 & 1 & 0 & 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \\ 0 & 2 & 0 & 2 & 1 & 1 & 1 & 1 & 2 & 0 & 2 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 2 & 1 & 0 & 1 & 0 & 1 & 2 & 1 & 2 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 & 2 & 0 & 2 & 0 & 0 & 2 & 0 & 2 & 2 & 0 & 2 & 0 \end{pmatrix}.$$

The k-equations are ordered in the vector K under the following rules: (1) the position of $k_{\vec{i}_1}$ is on the front of that of $k_{\vec{i}_2}$ if $\sum_{q=1}^{p} L_w(i_{1,q}) < \sum_{q=1}^{p} L_w(i_{2,q})$; (2) if $\sum_{q=1}^{p} L_w(i_{1,q}) = \sum_{q=1}^{p} L_w(i_{2,q})$, then the position of $k_{\vec{i}_1}$ is on the front of that of $k_{\vec{i}_2}$ if $i_{1,u} < i_{2,u}$ and $i_{1,q} = i_{2,q}$ for all $0 < q < u$, where $i_{1,q}$ and $i_{2,q}$ are the qth entries of $\vec{i}_1$ and $\vec{i}_2$, respectively. The frequency vector F is ordered in the ascending order of its quaternary-coded decimal counterpart. The k-matrix of higher-order-fraction QC designs ($p > 1$) will be discussed in the later part of this paper.

The aliasing index can be written in the form of $\rho = 2^{-\lfloor (a+\delta)/2 \rfloor}$, where a is a linear combination of frequencies. Therefore, we may write all a's into a matrix form $A = BF$, where A is the aliasing index equation vector or simply called a-equations, and B is the aliasing index equation coefficient matrix or simply called a-matrix. Generally speaking, the aliasing index of each $k_{\vec{w}}$ is $\rho_{\vec{w}(\mathrm{mod}\,2)}$, and $a_{\vec{w}(\mathrm{mod}\,2)}$ is a component of its order by definition. In addition, $\delta = 1$ if the sum of entries of $\vec{w}$ is even, or 0 otherwise. According to Phoa and Xu (2009), for (1/4)th-fractions, there is only one aliasing index for k_1, so $A = (a_1)$ and $B = (0101)$. For (1/16)th-fractions in Zhang et al. (2011), there are three aliasing indexes $A = (a_{01}, a_{10}, a_{11})$ and

$$B = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

In general, the a-equations are ordered in the vector A under similar rules as k-equations in K.

3. Some rules and theorems on the structure of quaternary-code designs. Given a general k-equation in $(1/4)^p$th-fraction QC designs $k_{\vec{w}} = \sum_{\vec{i} \in C(p)} c_{\vec{i}} f_{\vec{i}}$, where $c_{\vec{i}} = 0, 1$ or 2, all entries of $\vec{w}$ are quaternary and $\vec{i}$ is the ith row of $C(p)$, we denote $\vec{w} = (\vec{w}_l, \vec{w}_{p-l})$ as a partition into two segments: the first segment has length l and the second segment has length $p - l$. Similarly, we denote all $\vec{i} =$

$(\vec{i}_l, \vec{i}_{p-l})$. In addition, if c is quaternary constant, $\vec{c}_l$ represents a vector of length l that all entries are constant c.

The following rules suggest how a k-equation can be derived from another k-equation. Rule 1 extends the k-equations in $(1/4)^p$th-fraction QC designs to those in $(1/4)^{p+1}$th-fractions.

RULE 1. *Given a general k-equation in a $(1/4)^p$th-fraction QC design D_0, $k_{\vec{w}} = \sum_{\vec{i}\in C(p)} c_{\vec{i}} f_{\vec{i}}$. Then all k-equations with $w_{l+1} = 0$ in a $(1/4)^{p+1}$th-fraction QC design D can be expressed as $k_{(\vec{w}_l,0,\vec{w}_{p-l})} = \sum_{s=0}^{3} \sum_{\vec{i}\in C(p)} c_{\vec{i}} f_{(\vec{i}_l,s,\vec{i}_{p-l})}$.*

This result is obvious. If a k-equation consists of $w_{l+1} = 0$, the word described by this k-equation includes none of the $(2l + 1)$th and $(2l + 2)$th columns of D. It acts like considering the same k-equation in $(1/4)^p$th-fraction QC design D_0. This rule can be used to form the basic k-equations for QC designs, which are stated in the following corollaries.

COROLLARY 1. *For a general $(1/4)^{p+1}$th-fraction QC design, $k_{(\vec{0}_l,1,\vec{0}_{p-l})} = \sum_{\vec{i}\in C(p)}(f_{(\vec{i}_l,1,\vec{i}_{p-l})} + f_{(\vec{i}_l,3,\vec{i}_{p-l})} + 2f_{(\vec{i}_l,2,\vec{i}_{p-l})})$, where $(\vec{i}_l, \vec{i}_{p-l})$ represents the ith row of $C(p)$.*

COROLLARY 2. *For a general $(1/4)^{p+1}$th-fraction QC design, $k_{(\vec{0}_l,2,\vec{0}_{p-l})} = 2\sum_{\vec{i}\in C(p)}(f_{(\vec{i}_l,1,\vec{i}_{p-l})} + f_{(\vec{i}_l,3,\vec{i}_{p-l})})$, where $(\vec{i}_l, \vec{i}_{p-l})$ represents the ith row of $C(p)$.*

The proofs of two corollaries are given in the last section. Rule 2 considers the k-equations of a word that consists of only one out of two binary columns generated from every quaternary column in V.

RULE 2. *Given a k-equation in a $(1/4)^p$th-fraction QC design $k_{\vec{1}_p} = \sum_{\vec{i}\in C(p)} c_{\vec{i}} f_{(\vec{i}_{p-1},i_p)}$, where i_p represents the last entry of $\vec{i}$, then $k_{(1,\vec{3}_p)} = \sum_{s=0}^{3} \sum_{\vec{i}\in C(p)} c_{\vec{i}} f_{(s,\vec{i}_{p-1},(i_p+s) \bmod 4)}$.*

It provides a gateway to extend from the k-equations of $(1/4)^p$th-fraction to $(1/4)^{p+1}$th-fraction, where the subscript vectors of the k-equations are all odd entries. For examples, this rule helps to extend from k_1 of $(1/4)$th-fraction to k_{13} of $(1/16)$th-fraction, or k_{111} of $(1/64)$th-fraction to k_{1333} of $(1/256)$th-fraction.

Rule 3 provides a relationship between two k-equations of words with slight difference in the columns chosen.

RULE 3. *Given a general k-equation in a $(1/4)^{p+1}$th-fraction QC design $k_{\vec{w}} = k_{(\vec{w}_l,s_1,\vec{w}_{p-l})}$, then $k_{(\vec{w}_l,s_2,\vec{w}_{p-l})} = k_{\vec{w}} \oplus k_{(\vec{0}_l,2,\vec{0}_{p-l})}$, where $s_1 = (s_2 + 2) \bmod 4$.*

The addition of $k_{(\vec{0}_l,2,\vec{0}_{p-l})}$ implies that a new word is derived from the original word with additional inclusion of the $(2l-1)$th and $(2l)$th columns from D, plus some columns in I_n so that the termwise multiplication of these additional columns results in a vector of 1, that is, a complete aliased structure. Notice that the inclusion of a column twice is equivalent to the exclusion of the column. In the case when s_1 is odd, the exchange between 1 and 3 represents a derivation of different form of k-equations when the word includes either the $(2l-1)$th or $(2l)$th column only. On the other hand, when s_1 is even, the exchange between 0 and 2 represents a derivation of the k-equation of a new word that includes or excludes both the $(2l-1)$th or $(2l)$th columns.

Let $C_z(p)$ be a subset of $C(p)$ for $z = 0, 1, 2$ as follows. For z is even, $C_z(p) = \{\vec{i} \in C(p) : i_1 + \cdots + i_p = z(\text{mod}\, 4)\}$; otherwise, $C_1(p) = \{\vec{i} \in C(p) : i_1 + \cdots + i_p = 1 \text{ or } 3(\text{mod}\, 4)\}$. Then the general structure of a k-equation, where all entries of $\vec{w}$ are odd, can be derived in the following theorem.

THEOREM 1. *In a $(1/4)^p$th-fraction QC design, for all odd entries of $\vec{w} = \vec{1}_p$, a k-equation is expressed as $k_{\vec{w}} = 1\sum_{\vec{i}\in C_1(p)} f_{\vec{i}} + 2\sum_{\vec{i}\in C_2(p)} f_{\vec{i}}$.*

There are 2^{2p-1} frequencies with coefficients 1, 2^{2p-2} frequencies with coefficients 0 and 2^{2p-2} frequencies with coefficients 2. Furthermore, among those 2^{2p-2} frequencies with coefficients 2, there are 2^{p-1} frequencies that all entries of $\vec{i}_2$ are either 0 or 2. It is also the same for those 2^{2p-2} frequencies with coefficients 0.

EXAMPLE 2. We consider a k-equation k_{11} in a general (1/16)th-fraction QC design D. We can express $k_{11} = 1(f_{01} + f_{10} + f_{21} + f_{12} + f_{03} + f_{30} + f_{23} + f_{32}) + 2(f_{02} + f_{20} + f_{11} + f_{33})$, that is, $C_0 = \{(00), (22), (13), (31)\}$, $C_1 = \{(01), (10), (21), (12), (03), (30), (23), (32)\}$ and $C_2 = \{(02), (20), (11), (33)\}$. By counting the above frequencies, there are $2^{2p-1} = 8$ frequencies with coefficient 1, $2^{2p-2} = 4$ frequencies with coefficients 0 and $2^{2p-2} = 4$ frequencies with coefficients 2. Furthermore, among those four frequencies with coefficients 2, there are two frequencies (f_{02} and f_{20}) that all entries of $\vec{i}_2$ are either 0 or 2. It is also the same for those frequencies with coefficients 0 (f_{00} and f_{22}).

The last rule defines the a-equation of a word accompanied with a k-equation.

RULE 4. *Given a general k-equation in a $(1/4)^p$th-fraction QC design $k_{\vec{w}}$ as in Theorem 1, then the a-equation of the corresponding word is $a_{\vec{w}} = a_{\vec{w}\,\text{mod}\,2} = \sum_{\vec{i}\in C_1(p)} f_{\vec{i}}$.*

Rule 4 implies that the aliasing index of a word depends only on the number of odd entries in $\vec{w}$ and their positions, and the even entries basically have no effects.

For example, k_{10} and k_{12} are expected to share the same aliasing index a_{10}, but k_{110} and k_{011} are expected to have different aliasing indexes, the prior has aliasing index a_{110} and the latter has aliasing index a_{011}.

Among all 4^p k-equations for a general $(1/4)^p$th-fraction QC design D, some of them are equivalent to others and some are irrelevant. The following theorem considers these equivalences and irrelevance and specifies a list of k-equations that are necessary to be computed in order to obtain the properties of D.

THEOREM 2. *Consider a general* $(1/4)^p$*th-fraction QC design* D. *There exists* 4^p *possible combinations of* $\vec{w}$ *for* k*-equations. It is necessary and sufficient to consider the following* $\vec{w}$ *in order to obtain the properties of* D:

(1) $\vec{w}$ *that all entries are even, except all entries are* 0;
(2) $\vec{w}$ *that the first odd entry must be* 1 *for* $\vec{w}$ *that consists of odd entries.*

There are $2^p - 1$ k*-equations in the first group of* $\vec{w}$ *and* $2^{2p-1} - 2^{p-1}$ k*-equations in the second group.*

EXAMPLE 3. We consider a general (1/16)th-fraction QC design D and there are 16 possible combinations of $\vec{w}$ listed in Example 1. According to Theorem 2, the first group of $\vec{w}$ has only even entries. Except $\{0, 0\}$, there are three combinations that satisfy this situation and they are $\{0, 2\}$, $\{2, 0\}$ and $\{2, 2\}$. For the remaining 12 combinations (with at least one odd entry), these 6 combinations $\{0, 3\}$, $\{2, 3\}$, $\{3, 0\}$, $\{3, 1\}$, $\{3, 2\}$, $\{3, 3\}$ are not included in consideration because the k-equations of them are exactly equivalent to those with $\vec{w} = \{0, 1\}$, $\{2, 1\}$, $\{1, 0\}$, $\{1, 3\}$, $\{1, 2\}$, $\{1, 1\}$, respectively. Therefore, among all 16 possible combinations of $\vec{w}$, only 9 of them, 3 in the first group and 6 in the second group, are necessary and sufficient to be considered in order to determine the properties of D.

4. Code arithmetic (CA) approach for generating wordlength equations of (1/64)th-fraction QC designs. This section extends the results of (1/16)th-fraction QC designs that appeared in Zhang et al. (2011) and Phoa, Mukerjee and Xu (2012), and sets of k-equations and a-equations for (1/64)th-fractions QC designs are generated using the theorems above. These equations are applied to derive the design properties of (1/64)th-fraction QC designs.

Following Theorem 2, 35 k-equations are sufficient to determine the properties of a (1/64)th-fraction QC design. Specifically, seven of them belong to the first group and 28 of them belong to the second group. Using the CA approach, we derive these 35 k-equations and their corresponding a-equations from the k-equations of (1/4)th- and (1/16)th-fractions QC designs. First, we define $C(2)$ to be a 16×2

matrix consisting of all 16 possible combinations of quaternary entries. Throughout this section, we express all k-equations as a row in the k-matrix for clear and convenient notation.

Rule 1 and two corollaries are applied to obtain k-equations where $\vec{i}$ contains at least one 0. More explicitly, to obtain k-equations with two 0s in $\vec{i}$, that is, k_{100}, k_{010}, k_{001}, k_{200}, k_{020} and k_{002}, we apply Corollaries 1 and 2 with $l = 0, 1, 2$. For example, for k_{100}, we apply Corollary 1 with $l = 0$. This yields a k-equation where, for $j, k = 0, 1, 2, 3$, the coefficients of f_{0jk}, f_{1jk}, f_{2jk} and f_{3jk} are 0, 1, 2 and 1, respectively. Rule 4 suggests a_{100}, the a-equations of k_{100}, such that the coefficients of f_{0jk}, f_{1jk}, f_{2jk} and f_{3jk} are 0, 1, 0 and 1, respectively.

For all k-equations with one 0 in $\vec{i}$, we consider applying Rule 1 on k_{11}, k_{13}, k_{12}, k_{21} and k_{22} with different l. This leads to k_{011}, k_{013}, k_{012}, k_{021}, k_{022} when $l = 0$, k_{101}, k_{103}, k_{102}, k_{201}, k_{202} when $l = 1$ and k_{110}, k_{130}, k_{120}, k_{210}, k_{220} when $l = 2$. For example, for k_{101}, Rule 1 suggests that $\vec{w}_l = \vec{w}_{p-l} = 1$. Then for every row of $C(2)$, denoted as (c_1, c_2), the coefficients of $f_{(c_1,0,c_2)}, f_{(c_1,1,c_2)}, f_{(c_1,2,c_2)}, f_{(c_1,3,c_2)}$ in k_{101} are all equal to the coefficient of $f_{(c_1,c_2)}$ in k_{11}.

It is straightforward to substitute 0s in all k-equations mentioned above with 2 by Rule 3. By changing one 0 into 2 in $\vec{i}$, we obtain k_{102}, k_{120}, k_{012}, k_{210}, k_{021}, k_{201}, k_{202}, k_{220}, k_{022}, k_{112}, k_{132}, k_{122}, k_{212}, k_{222}, k_{121}, k_{123}, k_{221}, k_{211} and k_{213}. For example, in order to obtain k_{121}, Rule 3 suggests performing a CA operation $k_{121} = k_{101} \oplus k_{020}$. The a-equation of k_{121} is equal to a_{101}.

Rule 2 is applied in order to obtain the k-equations with all odd entries in $\vec{i}$, including k_{111}, k_{113}, k_{131} and k_{133}. According to Rule 2, k_{133} can be derived from k_{11}. For every row of $C(2)$, the first and second entries are considered as $\vec{i}_{p-1}$ and i_p, respectively. For example, $\vec{i}_{p-1} = 1$ and $i_p = 0$ for f_{10}. Then we can determine the coefficients of frequency vectors in k_{133} from those in k_{11}. Consider $\vec{i} = (10)$, for example. The coefficient of f_{10} in k_{11} is 1. This implies $f_{010} = f_{111} = f_{212} = f_{313} = 1$ in k_{133} for $s = 0, 1, 2, 3$. Consider $\vec{i} = (02)$ as another example. The coefficient of f_{02} in k_{11} is 2. This implies $f_{002} = f_{103} = f_{200} = f_{301} = 2$ in k_{133} for $s = 0, 1, 2, 3$. The other three k-equations without 0s in $\vec{i}$ can be derived from k_{133} via the CA operations suggested in Rule 3: $k_{111} = (k_{133} \oplus k_{020}) \oplus k_{002}$, $k_{113} = k_{133} \oplus k_{020}$, and $k_{131} = k_{133} \oplus k_{002}$. The a-equations of k_{111}, k_{113}, k_{131} and k_{133} are the same.

There are in total 35 k-equations and 7 a-equations in K and A, respectively. Similar to (1/16)th-fraction QC designs, we may rewrite these k-equations and a-equations into matrix forms where

$$K = (k_{001}, k_{010}, k_{100}, k_{002}, k_{011}, k_{013}, k_{020}, k_{101}, k_{103}, k_{110}, k_{130}, k_{200}, k_{012}, k_{021}, k_{102}, k_{111}, k_{113}, k_{131}, k_{133}, k_{120}, k_{201}, k_{210}, k_{022}, k_{112}, k_{132}, k_{121}, k_{123}, k_{202}, k_{211}, k_{213}, k_{220}, k_{122}, k_{212}, k_{221}, k_{222})^T,$$

$$F = (f_{000}, f_{001}, f_{002}, f_{003}, f_{010}, f_{011}, f_{012}, f_{013}, f_{020}, f_{021}, f_{022}, f_{023}, f_{030}, f_{031}, f_{032}, f_{033}, f_{100}, f_{101}, f_{102}, f_{103}, f_{110}, f_{111}, f_{112}, f_{113}, f_{120}, f_{121}, f_{122}, f_{123}, f_{130}, f_{131}, f_{132}, f_{133}, f_{200}, f_{201}, f_{202}, f_{203}, f_{210}, f_{211}, f_{212}, f_{213}, f_{220}, f_{221}, f_{222}, f_{223}, f_{230}, f_{231}, f_{232}, f_{233}, f_{300}, f_{301}, f_{302}, f_{303}, f_{310}, f_{311}, f_{312}, f_{313}, f_{320}, f_{321}, f_{322}, f_{323}, f_{330}, f_{331}, f_{332}, f_{333})^T,$$

$$C = \begin{pmatrix}
0121012101210121012101210121012101210121012101210121012101210121\\
0000111122221111000011112222111100001111222211110000111122221111\\
0000000000000000111111111111111122222222222222221111111111111111\\
0202020202020202020202020202020202020202020202020202020202020202\\
0121121021011012012112102101101201211210210110120121121021011012\\
0121101221011210012110122101121001211012210112100121101221011210\\
0000222200002222000022220000222200002222000022220000222200002222\\
0121012101210121121012101210121021012101210121011012101210121012\\
0121012101210121101210121012101221012101210121011210121012101210\\
0000111122221111111111112222111100002222111100001111111100001111\\
0000111122221111111111110000111122222222111100001111111122221111\\
0000000000000000222222222222222200000000000000002222222222222222\\
0202111120201111020211112020111102021111202011110202111120201111\\
0121210101212101012121010121210101212101012121010121210101212101\\
0202020202020202111111111111111120202020202020201111111111111111\\
0121121021011012121210210110120121210110120112101012012112102101\\
0121101221011210101221011210012121011210012110121210012110122101\\
0121101221011210121001211012210121011210012110121012210112100121\\
0121121021011012101201211210210121011012012112101210210110120121\\
0000222200002222111111111111111122220000222200001111111111111111\\
0121012101210121210121012101210101210121012101212101210121012101\\
0000111122221111222211110000111100001111222211112222111100001111\\
0202202002022020020220200202202002022020020220200202202002022020\\
0202111120201111111120201111020220201111020211111111020211112020\\
0202111120201111111102021111202020201111020211111111202011110202\\
0121210101212101121010121210101221010121210101211012121010121210\\
0121210101212101101212101012121021010121210101211210101212101012\\
0202020202020202202020202020202002020202020202022020202020202020\\
0121121021011012210110120121121001211210210110122101101201211210\\
0121101221011210210112100121101201211012210112102101121001211012\\
0000222200002222222200002222000000002222000022222222000022220000\\
0202202002022020111111111111111120200202202002021111111111111111\\
0202111120201111202011110202111102021111202011112020111102021111\\
0121210101212101210101212101012101212101012121012101012121010121\\
0202202002022020202002022020020202022020020220202020020220200202
\end{pmatrix},$$

$$A = (a_{001}, a_{010}, a_{100}, a_{011}, a_{101}, a_{110}, a_{111}),$$

$$B = \begin{pmatrix} 0101010101010101010101010101010101010101010101010101010101010101 \\ 0000111100001111000011110000111100001111000011110000111100001111 \\ 0000000000000000111111111111111100000000000000001111111111111111 \\ 0101101001011010010110100101101001011010010110100101101001011010 \\ 0101010101010101101010101010101001010101010101011010101010101010 \\ 0000111100001111111100001111000000001111000011111111000011110000 \\ 0101101001011010101010101010010110100101010110100101101010100101 \end{pmatrix}.$$

The constants for calculating wordlengths are 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 4, 4, 5, 5, 5, 6 and δ for seven a-equations are 0, 0, 0, 1, 1, 1, 0. Then we apply the K and A matrix to the design properties of a general (1/64)th-fraction QC design D with an even number of factors. Assume D is constructed from a generator matrix $G = (u, v, w, I_n)$. Theorem 3 presented below gives an account of words of all possible types.

THEOREM 3. *With reference to the* $2^{(2n+6)-6}$ *QC design* D, *assuming* $\sum_{i=1,3;j=1,3;k=0,2} f_{ijk}$, $\sum_{i=1,3;j=0,2;k=1,3} f_{ijk}$ *and* $\sum_{i=0,2;j=1,3;k=1,3} f_{ijk}$ *are all greater than* 0, *the following hold*:

(a) *There are* $8/\rho_{100}^2$ *words each with aliasing index* ρ_{100}; *each* 1/4 *of them have lengths* $k_{100}+1$, $k_{120}+3$, $k_{102}+3$ *and* $k_{122}+5$.

(b) *There are* $8/\rho_{010}^2$ *words each with aliasing index* ρ_{010}; *each* 1/4 *of them have lengths* $k_{010}+1$, $k_{210}+3$, $k_{012}+3$ *and* $k_{212}+5$.

(c) *There are* $8/\rho_{001}^2$ *words each with aliasing index* ρ_{001}; *each* 1/4 *of them have lengths* $k_{001}+1$, $k_{201}+3$, $k_{021}+3$ *and* $k_{221}+5$.

(d) *There are* $8/\rho_{110}^2$ *words each with aliasing index* ρ_{110}; *each* 1/4 *of them have lengths* $k_{110}+2$, $k_{130}+2$, $k_{112}+4$ *and* $k_{132}+4$.

(e) *There are* $8/\rho_{101}^2$ *words each with aliasing index* ρ_{101}; *each* 1/4 *of them have lengths* $k_{101}+2$, $k_{103}+2$, $k_{121}+4$ *and* $k_{123}+4$.

(f) *There are* $8/\rho_{011}^2$ *words each with aliasing index* ρ_{011}; *each* 1/4 *of them have lengths* $k_{011}+2$, $k_{013}+2$, $k_{211}+4$ *and* $k_{213}+4$.

(g) *There are* $8/\rho_{111}^2$ *words each with aliasing index* ρ_{111}; *each* 1/4 *of them have lengths* $k_{111}+3$, $k_{113}+3$, $k_{131}+3$ *and* $k_{133}+3$.

(h) *There are* 7 *words each with aliasing index* 1; *they have lengths* $k_{200}+2$, $k_{020}+2$, $k_{002}+2$, $k_{220}+4$, $k_{202}+4$, $k_{022}+4$ *and* $k_{222}+6$, *respectively.*

All ρ_{ijk} *are defined as* $2^{-\lfloor (a_{ijk}+\delta)/2 \rfloor}$, *where* $\delta = 1$ *for* ρ_{110}, ρ_{101} *and* ρ_{011}, *and* $\delta = 0$ *otherwise.*

The proof of Theorem 3 can be done in a similar way as either the matrix expansion method in the proof of Theorem 1 of Phoa and Xu (2009) or the trigonometric approach in the proof of Theorem 2 of Zhang et al. (2011) and omitted here. Theorem 3, in conjunction with equations of K and A, shows that the resolution and wordlength pattern of the design D depend on u, v and w only. The following example illustrates the calculations of the generalized resolution and generalized wordlength pattern of D.

EXAMPLE 4. Given the generating matrix of a 256×14 quaternary-code design D,

$$G = (u, v, w, I_4) = \begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 3 & 3 & 0 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

D can be represented by a frequency vector $F = (\vec{0}_{22}, 1, \vec{0}_2, 1, \vec{0}_5, 1, \vec{0}_7, 1, \vec{0}_{24})$, where $\vec{0}_n$ is a vector of 0 with length n. So $K = CF = (5, 5, 5, 6, 4, 4, 6, 4, 4, 4, 4, 6, 3, 3, 3, 3, 3, 3, 7, 3, 3, 3, 4, 2, 6, 2, 6, 4, 6, 2, 4, 5, 5, 5, 2)$ and $A = BF = (3, 3, 3, 2, 2, 2, 1)$. It leads to 35 wordlengths with lengths 6, 6, 6, 8, 6, 6, 8, 6, 6, 6, 6, 8, 6, 6, 6, 6, 6, 6, 10, 6, 6, 6, 8, 6, 10, 6, 10, 8, 10, 6, 8, 10, 10, 10, 8 and seven aliasing indexes all equal to 1/2. Theorem 3 entails 224 partial words each with aliasing index 1/2; of these, 168 have length six and 56 have length ten. In addition, Theorem 3 entails seven complete words of length eight. Hence, in this case the QC design D, which is a 2^{14-6} design, has resolution 6.5 and wordlength pattern (0, 0, 0, 0, 0, 42, 0, 7, 0, 14, 0, 0, 0, 0). Comparing to the regular design of the same size, this QC design has a higher resolution (6.5 versus 5.0) and it has better aberration ($A_5 = 0$ for QC design versus $A_5 \neq 0$ for regular design). Therefore, this QC design is more favorable than its corresponding regular design.

Instead of performing a complete enumeration, a periodic structure for a class of good (1/64)th-fraction QC designs with high resolution is presented in the following theorem.

THEOREM 4. *Given a $2^{(2n+6)-6}$ QC design D_0 defined by a frequency vector F_0, assume D_0 satisfies the conditions in Theorem 3 and it has generalized resolution $R_0 = r_0 + 1 - \rho_0$. Then for $t \geq 0$, a $2^{(2n+126t+6)-6}$ QC design D_t defined by $F_t = F_0 + (0, \vec{1}_{63})t$ has generalized resolution $R_t = r_t + 1 - \rho_t$, where $r_t = r_0 + 64t$ and $\rho_t = \rho_0(2^{-16t})$ if $\rho_0 < 1$ and $\rho_t = 1$ if $\rho_0 = 1$.*

EXAMPLE 5. Following Example 4, let $F_0 = (\vec{0}_{22}, 1, \vec{0}_2, 1, \vec{0}_5, 1, \vec{0}_7, 1, \vec{0}_{24})$ and the 256×14 QC design D_0 has generalized resolution 6.5. Then Theorem 4 suggests that for $t = 1$, a 2^{140-6} QC design D_1 defined by $F_t = (0, \vec{1}_{21}, 2, \vec{1}_2, 2, \vec{1}_5, 2, \vec{1}_7, 2, \vec{1}_{24})$ has $r_t = 6 + 64(1) = 70$ and $\rho_t = (1/2) \times (2^{-16(1)}) = 2^{-17}$, that is, generalized resolution 70.9999924.

5. Summary. This work provides some theoretical understandings of the structure of a general $(1/4)^p$th-fraction QC design. In Section 2 we show via the Code Arithmetic approach how the k-equations and a-equations of a general $(1/4)^p$th-fraction QC design are developed from those of other $(1/4)^h$th-fraction QC designs, where $p > h$. Section 3 lists four rules on the structure of k-equations and a-equations when some entries of $\vec{w}$ are added and/or changed. In addition, Theorem 1 describes the general structure of k-equations when all entries are odd and Theorem 2 suggests which k-equations are sufficient to be considered so that the design properties can be determined. In Section 4 these rules and theorems are applied to determine the properties of (1/64)th-fraction QC designs and the periodic structure regarding resolution is derived.

6. Proofs.

6.1. *Proof of Corollaries 1 and 2*. We prove Corollary 1 via induction. It is trivial for $p=1$, because it leads to k_{10} and k_{01} for $l=0,1$. Assume $p=z$ is true, that is, $k_{(\vec{0}_l,1,\vec{0}_{z-l})}=\sum_{\vec{i}\in C(z)}(f_{(\vec{i}_l,1,\vec{i}_{z-l})}+f_{(\vec{i}_l,3,\vec{i}_{z-l})}+2f_{(\vec{i}_l,2,\vec{i}_{z-l})})$. For $p=z+1$, we rewrite $\vec{w}$ as $(\vec{0}_l,1,0,\vec{0}_{z-l})$, that is, insert a 0 in the $(l+2)$th entry of $\vec{w}$. Applying Rule 1, we have $k_{(\vec{0}_l,1,0,\vec{0}_{z-l})}=\sum_{s=0}^{3}\sum_{\vec{i}\in C(z)}(f_{(\vec{i}_l,1,s,\vec{i}_{z-l})}+f_{(\vec{i}_l,3,s,\vec{i}_{z-l})}+2f_{(\vec{i}_l,2,s,\vec{i}_{z-l})})$. Notice that $(\vec{i}_l,s,\vec{i}_{z-l})$ represents the ith row of $C(z+1)$ for $s=0,1,2,3$, and the above equation becomes $k_{(\vec{0}_l,1,\vec{0}_{(z+1)-l})}=\sum_{s=0}^{3}\sum_{\vec{i}\in C(z+1)}(f_{(\vec{i}_l,1,\vec{i}_{(z+1)-l})}+f_{(\vec{i}_l,3,\vec{i}_{(z+1)-l})}+2f_{(\vec{i}_l,2,\vec{i}_{(z+1)-l})})$. This completes the proof of Corollary 1. The proof of Corollary 2 follows the same induction except the formula is different.

6.2. *Proof of Theorem 1*. We prove Theorem 1 via induction. The cases of $p=1$ and $p=2$ are true from the results of Phoa and Xu (2009) and Zhang et al. (2011). Assume it is true for $p=z$ is true, that is, for $k_{\vec{w}}=1\sum_{\vec{i}\in C_1(z)}f_{\vec{i}}+2\sum_{\vec{i}\in C_2(z)}f_{\vec{i}}$, the sum of entries of all $\vec{i}$ in $C_1(z)$ are odd and the sum of entries of all $\vec{i}$ in $C_2(z)$ are even. Consider $p=z+1$. We start from rewriting $k_{\vec{1}_z}=0(\sum_{\vec{i}\in C_0(z)}f_{(\vec{i}_{z-1},i_z)})+1(\sum_{\vec{i}\in C_1(z)}f_{(\vec{i}_{z-1},i_z)})+2(\sum_{\vec{i}\in C_2(z)}f_{(\vec{i}_{z-1},i_z)})$. The application of Rule 2 suggests that $k_{(1,\vec{3}_z)}=\sum_{s=0}^{3}0(\sum_{\vec{i}\in C_0(z)}f_{(s,\vec{i}_{z-1},i_z+s)})+1(\sum_{\vec{i}\in C_1(z)}f_{(s,\vec{i}_{z-1},i_z+s)})+2(\sum_{\vec{i}\in C_2(z)}f_{(s,\vec{i}_{z-1},i_z+s)})$. Notice that if the sum of entries of $(\vec{i}_{z-1})$ plus i_z is odd, then s plus the sum of entries of $\sum(\vec{i}_{z-1})$ plus (i_z+s) is still odd for $s=0,1,2,3$. It is also true for the even case.

Applying Rule 3, $k_{(\vec{w}_l,s_2,\vec{w}_{p-l})}=k_{(\vec{w}_l,s_1,\vec{w}_{p-l})}\oplus k_{(\vec{0}_l,2,\vec{0}_{p-l})}$, where $s_2=(s_1+2)\bmod 4$. $L_w(1+2)=L_w(3+2)=1$ implies that the frequencies with an odd sum of entries of $\vec{i}$ have odd coefficients. Similarly, $L_w(0+2)=2$ and $L_w(2+2)=0$ imply that the frequencies with an even sum of entries $\vec{i}$ have even coefficients. Therefore, by repeatedly applying Rule 3 to change all entries of 3 into 1 in $\vec{w}$, we can express $k_{\vec{1}_{z+1}}=0\sum_{\vec{i}\in C_0(z+1)}f_{\vec{i}}+1\sum_{\vec{i}\in C_1(z+1)}f_{\vec{i}}+2\sum_{\vec{i}\in C_2(z+1)}f_{\vec{i}}$. This completes the proof.

6.3. *Proof of Theorem 2*. Consider a general $(1/4)^p$th-fraction QC design D. There are 4^p different combinations of $\vec{w}$ with entries in $Z_4\in\{0,1,2,3\}$. Among these $\vec{w}$, there are 2^p of them where their entries are all even. Then it is obvious that $k_{\vec{0}}$ is obviously irrelevant to any properties of D because this k-equation does not include any columns from V and the columns from I_n are complete. This leads to the first group of $\vec{w}$ with a total of $2^p - 1$ possible combinations.

Eliminating the choice with all even entries, there are $4^p - 2^p$ different $\vec{w}$ that consist of at least one odd entry. If we focus on the first odd entry of $\vec{w}$, half of these $\vec{w}$ start with 1 and another half start with 3. Notice that $k_{\vec{w}}$ and $k_{\vec{w}'}$ are equivalent if all 1 entries in $\vec{w}$ become 3 entries in $\vec{w}'$ and vice versa. It is proved as follows.

Using the expression in Theorem 1, without loss of generality, $k_{\vec{w}} = 1\sum_{\vec{i}\in C_1(p)} f_{\vec{i}} + 2\sum_{\vec{i}\in C_2(p)} f_{\vec{i}}$. A repeated use of Rule 3 on every odd entry of $\vec{w}$ in $k_{\vec{w}}$ leads to $k_{\vec{w}'} = k_{\vec{w}} \oplus k_{\vec{w}_2}$, where the entries of $\vec{w}_2$ are 2 if the corresponding entry of $\vec{w}$ is odd, and 0 otherwise. We can express $k_{\vec{w}_2}$ easily by the CA operation on the expressions of Corollary 2 and it results in $k_{\vec{w}_2} = 2\sum_{\vec{i}\in C_1(p)} f_{\vec{i}} + 0\sum_{\vec{i}\in C_2(p)} f_{\vec{i}}$. Then $k_{\vec{w}'}$ can be expressed in the same way as $k_{\vec{w}}$ due to the Lee weight $L_w(3) = 1$.

Therefore, for all $\vec{w}$ that consist of odd entries, it is sufficient and necessary to consider the k-equations that the first odd entry of $\vec{w}$ is 1, and there are $(4^p - 2^p)/2$ or $2^{2p-1} - 2^{p-1}\vec{w}$ in total.

6.4. *Proof of Theorem* 4. About the periodicities of r_t, we start from the original k-matrix $K_0 = CF_0$. If $F_t = F_0 + (0, \vec{1}_{63})t$, then $K_t = CF_t = C(F_0 + (0, \vec{1}_{63})t) = K_0 + C(0, \vec{1}_{63})t$. Since the second term results in a vector of length 35 and all entries are $64t$, and the constants for calculating wordlengths are invariant to t, $r_t = r_0 + 64t$.

About the periodicities of ρ_t, we start from the original a-matrix $A_0 = BF_0$. Similar to the k-matrix, $A_t = A_0 + B(0, \vec{1}_{63})t$. Since the second term results in a vector of length 7 and all entries are $32t$, and the constants for calculating aliasing indexes are fixed at $(0, 0, 0, 1, 1, 1, 0)$, $\rho_t = 2^{-\lfloor (a_t+\delta)/2\rfloor} = 2^{-\lfloor (a_0+32t+\delta)/2\rfloor} = 2^{-\lfloor (a_0+\delta)/2\rfloor} 2^{-32t/2} = \rho_0(2^{-16t})$.

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