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INFLUENCE OF SPATIAL CORRELATION FOR DIRECTED POLYMERS

BY HUBERT LACOIN[1]

Università di Roma Tre

In this paper, we study a model of a Brownian polymer in $\mathbb{R}_+ \times \mathbb{R}^d$, introduced by Rovira and Tindel [*J. Funct. Anal.* **222** (2005) 178–201]. Our investigation focuses mainly on the effect of strong spatial correlation in the environment in that model in terms of free energy, fluctuation exponent and volume exponent. In particular, we prove that under some assumptions, very strong disorder and superdiffusivity hold at all temperatures when $d \geq 3$ and provide a novel approach to Petermann's superdiffusivity result in dimension one [Superdiffusivity of directed polymers in random environment (2000) Ph.D. thesis]. We also derive results for a Brownian model of pinning in a nonrandom potential with power-law decay at infinity.

1. Introduction.

1.1. *Motivation and description of the model.* Much progress has been made lately in the understanding of localization and delocalization phenomena for random polymer models and especially for a directed polymer in a random environment (see [6, 13] for reviews on the subject). The directed polymer in random environment was first introduced in a discrete setup, where the polymer is modeled by the graph of a random walk in $\mathbb{Z}^d$ and the polymer measure is a modification of the law of a simple random walk on $\mathbb{Z}^d$. Recently, though, there has been much interest in the corresponding continuous models, involving Brownian motion rather than simple random walk (see [2, 8, 22, 26]), or semicontinuous models (continuous time and discrete space [17], discrete time and continuous space [18, 20]).

The advantage of these continuous or semicontinuous models is that they allow the use of techniques from stochastic calculus to derive results in a simple way. Another advantage is that they are a natural framework in which to study the influence of spatial correlation in the environment. In this paper, we investigate the influence of slowly vanishing spatial correlation for the directed polymer in Brownian environment, first introduced by Rovira and Tindel [22], which we now describe.

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Let $(\omega(t,x))_{(t\in\mathbb{R}_+,x\in\mathbb{R}^d)}$ be a real centered Gaussian field (under the probability law $\mathbf{P}$) with covariance function

$$\mathbf{P}[\omega(t,x)\omega(s,y)] =: (t\wedge s)Q(x-y), \tag{1.1}$$

where Q is a continuous nonnegative covariance function going to zero at infinity (here, and throughout the paper, $\mathbf{P}[f(\omega)]$ denotes expectation with respect to $\mathbf{P}$; analogous notation is used for other probability laws). Informally, the field can be seen as a summation in time of independent, infinitesimal translation-invariant fields $\omega(\mathrm{d}t,x)$ with covariance function $Q(x-y)\,\mathrm{d}t$. To avoid normalization, we assume $Q(0)=1$. We define the random Hamiltonian formally as

$$H_{\omega,t}(B)=H_t(B):=\int_0^t \omega(\mathrm{d}s,B_s).$$

For a more precise definition of H_t, we refer to [2], Section 2, where a rigorous meaning is given for the above formula. Notice that with this definition, $(H_t(B))$ is a centered Gaussian field indexed by the continuous function $B\in C[0,t]$, with covariance matrix

$$\mathbf{P}\big[H_t\big(B^{(1)}\big)H_t\big(B^{(2)}\big)\big]:=\int_0^t Q\big(B_s^{(1)}-B_s^{(2)}\big)\,\mathrm{d}s.$$

For most of the purposes of this article, this could be considered as the definition of H_t.

One defines the (random) polymer measure for inverse temperature β as a transformation of the Wiener measure P as follows:

$$\mathrm{d}\mu_t^{\beta,\omega}(B):=\frac{1}{Z_t^{\beta,\omega}}\exp(\beta H_t(B))\,\mathrm{d}P(B),$$

where $Z_t^{\beta,\omega}$ is the partition function of the model

$$Z_t^{\beta,\omega}:=P[\exp(\beta H_t)].$$

The aim of studying a directed polymer is to understand the behavior of $(B_s)_{s\in[0,t]}$ under μ_t when t is large for a typical realization of the environment ω.

1.2. *Very strong disorder and free energy.* To study some characteristic properties of the system, it is useful to consider the renormalized partition function

$$W_t^{\beta,\omega}=W_t:=P\left[\exp\left(\int_0^t \beta\omega(\mathrm{d}s,B_s)-\frac{\beta^2}{2}\,\mathrm{d}s\right)\right]=\frac{Z_t^{\beta,\omega}}{\mathbf{P}Z_t^{\beta,\omega}}.$$

It can be checked, without much effort, that W_t is a positive martingale with respect to

$$\mathcal{F}_t:=\sigma\{\omega_s,s\le t\}.$$

Therefore, it converges to a limit W_∞. It follows from a standard argument (checking that the event is in the tail σ-algebra) that

$$\mathbf{P}\{W_\infty := 0\} \in \{0, 1\}.$$

Bolthausen first had the idea of studying this martingale for polymers in a discrete setup [3]. He used it to prove that when the transversal dimension d is larger than 3 and β is small enough, the behavior of the polymer trajectory B is diffusive under μ_t. The technique has since been improved by Comets and Yoshida [9] to prove that whenever W_∞ is nondegenerate, diffusivity holds. The argument of [9] can be adapted to our Brownian case. When $\mathbf{P}\{W_\infty := 0\} = 0$, we say that *weak disorder* holds; the situation where $\mathbf{P}\{W_\infty := 0\} = 1$ is referred to as *strong disorder*.

In the Gaussian setup, a partial annealing argument shows that increasing β increases the influence of disorder. Indeed, for any $t \geq 0$,

$$\begin{aligned}
W_t^{\beta+\beta',\omega} &= P\left[\exp\left(\int_0^t (\beta+\beta')\omega(\mathrm{d}s, B_s) - \frac{\beta^2+\beta'^2}{2}\,\mathrm{d}s\right)\right] \\
&\overset{(\mathcal{L})}{=} P\left[\exp\left(\int_0^t \beta\omega^{(1)}(\mathrm{d}s, B_s) + \sqrt{\beta'^2 + 2\beta\beta'}\,\omega^{(2)}(\mathrm{d}s, B_s)\right.\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left.\left. - \frac{(\beta+\beta')^2}{2}\,\mathrm{d}s\right)\right] \\
&=: \widehat{W}_t^{\omega^{(1)},\omega^{(2)}},
\end{aligned}$$

where the equality holds in law and $\omega^{(1)}$, $\omega^{(2)}$ are two independent Gaussian fields distributed like ω (we denote by $\mathbf{P}^{(1)} \otimes \mathbf{P}^{(2)}$ the associated probability). This is also valid for $t = \infty$. Averaging with respect to $\omega^{(2)}$ on the right-hand side gives

$$\mathbf{P}^{(2)} \widehat{W}_t^{\omega^{(1)},\omega^{(2)}} = W_t^{\beta,\omega^{(1)}}.$$

Moreover, for a given realization of $\omega^{(1)}$, $\mathbf{P}^{(2)} \widehat{W}_\infty^{\omega^{(1)},\omega^{(2)}} = 0$ implies that

$$\widehat{W}_\infty^{\omega^{(1)},\omega^{(2)}} = 0, \qquad \mathbf{P}^{(2)}\text{-a.s.}$$

and, therefore,

$$\begin{aligned}
\mathbf{P}\{W_\infty^{\beta+\beta',\omega} = 0\} &\geq \mathbf{P}^{(1)}\{\mathbf{P}^{(2)} \widehat{W}_\infty^{\omega^{(1)},\omega^{(2)}} = 0\} \\
&= \mathbf{P}^{(1)}\{W_\infty^{\beta,\omega^{(1)}} = 0\} = \mathbf{P}\{W_\infty^{\beta,\omega} = 0\}.
\end{aligned}$$

As a consequence, there exists a critical value β_c separating the two phases, that is, there exists $\beta_c \in [0, \infty)$ such that

$$\begin{aligned}
\beta \in (0, \beta_c) \quad &\Rightarrow \quad \text{weak disorder holds,} \\
\beta > \beta_c \quad &\Rightarrow \quad \text{strong disorder holds.}
\end{aligned}$$

From the physicist's point of view, it is, however, more natural to have a definition of strong disorder using free energy. The quantity to consider is the difference between quenched and annealed free energy.

PROPOSITION 1.1. *The a.s. limit*

$$p(\beta) := \lim_{t\to\infty} \frac{1}{t} \log W_t = \lim_{t\to\infty} \frac{1}{t}\mathbf{P}[\log W_t] =: \lim_{t\to\infty} p_t(\beta) \tag{1.2}$$

exists and is almost surely constant. The function $\beta \mapsto p(\beta)$ is nonpositive and nonincreasing.

We can define

$$\overline{\beta_c} := \sup\{\beta > 0 \text{ such that } p(\beta) = 0\}.$$

It is obvious from the definitions that $\beta_c \le \overline{\beta_c}$.

For a proof of the existence of the limits above and their equality, we refer to [22], Lemma 2.4, Proposition 2.6. The nonpositivity follows from Jensen's inequality

$$\mathbf{P}[\log W_t] \le \log \mathbf{P}[W_t] = 0.$$

It can be shown (for results in the discrete setup, see [4, 5]) that an exponential decay of W_t corresponds to a significant localization property of the trajectories. More precisely, under this condition, it can be shown that two paths chosen independently with law $\mu_t^{\beta,\omega}$ tend to spend a positive fraction of the time in the same neighborhood. For example, whenever the left-hand side exists [i.e., everywhere except for perhaps countably many β, as $p(\beta) + \beta^2/2$ is a convex function], we have

$$\frac{\partial p}{\partial \beta}(\beta) := -\lim_{t\to\infty} \frac{1}{t\beta}\mathbf{P}\left[\mu_{\beta,t}^{\otimes 2}\left(\int_0^t Q\big(B_s^{(1)} - B_s^{(2)}\big)\right)\right];$$

see [4], Section 7, where this equality is proved for directed polymers in $\mathbb{Z}_+ \times \mathbb{Z}^d$. It has become customary to refer to this situation as *very strong disorder*.

It is widely expected that the two notions of strong disorder coincide outside the critical point and that we have $\beta_c = \overline{\beta_c}$. However, it remains an open and challenging conjecture. In [7] and [15], it has been shown that for the directed polymer in $\mathbb{Z}_+ \times \mathbb{Z}^d$ with $d = 1, 2$ and i.i.d. site disorder, very strong disorder holds at all temperatures, and it was previously well known that there is a nontrivial phase transition when $d \ge 3$. The same is expected to hold in continuous space and time if the correlation function Q decays sufficiently fast at infinity.

1.3. *Superdiffusivity.* Another widely studied issue for directed polymers is the superdiffusivity phenomenon [1, 2, 14, 18, 20, 21, 24]. As mentioned earlier, in the weak disorder phase, the trajectory of the polymer conserves all the essential features of the nondisordered model (i.e., standard Brownian motion). Therefore, if one looks at a trajectory up to time t, the end position of the chain, the maximal distance to the origin and the typical distance of a point in the chain to the origin are all of order $t^{1/2}$. This is one of the features of a *diffusive* behavior. It is believed that in the *strong disorder* phase, this property is changed, and that the quantities mentioned earlier are greater than $t^{1/2}$ (the chain tends to go farther from the origin to reach a more favorable environment). Physicists have conjectured that there exists a positive real number $\xi > 1/2$ such that, under μ_t for large t,

$$\max_{s\in[0,t]} \|B_s\| \approx t^{\xi}.$$

We refer to ξ as the *volume exponent*. It is believed that in the strong disorder phase, ξ does not depend on the temperature and is equal to the exponent of the associated oriented last-passage percolation model which corresponds to zero temperature (see [21]). Moreover, the volume exponent should be related to the *fluctuation exponent*, $\chi > 0$, which describes the fluctuation of $\log Z_t$ around its average and is defined (in an informal way) by

$$\mathrm{Var}_{\mathbf{P}} \log Z_t \approx t^{2\chi} \qquad \text{for large } t.$$

The two exponents should satisfy the scaling relation

$$\chi = 2\xi - 1.$$

Moreover, in dimension 1, an additional *hyperscaling* relation, $\xi = 2\chi$, should hold and it is therefore widely believed that $\xi = 2/3$ and $\chi = 1/3$. In larger dimensions, there is no consensus in the physics literature regarding the exponent values.

Superdiffusivity remains, however, a very challenging issue since, in most cases, the existence of ξ and χ has not been rigorously established.

However, some mathematical results have been obtained in various contexts related to directed polymers and can be translated informally as inequalities involving ξ and χ.

- For undirected first-passage percolation, Newman and Piza proved that $\xi \leq 3/4$ in every dimension [19] and, in collaboration with Licea [16], that $\xi \geq 3/5$ in dimension 2 (corresponding to $d = 1$ for a directed polymer), using geometric arguments.
- Johansson proved [14] that $\xi = 2/3$ and $\chi = 1/3$ for last-passage oriented percolation with i.i.d. exponential variables on $\mathbb{N} \times \mathbb{Z}$ (this corresponds to the discrete directed polymer with $\beta = \infty$). The method he employed relies on exact calculation and it is probably difficult to adapt to other cases.
- For a discrete-time continuous-space directed polymer model, Petermann [20] proved that for $d = 1$, $\xi \geq 3/5$. Méjane [18] proved for the same model that $\xi \leq 3/4$ in every dimension. The result of Petermann has recently been adapted for Brownian polymer in Brownian environment by Bezerra, Tindel and Viens [2].
- Very recently, Balazs, Quastel and Seppäläinen [1] computed the scaling exponent for the Hopf–Cole solution of the KPZ/stochastic Burgers equation, a problem that can be interpreted as a $(1 + 1)$-dimensional directed polymer in a random

environment given by space–time white noise. Their result is coherent with physical predictions, that is, $\chi = 1/3$ and $\xi = 2/3$, and may lead to exact results for other models.

• Seppäläinen [24] proved, for directed polymers with log-gamma distributed weight, that $\xi = 2/3$. As in Johansson's case, his result relies on exact calculations that are specific to the particular distribution of the environment.

In addition, the following relations linking ξ and χ have been proven to hold in various contexts:

$$\begin{aligned} \chi &\geq 2\xi - 1, \\ \chi &\geq \frac{1 - d\xi}{2}, \\ \chi &\leq 1/2 \end{aligned}$$

(see, e.g., [8] in the case of Brownian polymer in Poissonian environment), leading, for example, to $\chi \geq 1/8$ in dimension $1+1$.

1.4. *Presentation of the main results.* In this paper, we focus (mainly) on the case where Q has power-law decay [recall (1.1)]. Unless otherwise stated, we will consider that there exists $\theta > 0$ such that

$$Q(x) \asymp \|x\|^{-\theta} \qquad \text{as } \|x\| \to \infty, \tag{1.3}$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$. By $f(x) \asymp g(x)$ as $\|x\| \to \infty$, we mean that there exist positive constants R and c such that

$$c^{-1} f(x) \leq g(x) \leq c f(x) \qquad \forall x,\ \|x\| \geq R. \tag{1.4}$$

In the sequel, we also write, for functions of one real variable, $f(t) \asymp g(t)$ as $t \to \infty$ and $f(t) \asymp g(t)$ as $t \to 0_+$ [with definitions similar to (1.4)]. In this setup, we obtain various results concerning free energy, volume exponent and fluctuation exponent. These results show that when the spatial correlation decays sufficiently slowly ($d \geq 2$, $\theta \leq 2$ or $d = 1$, $\theta < 1$), the essential properties of the system are changed, even in a spectacular way for $d \geq 3$, where the weak disorder phase disappears and we can prove superdiffusivity.

THEOREM 1.2. *We have the following characterization of weak/strong disorder regimes*:

(i) *if* $d \geq 3$ *and* $\theta > 2$, *then* $\overline{\beta_c} \geq \underline{\beta_c} > 0$;
(ii) *if* $d \geq 2$ *and* $\theta < 2$, *then* $\beta_c = \overline{\beta_c} = 0$;
(iii) $d = 1$, $\beta_c = \overline{\beta_c} = 0$ *for any value of* θ.

In the cases where $\overline{\beta_c} = 0$, we obtain sharp bounds on both sides for the free energy.

THEOREM 1.3. *For $d \geq 2$, $\theta < 2$ or $d = 1$, $\theta < 1$, we have*

$$p(\beta) \asymp -\beta^{4/(2-\theta)}.$$

For $d = 1$, $Q \in \mathbb{L}_1(\mathbb{R})$ (with no other assumption on the decay), we have

$$p(\beta) \asymp -\beta^4.$$

REMARK 1.4. For $d = 1$, $\theta > 1$, one can see that Theorem 1.3 is identical to [15], Theorem 1.5, suggesting that, in this case, the Brownian model is in the same *universality class* as the discrete model. One would have the same conclusion $d = 2$, $\theta > 2$, suggesting that the system does not feel the correlation if Q is in $\mathbb{L}_1(\mathbb{R}^d)$.

REMARK 1.5. In the cases we have left unanswered, namely $d = 2, \theta \geq 2$ and $d = 3$, $\theta = 2$, the technique used for the two-dimensional discrete case (see [15]) can be adapted to prove that $\overline{\beta_c} = 0$. Since the method is relatively complicated and very similar to that which is applied in the discrete case, we do not develop it here. In these cases, $p(\beta)$ decays faster than any polynomial around zero. For $d = 2$, $\theta > 2$ or $d = 3$, $\theta = 2$, one would expect to have

$$p(\beta) \asymp -\exp\left(-\frac{c}{\beta^2}\right),$$

while for $d = 2$, $\theta = 2$, one should have

$$p(\beta) \asymp -\exp\left(-\frac{c}{\beta}\right).$$

However, in both cases, one cannot get a lower bound and an upper bound that match.

For $d \geq 3$, Theorem 1.2 ensures that diffusivity holds at high temperatures when $\theta > 2$. We have proved that, on the other hand, superdiffusivity holds (in every dimension) for $\theta < 2$.

THEOREM 1.6. *When $d \geq 2$ and $\theta < 2$ or $d = 1$ and $\theta < 1$, we have*

$$\lim_{\varepsilon \to 0} \liminf_{t \to \infty} \mathbf{P}\mu_t^{\beta,\omega}\left\{\sup_{0 \leq s \leq t} \|B_s\| \geq \varepsilon t^{3/(4+\theta)}\right\} = 1.$$

For $d = 1$, $Q \in \mathbb{L}_1(\mathbb{R})$, we have

$$\lim_{\varepsilon \to 0} \liminf_{t \to \infty} \mathbf{P}\mu_t^{\beta,\omega}\left\{\sup_{0 \leq s \leq t} \|B_s\| \geq \varepsilon t^{3/5}\right\} = 1.$$

In the development, we will not go into the details of the proof of the case $Q \in \mathbb{L}_1(\mathbb{R})$ as it is very similar to the proofs of the other cases and, furthermore, because it is just a minor improvement of the result of [2].

REMARK 1.7. The argument we use for our proof uses change-of-measure and coupling arguments instead of computation on Gaussian covariance matrices as introduced by Petermann and later adapted by Bezerra, Tindel and Viens (see [2, 20]). In our view, this makes the computation much clearer. Besides, our proof is shorter and goes substantially further.

REMARK 1.8. Another polymer model, namely: Brownian polymer in a Poissonian environment, has been introduced and studied by Comets and Yoshida [8]. We would like to stress that our proofs do not rely on the Gaussian nature of the environment and that superdiffusivity with exponent $3/5$, as well as very strong disorder in dimension 1 and 2 (in dimension 2 one needs to adapt the method used in [15]), can also be proven for this model by using methods developed in the present paper. We focus on Brownian polymer mainly because it is the natural model to study the effect of long-range spatial memory.

On the other hand, the bound of Méjane also holds for this model and so we present a short proof for it.

PROPOSITION 1.9. *For any values of β, d and arbitrary Q, and any $\alpha > 3/4$, we have*

$$\lim_{t\to\infty} \mathbf{P}\mu_t\Big\{\max_{s\in[0,t]} \|B_s\| \geq t^\alpha\Big\} = 0.$$

REMARK 1.10. The two previous results can be interpreted as

$$\frac{3}{4+(\theta\wedge d)} \leq \xi \leq 3/4.$$

Taking θ close to zero ensures that the upper bound for ξ is optimal as a bound which holds for any correlation function Q. To get a better upper bound (e.g., $\xi \leq 2/3$ in the one-dimensional case), one would have to use explicitly the lack of correlation in the environment.

Our final result concerns the lower bound on the variance of $\log Z_t$.

THEOREM 1.11. *For any values of β, d and Q such that* (1.3) *holds, if α is such that*

$$\lim_{t\to\infty} \mathbf{P}\mu_t\Big\{\max_{s\in[0,t]} \|B_s\| \geq t^\alpha\Big\} = 0,$$

then there exists a constant c (depending on β, d and Q) such that

$$\mathrm{Var}_{\mathbf{P}} \log Z_t \geq ct^{1-(\theta\wedge d)\alpha}.$$

In particular, for every ε, one can find c (depending on β, d, Q and ε) such that

$$\mathrm{Var}_{\mathbf{P}} \log Z_t \geq ct^{(4-3\theta)/4-\varepsilon}.$$

The previous result can be informally written as

$$\chi \geq \frac{1-(\theta \wedge d)\xi}{2}.$$

The paper is organized as follows.

- In Section 2, we study a homogeneous pinning model in order to derive results that will be of use for the study of $p(\beta)$.
- In Section 3, we prove all the results concerning weak and strong disorder and the free energy of the directed polymer, that is, Theorems 1.2 and 1.3.
- In Section 4, we prove all the results concerning volume exponent and fluctuation exponent, that is, Theorem 1.6, Proposition 1.9 and Theorem 1.11.

2. Brownian homogeneous pinning in a power-law tailed potential.

2.1. *The model and presentation of the results.* In this section, we study a deterministic Brownian pinning model. Most of the results obtained in this section will be used as tools to prove lower bounds on the free energy for the directed polymer model, but they are also of interest in their own right. This pinning model was recently introduced and studied in a paper by Cranston et al. [10] in the case of a smooth and compactly supported potential—various results were obtained using the tools of functional analysis. Our main interest here is in potentials with power-law decay at infinity.

V is either a bounded continuous nonnegative function of $\mathbb{R}^d$ such that $V(x)$ tends to zero when x goes to infinity or else $V(x) = \mathbf{1}_{\{\|x\| \leq 1\}}$.

We define the energy of a continuous trajectory up to time t, $(B_s)_{s\in\mathbb{R}}$, to be the Hamiltonian

$$G_t(B) := \int_0^t V(B_s)\,\mathrm{d}s.$$

We define $\nu_t^{(h)}$, the Gibbs measure associated with that Hamiltonian and pinning parameter (or inverse temperature) $h \in \mathbb{R}$ to be

$$\mathrm{d}\nu_t^{(h)}(B) := \frac{\exp(hG_t(B))}{Y_t^{(h)}}\,\mathrm{d}P(B),$$

where $Y_t^{(h)}$ denotes the partition function

$$Y_t^{(h)} := P[\exp(hG_t(B))].$$

As for the Brownian polymer, the aim is to investigate the typical behavior of the chain under ν_t for large t. The essential question for this model is whether or not the pinning potential hV is sufficient to keep the trajectory of the polymer near the origin, where V takes larger values. It is intuitively clear that for large h, the potential localizes the polymer near the origin [the distance remains $O(1)$

as t grows] and that $h \leq 0$ has no chance to localize the polymer. Therefore, one must determine whether the polymer is in the localized phase for all positive $h > 0$ or if the phase transition occurs for some critical value of $h_c > 0$. This question was answered in [10] in the case of compactly supported smooth V, where it was shown that localization holds for all $h > 0$ if and only if $d \leq 2$ (in fact, this issue is strongly related to recurrence/transience of the Brownian motion).

Answering this question relies on studying the free energy.

PROPOSITION 2.1. *The limit*

$$\mathrm{F}(h) := \lim_{t\to\infty} \frac{1}{t} \log Y_t^{(h)}$$

exists and is nonnegative. $h \mapsto \mathrm{F}(h)$ *is a nondecreasing*, *convex function*.

We call $\mathrm{F}(h)$ the *free energy* of the model. We define

$$h_c = h_c(V) := \inf\{h : \mathrm{F}(h) > 0\} \geq 0.$$

The existence of the limit is not straightforward. Cranston et al. proved it in [10], Section 7, in the case of C^∞ compactly supported V; we will adapt their proof to our case. To understand why $\mathrm{F}(h) > 0$ corresponds to the localized phase, we note the following point: convexity allows us to interchange limit and derivative; therefore, at points where F has a derivative,

$$\frac{\partial \mathrm{F}}{\partial h}(h) = \lim_{t\to\infty} \frac{1}{t} \nu_t \left[\int_0^t V(B_s)\,\mathrm{d}s \right].$$

Cranston et al. proved that when V is smooth and compactly supported, $h_c(V) = 0$ for $d = 1, 2$ and $h_c(V) > 0$ when $d \geq 3$, with some estimate on the free energy around h_c. We want to see how this result can be modified when V has power-law decay at infinity. To do so, we will use [10], Theorem 6.1, for which we present a simplified version for $V(x) = \mathbf{1}_{\{\|x\|\leq 1\}}$ (which is not smooth, but for which the result still holds by monotonicity of the free energy).

THEOREM 2.2 (From [10], Theorem 6.1). *Let* $V : \mathbb{R}^d \to \mathbb{R}_+$ *be defined as*

$$V(x) = \mathbf{1}_{\{\|x\|\leq 1\}}.$$

Then, *for* $d = 1, 2$, *we have* $h_c = 0$ *and*, *as* $h \to 0+$,

$$\begin{aligned} &\mathrm{F}(h) = 2h^2\big(1 + o(1)\big) \qquad \textit{for } d = 1, \\ &\mathrm{F}(h) = \exp\big(-h^{-1}\big(1 + o(1)\big)\big) \qquad \textit{for } d = 2. \end{aligned} \tag{2.1}$$

For $d \geq 3$, *we have* $h_c > 0$.

REMARK 2.3. Although Theorem 2.2 is completely analogous to results obtained for discrete random walk pinning on a defect line (see [11]), the methods used to prove it are very different. Indeed, the technique used in the discrete setup uses discrete renewal theory and cannot be adapted here.

Unless otherwise stated, we assume from now on that V has power-law decay at infinity, that is, that there exists $\theta > 0$ such that

$$V(x) \asymp \|x\|^{-\theta} \qquad \text{as } \|x\| \to \infty \tag{2.2}$$

[the notation $\asymp$ was introduced in (1.3)]. We prove that in dimension $d \geq 3$, whether or not h_c is equal to zero depends only on the exponent θ. Furthermore, when the value of θ varies, so does the critical exponent, which can take any value in $(1, \infty)$.

THEOREM 2.4. *For $d \geq 3$, we have*

$$\theta > 2 \quad \Rightarrow \quad h_c(V) > 0 \tag{2.3}$$

and when $h > 0$ is small enough, we have

$$\lim_{t\to\infty} Y_t^{(h)} < \infty.$$

Moreover, when $\theta < 2$,

$$\textsc{f}(h) \asymp h^{2/(2-\theta)} \qquad \text{as } h \to 0_+. \tag{2.4}$$

For the lower-dimensional cases ($d = 1, 2$), it follows from [10], Theorem 6.1, and monotonicity of $\textsc{f}$ in V that $h_c = 0$ for all θ. However, some of the features of Theorem 2.4 still hold.

THEOREM 2.5. *For $d = 2$, $\theta < 2$ and $d = 1$, $\theta < 1$, we have*

$$\textsc{f}(h) \asymp h^{2/(2-\theta)} \qquad \text{as } h \to 0_+. \tag{2.5}$$

For the sake of completeness, we also present the result for the case $d = 1$, $\theta > 1$. The following is the generalization of a result proved for compactly supported smooth function in [10], Theorem 6.1.

PROPOSITION 2.6. *For $d = 1$ and $V \in \mathbb{L}_1(\mathbb{R})$ continuous and nonnegative,*

$$\textsc{f}(h) = \frac{\|V\|^2_{\mathbb{L}_1(\mathbb{R})}}{2} h^2\bigl(1 + o(1)\bigr) \qquad \text{as } h \to 0_+. \tag{2.6}$$

In the case $d = 2$, $\theta \geq 2$, it can be checked that monotonicity in V of the free energy and (2.1) together imply that $h_c > 0$ and that $\textsc{f}(h)$ decays faster that any power of h around $h = 0$. In the case $d = 1$, $\theta = 1$, monotonicity again implies that $\textsc{f}(h)$ behaves like $h^{2+o(1)}$ around $0+$.

REMARK 2.7. In [10], the critical behavior of the free energy is computed (in a sharp way) for every dimension, even in the case where $h_c > 0$ ($d \geq 3$), as it is done in the discrete case. As will be seen in the proof, the method used to obtain critical exponents in the present paper fails to give any result when $h_c > 0$. However, it would be natural to think that for some value of $\theta > 2$, the critical exponent depends on θ and is different from the one obtained in [10], Theorem 6.1.

2.2. *Preparatory proof.* In this subsection, we give the proofs of results that are easy consequences of results in [10]: existence of the free energy and an estimate of the free energy in the case $V \in \mathbb{L}^1(\mathbb{R})$.

PROOF OF PROPOSITION 2.1. Given V and ε, one can find a compactly supported C^∞ function $\check{V}$ such that

$$\check{V} \leq V \leq \check{V} + \varepsilon.$$

We write $\check{Y}_t^{(h)}$ for the partition function corresponding to $\check{V}$. Trivially, we have, for every t,

$$\frac{1}{t} \log \check{Y}_t^{(h)} \leq \frac{1}{t} \log Y_t^{(h)} \leq \frac{1}{t} \log \check{Y}_t^{(h)} + h\varepsilon.$$

As proved in [10], Section 7, $\log \check{Y}_t$ converges as t goes to infinity so that

$$\limsup \frac{1}{t} \log Y_t^{(h)} - \liminf \frac{1}{t} \log Y_t^{(h)} \leq h\varepsilon.$$

The proof can also be adapted to prove the existence of the free energy for the potential

$$V(x) := \mathbf{1}_{\{\|x\| \leq 1\}}.$$

We omit the details here. □

PROOF OF PROPOSITION 2.6. First, we prove the upper bound. By the occupation times formula (see, e.g., [23], page 224), we have

$$\int_0^t V(B_s)\, \mathrm{d}s = \int_{\mathbb{R}} L_t^x V(x)\, \mathrm{d}x,$$

where L_t^x is the local time of the Brownian motion in x at time t. By Jensen's inequality, we then have

$$\exp\left(h \int_{\mathbb{R}} L_t(x) V(x)\, \mathrm{d}x\right) \leq \int_{\mathbb{R}} \frac{V(x)\, \mathrm{d}x}{\|V\|_{\mathbb{L}_1(\mathbb{R})}} \exp\left(h \|V\|_{\mathbb{L}_1(\mathbb{R})} L_t^x\right).$$

Moreover, under the Wiener measure with initial condition zero, L_t^x is stochastically bounded (from above) by L_t^0 for all x so that

$$
\begin{aligned}
Y_t^{(h)} &\le \int_{\mathbb{R}} \frac{V(x)\,\mathrm{d}x}{\|V\|_{\mathbb{L}_1(\mathbb{R})}} P\big[\exp\big(h\|V\|_{\mathbb{L}_1(\mathbb{R})} L_t^x\big)\big] \\
&\le P\big[\exp\big(h\|V\|_{\mathbb{L}_1(\mathbb{R})} L_t^0\big)\big] \le 2\exp\biggl(\frac{th^2\|V\|^2_{\mathbb{L}_1(\mathbb{R})}}{2}\biggr).
\end{aligned} \tag{2.7}
$$

In the last inequality, we have used the fact that $L_t^0 \overset{(\mathcal{L})}{=} \sqrt{t}|\mathcal{N}(0,1)|$. Taking the limit as t tends to infinity gives the upper bound. For the lower bound, the assumption we have on V guarantees that, given $\varepsilon > 0$, we can find $\check{V}$, smooth and compactly supported, such that

$$
\begin{gathered}
\check{V} \le V, \\
\|\check{V}\|_{\mathbb{L}_1(\mathbb{R})} \ge \|V\|_{\mathbb{L}_1(\mathbb{R})} - \varepsilon.
\end{gathered}
$$

Let $\check{\mathrm{F}}$ be the free energy associated with $\check{V}$. By [10], Theorem 6.1, and monotonicity, we have that for h small enough (how small depends on ε),

$$
\mathrm{F}(h) \ge \check{\mathrm{F}}(h) \ge \biggl(\frac{\|\check{V}\|^2_{\mathbb{L}_1(\mathbb{R})}}{2} - \varepsilon\biggr)h^2.
$$

Since ε was chosen arbitrarily, this gives the lower bound. □

2.3. *Proof of upper bound results on the free energy.* In this subsection, we prove the upper bounds corresponding to (2.3), (2.4) and (2.5). We will give a brief sketch of the proof. To do so, we will repeatedly use the following result.

LEMMA 2.8. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers and $(p_n)_{n\in\mathbb{N}}$ a sequence of strictly positive real numbers satisfying $\sum_{n\in\mathbb{N}} p_n = 1$. We then have*

$$
\prod_{n\in\mathbb{N}} a_n \le \sum_{n\in\mathbb{N}} p_n a_n^{1/p_n} \tag{2.8}
$$

provided the left-hand side is defined. This formula is also valid for a product with finitely many terms.

PROOF. Let X be the random variable whose distribution P is defined as follows:

$$
P\{X = x\} = \sum_{\{n:(\log a_n)/p_n = x\}} p_n.
$$

The formula considered is just Jensen's inequality:

$$
\exp(P[X]) \le P\exp(X). \qquad \square
$$

We now proceed to the proof of delocalization at high temperature for $\theta > 2$ and $d \geq 3$. The strategy is to use Lemma 2.8 to bound the partition function by a convex combination of countably many partition functions of pinning systems with compactly supported potential and then to use rescaling of Brownian motion to get that all of these partition functions are uniformly bounded as $t \to \infty$.

PROOF OF (2.3). Let $\theta > 2$ and $\varepsilon > 0$. We define

$$\overline{V}(x) := \sum_{n=0}^{\infty} \mathbf{1}_{\{\|x\| \leq 2^n\}} 2^{-n\theta} = \frac{1}{1-2^{-\theta}} \left[\mathbf{1}_{\{\|x\| \leq 1\}} + \sum_{n=1}^{\infty} \mathbf{1}_{\{2^{n-1} < \|x\| \leq 2^n\}} 2^{-n\theta} \right]. \tag{2.9}$$

Assuming that $V > 0$, it follows from assumption (2.2) that there exist constants c_1 and C_1 such that

$$c_1 \overline{V}(x) \leq V(x) \leq C_1 \overline{V}(x). \tag{2.10}$$

The proof would also work for $V \geq 0$, by defining $\overline{V}$ differently, that is, dropping the $\mathbf{1}_{\{\|x\| \leq 1\}}$ term and starting the sum from some large n_0 at the left-hand side of (2.9). We restrict our attention to the case $V > 0$ here, only for notational reasons: sums on balls are cleaner (to write) than sums on annuli.

Hence, for any $p \in (0, 1)$ and $h > 0$, we have

$$\begin{aligned} Y_t^{(h)} &\leq P\left[\exp\left(hC_1 \int_0^t \overline{V}(B_s)\, \mathrm{d}s\right)\right] \\ &\leq (1-p) \sum_{n=0}^{\infty} p^n P\left[\exp\left((1-p)^{-1} p^{-n} h C_1 \int_0^t \mathbf{1}_{\{\|B_s\| \leq 2^n\}} 2^{-n\theta}\, \mathrm{d}s\right)\right] \\ &= (1-p) \sum_{n=0}^{\infty} p^n P\left[\exp\left((1-p)^{-1} p^{-n} 2^{n(2-\theta)} h C_1 \int_0^{2^{-2n}t} \mathbf{1}_{\{\|B_s\| \leq 1\}}\, \mathrm{d}s\right)\right], \end{aligned}$$

where the second inequality uses Lemma 2.8 with $p_n := (1-p)p^n$ and the last equality is just a rescaling of the Brownian motion. We choose p such that $p 2^{\theta-2} = 1$. We get

$$Y_t^{(h)} \leq P\left[\exp\left((1-p)^{-1} h C_1 \int_0^t \mathbf{1}_{\{\|B_s\| \leq 1\}}\, \mathrm{d}s\right)\right].$$

For h small enough, Theorem 2.2 for $d \geq 3$ allows us to conclude. Moreover (see [8], Proposition 4.2.1), in this case, we have

$$\lim_{t \to \infty} Y_t^{(h)} := Y_\infty^{(h)} < \infty. \qquad \square$$

To prove an upper bound on the free energy when $\theta < 2$, we start by putting aside the contribution given by $V(x)$ when x is large. This way, we just have to estimate the partition function associated with a compactly supported potential. In dimension $d \geq 3$, what was done in the preceding proof will be sufficient to obtain the result. For $d = 1$ and $d = 2$, we will have to make good use of Theorem 2.2.

PROOF OF THE UPPER BOUNDS IN (2.4) AND (2.5). We start with the case $d = 1$, $\theta < 1$. From assumption (2.2), there exists a constant C_2 such that for any $h \leq 1$,

$$
\begin{aligned}
&V(x) \leq C_2 h^{\theta/(2-\theta)} \qquad \forall x,\ |x| \geq h^{-1/(2-\theta)},\\
&\int_{|x| \leq h^{-1/(2-\theta)}} V(x)\,\mathrm{d}x \leq C_2 h^{-(1-\theta)/(2-\theta)}.
\end{aligned} \tag{2.11}
$$

We write $\check{V}(x) := V(x)\mathbf{1}_{\{|x| \leq h^{-1/(2-\theta)}\}}$. We have

$$
V(B_s) \leq \check{V}(B_s) + C_2 h^{\theta/(2-\theta)}
$$

so that

$$
\log Y_t^{(h)} \leq t C_2 h^{2/(2-\theta)} + \log P\left[\exp\left(h \int_0^t \check{V}(B_s)\,\mathrm{d}s\right)\right].
$$

We know from (2.7) and (2.11) that the second term is smaller than

$$
\log 2 + t \frac{\|\check{V}\|^2_{\mathbb{L}_1(\mathbb{R})}}{2} \leq \log 2 + t C_2^2 h^{2/(2-\theta)},
$$

which is the desired result.

Now, we consider the case $d \geq 2$, $\theta < 2$. Define $\overline{n} = \overline{n}_h := \lceil |\log h|/[(2-\theta)\log 2] \rceil - K$ for some large integer K. We have

$$
\sum_{n > \overline{n}} 2^{-n\theta} \mathbf{1}_{\{\|B_s\| \leq 2^n\}} \leq \sum_{n > \overline{n}} 2^{-n\theta} = \frac{2^{-(\overline{n}+1)\theta}}{1 - 2^{-\theta}}.
$$

Therefore [using formula (2.10)], we can find a constant C_3 (depending on K and C_1) such that

$$
\begin{aligned}
Y_t^{(h)} &\leq P\left[\exp\left(C_1 h \int_0^t \overline{V}(B_s)\,\mathrm{d}s\right)\right]\\
&\leq e^{C_3 t h^{2/(2-\theta)}} P\left[\exp\left(C_1 h \int_0^t \sum_{n=0}^{\overline{n}} 2^{-n\theta} \mathbf{1}_{\{\|B_s\| \leq 2^n\}}\,\mathrm{d}s\right)\right],
\end{aligned}
$$

so

$$
\frac{1}{t}\log Y_t^{(h)} \leq C_3 h^{2/(2-\theta)} + \frac{1}{t}\log P\left[\exp\left(C_1 h \int_0^t \sum_{n=0}^{\overline{n}} 2^{-n\theta} \mathbf{1}_{\{\|B_s\| \leq 2^n\}}\,\mathrm{d}s\right)\right]. \tag{2.12}
$$

We now have to check that the second term in (2.12) does not give a contribution larger than $h^{2/(2-\theta)}$. For any p [we choose $p = 2^{(\theta/2)-1}$], as a consequence of Lemma 2.8, we have

$$
\begin{aligned}
&P\left[\exp\left(C_1 h \int_0^t \sum_{n=0}^{\overline{n}} 2^{-n\theta}\mathbf{1}_{\{\|B_s\|\le 2^n\}}\,\mathrm{d}s\right)\right]\\
&\quad = P\left[\exp\left(C_1 h \int_0^t \sum_{n=0}^{\overline{n}} 2^{-(\overline{n}-n)\theta}\mathbf{1}_{\{\|B_s\|\le 2^{\overline{n}-n}\}}\,\mathrm{d}s\right)\right]\\
&\quad \le \sum_{n=0}^{\overline{n}} \frac{(1-p)p^n}{1-p^{\overline{n}+1}} P\left[\exp\left(C_1 h \int_0^t \frac{p^{-n}(1-p^{\overline{n}+1})}{1-p} 2^{(n-\overline{n})\theta}\mathbf{1}_{\{\|B_s\|\le 2^{\overline{n}-n}\}}\,\mathrm{d}s\right)\right]\\
&\quad = \sum_{n=0}^{\overline{n}} \frac{(1-p)p^n}{1-p^{\overline{n}+1}}\\
&\qquad\qquad \times P\left[\exp\left(C_1 h \int_0^{t2^{-2(\overline{n}-n)}} \frac{p^{-n}(1-p^{\overline{n}+1})}{1-p} 2^{(\overline{n}-n)(2-\theta)}\mathbf{1}_{\{\|B_s\|\le 1\}}\,\mathrm{d}s\right)\right],
\end{aligned}
$$

where the last line is obtained simply by rescaling the Brownian motion in the expectation. Now, observe that for any $\varepsilon > 0$, one can find a value of K such that

$$C_1 h \le (1-p)\varepsilon 2^{-\overline{n}(2-\theta)},$$

so

$$C_1 h \frac{p^{-n}(1-p^{\overline{n}+1})}{1-p} 2^{(\overline{n}-n)(2-\theta)} \le \varepsilon 2^{n((\theta/2)-1)}.$$

Therefore, we have

$$
\begin{aligned}
&P\left[\exp\left(C_1 h \int_0^t \sum_{n=0}^{\overline{n}} 2^{-n\theta}\mathbf{1}_{\{\|B_s\|\le 2^n\}}\,\mathrm{d}s\right)\right]\\
&\qquad \le \max_{n\in\{0,\dots,\overline{n}\}} P\left[\exp\left(\int_0^{t2^{-2(\overline{n}-n)}} \varepsilon 2^{n(\theta/2-1)}\mathbf{1}_{\{\|B_s\|\le 1\}}\,\mathrm{d}s\right)\right].
\end{aligned}
$$

For $d \ge 3$, the right-hand side is less than

$$P\left[\exp\left(\int_0^t \varepsilon\mathbf{1}_{\{\|B_s\|\le 1\}}\,\mathrm{d}s\right)\right],$$

which stays bounded as t goes to infinity. For $d = 2$, if t is sufficiently large and ε small enough, then Theorem 2.2 allows us to write, for all $n \in \{0,\dots,\overline{n}\}$,

$$
\begin{aligned}
&\log P\left[\exp\left(\int_0^{t2^{-2(\overline{n}-n)}} \varepsilon 2^{n((\theta/2)-1)}\mathbf{1}_{\{\|B_s\|\le 1\}}\,\mathrm{d}s\right)\right]\\
&\qquad \le t2^{-2\overline{n}}2^{2n}\exp(-2\varepsilon^{-1}2^{n(1-\theta/2)}).
\end{aligned}
$$

The maximum over n of the right-hand side is attained for $n = 0$. Therefore,

$$\log P\left[\exp\left(C_1 h \int_0^t \sum_{n=0}^{\overline{n}} 2^{-n\theta} \mathbf{1}_{\{\|B_s\| \le 2^n\}}\, \mathrm{d}s\right)\right] \le t 2^{-2\overline{n}}.$$

Inserting this into (2.12) ends the proof. □

2.4. *Proof of lower bounds on the free energy.* In this section, we prove the lower bounds for the asymptotics (2.4) and (2.5). This is substantially easier than what has been done for upper bounds. Here, one just needs to find a compactly supported potential which is bounded from above by V and that gives the appropriate contribution.

For any $n \in \mathbb{N}$,

$$Y_t^{(h)} \ge P\left[\exp\left(c_1 h \int_0^t \overline{V}(B_s)\, \mathrm{d}s\right)\right] \ge P\left[\exp\left(c_1 h 2^{-n\theta} \int_0^t \mathbf{1}_{\{\|B_s\| \le 2^{-n}\}}\, \mathrm{d}s\right)\right].$$

Rescaling the Brownian motion, we get

$$Y_t^{(h)} \ge P\left[\exp\left(c_1 h 2^{n(2-\theta)} \int_0^{t2^{-2n}} \mathbf{1}_{\{\|B_s\| \le 1\}}\, \mathrm{d}s\right)\right].$$

We can choose $n = n_h = \lceil |\log h|/[(2-\theta)\log 2]\rceil + K$ for some integer K. Let $C_4 > 0$ be such that

$$\lim_{t\to\infty} \frac{1}{t} \log P\left[\exp\left(\int_0^t C_4 \mathbf{1}_{\{\|B_s\| \le 1\}}\, \mathrm{d}s\right)\right] \ge 1.$$

By choosing K large enough, we can get

$$\log Y_t^{(h)} \ge \log P\left[\exp\left(C_4 \int_0^{t2^{-2n_h}} \mathbf{1}_{\{\|B_s\| \le 1\}}\, \mathrm{d}s\right)\right] \ge t 2^{-2n_h}.$$

From this, we get that $h_c = 0$ and that

$$\mathrm{F}(h) \ge 2^{-2(K+1)} h^{2/(2-\theta)}. \qquad \square$$

REMARK 2.9. The above proofs indicate that under the measure $\nu_t(h)$, when t is large and $\theta < 2$, $d \ge 2$ or $\theta < 1$, $d = 1$, the typical distance of the polymer chain $(B_s)_{s\in[0,t]}$ to the origin is of order $h^{-1/(2-\theta)}$.

3. The directed polymer free energy.

3.1. *Lower bounds on the free energy, the second moment method and replica coupling.* In this section, we make use of the result obtained for homogeneous pinning models to get some lower bounds on the free energy and prove the corresponding halves of Theorems 1.2 and 1.3. The partition function of a homogeneous pinning model appears naturally when one computes the second moment of W_t.

We start with a short proof of the fact that weak disorder holds for small β if $d \geq 3, \theta > 2$.

PROOF. It is sufficient to show that W_t converges in L_2 for β sufficiently small. We have

$$\begin{aligned}\mathbf{P}[W_t^2] &= \mathbf{P}\left[P^{\otimes 2}\exp\left(\int_0^t [\beta\omega(\mathrm{d}s, B_s^{(1)}) + \beta\omega(\mathrm{d}s, B_s^{(2)})] - \beta^2\,\mathrm{d}s\right)\right] \\ &= P^{\otimes 2}\left[\exp\left(\beta^2\int_0^t Q(B_s^{(1)} - B_s^{(2)})\,\mathrm{d}s\right)\right].\end{aligned}$$

The left-hand side is the partition function of the homogeneous pinning model described in the first section. Therefore, the result is a simple consequence of Theorem 2.4. □

We now prove the lower bound on the free energy corresponding to Theorem 1.3. We use a method called *replica coupling*. The idea of using such a method for directed polymers came in [15] and was inspired by a work on the pinning model of Toninelli [25].

PROOF. Define, for $\beta > 0$, $r \in [0, 1]$,

$$\Phi_t(r, \beta) := \frac{1}{t}\mathbf{P}\left[\log P\exp\left(\int_0^t \sqrt{r}\beta\omega(\mathrm{d}s, B_s) - r\beta^2/2\,\mathrm{d}s\right)\right]$$

and for $\beta > 0$, $r \in [0, 1]$, $\lambda > 0$,

$$\begin{aligned}\Psi_t(r, \lambda, \beta) &:= \frac{1}{2t}\mathbf{P}\left[\log P^{\otimes 2}\exp\left(\int_0^t \sqrt{r}\beta[\omega(\mathrm{d}s, B_s^{(1)}) + \omega(\mathrm{d}s, B_s^{(2)})]\right.\right. \\ &\qquad\qquad \left.\left. + \beta^2[\lambda Q(B_s^{(1)} - B_s^{(2)}) - r]\,\mathrm{d}s\right)\right] \\ &=: \frac{1}{2t}\mathbf{P}[\log P^{\otimes 2}\exp(\widehat{H}_t(B^{(1)}, B^{(2)}, r, \lambda))].\end{aligned}$$

The function $r \mapsto \Phi_t(r, \beta)$ satisfies [recall the definition of p_t in (1.2)]

$$\Phi_t(0, \beta) = 0 \quad \text{and} \quad \Phi_t(1, \beta) = p_t(\beta).$$

In the sequel, we use the following version of the Gaussian integration by parts formula. The proof is straightforward.

LEMMA 3.1. *Let (ω_1, ω_2) be a centered two-dimensional Gaussian vector. We have*

$$\mathbf{P}[\omega_1 f(\omega_2)] := \mathbf{P}[\omega_1\omega_2]\mathbf{P}[f'(\omega_2)].$$

Using this formula we get that

$$\frac{\mathrm{d}}{\mathrm{d}r}\Phi_t(r,\beta) = -\frac{\beta^2}{2t}\mathbf{P}\Big[\big(\mu_t^{(\sqrt{r}\beta)}\big)^{\otimes 2}\Big[\int_0^t Q\big(B_s^{(1)} - B_s^{(2)}\big)\,\mathrm{d}s\Big]\Big]. \tag{3.1}$$

Doing the same for Ψ_t, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}r}\Psi_t(r,\lambda,\beta)\\
&\quad = \frac{\beta^2}{2t}\mathbf{P}\Big[\frac{P^{\otimes 2}e^{\widehat{H}_t(B^{(1)},B^{(2)},r,\lambda)}\int_0^t Q(B_s^{(1)} - B_s^{(2)})\,\mathrm{d}s}{P^{\otimes 2}e^{\widehat{H}_t(B^{(1)},B^{(2)},r,\lambda)}}\Big]\\
&\qquad - \frac{\beta^2}{t}\mathbf{P}\Big[\frac{P^{\otimes 4}e^{\widehat{H}_t(B^{(1)},B^{(2)},r,\lambda)+\widehat{H}_t(B^{(3)},B^{(4)},r,\lambda)}\int_0^t Q(B_s^{(1)} - B_s^{(3)})\,\mathrm{d}s}{P^{\otimes 4}e^{\widehat{H}_t(B^{(1)},B^{(2)},r,\lambda)+\widehat{H}_t(B^{(3)},B^{(4)},r,\lambda)}}\Big]\\
&\quad \le \frac{\beta^2}{2t}\mathbf{P}\Big[\frac{P^{\otimes 2}e^{\widehat{H}_t(B^{(1)},B^{(2)},r,\lambda)}\int_0^t Q(B_s^{(1)} - B_s^{(2)})\,\mathrm{d}s}{P^{\otimes 2}e^{\widehat{H}_t(B^{(1)},B^{(2)},r,\lambda)}}\Big] = \frac{\mathrm{d}}{\mathrm{d}\lambda}\Psi_t(r,\lambda,\beta).
\end{aligned} \tag{3.2}$$

This implies that for every $r \in [0,1]$, we have

$$\Psi_t(r,\lambda,\beta) \le \Psi(0,\lambda + r,\beta).$$

In view of (3.1) and (3.2), using convexity and monotonicity of Ψ_t with respect to λ and $\Psi_t(r,0,\beta) = \Phi_t(r,\beta)$, we have

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}r}\Phi_t(r,\beta) &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\Psi_t(r,\lambda,\beta)\Big|_{\lambda=0}\\
&\le \frac{\Psi_t(r,2-r,\beta) - \Phi_t(r,\beta)}{2-r} \le \Psi_t(0,2,\beta) - \Phi_t(r,\beta),
\end{aligned}$$

where the last inequality uses the fact that $r \le 1$. Integrating this inequality between zero and one, we get

$$p_t(\beta) \ge (1-e)\Psi_t(0,2,\beta), \tag{3.3}$$

where

$$\begin{aligned}
\Psi_t(0,2,\beta) &= \frac{1}{2t}\log P^{\otimes 2}\exp\Big[2\beta^2\int_0^t Q\big(B_s^{(1)} - B_s^{(2)}\big)\,\mathrm{d}s\Big]\\
&= \frac{1}{2t}\log P\Big[\exp\Big(2\beta^2\int_0^t Q\big(\sqrt{2}B_s\big)\,\mathrm{d}s\Big)\Big] =: \frac{1}{2t}\log Y_t.
\end{aligned}$$

Here, Y_t is the partition function of a homogeneous pinning model with potential $Q(\sqrt{2}\cdot)$ and pinning parameter $2\beta^2$. Therefore, we know from Theorem 2.4 and Proposition 2.6 that

$$\lim_{t\to\infty}\frac{1}{t}\log Y_t \asymp \beta^{4/(2-\theta)} \quad \text{or} \quad \lim_{t\to\infty}\frac{1}{t}\log Y_t \asymp \beta^4$$

(where the case to be considered depends on the assumption we have on Q). This, combined with (3.3), gives the desired bound. This completes the proof. □

3.2. *Proof of upper bounds on the free energy* (*Theorem* 1.3). The technique of the proof is mainly based on a change-of-measure argument. This method was developed and first usedt for pinning models [12] and adapted for directed polymer in [15]. Here, we have to adapt it to the continuous case and take advantage of the occurrence of spatial memory in the environment. We briefly sketch an outline of the proof.

- First, we use Jensen's inequality to reduce the proof to estimating a fractional moment (a noninteger moment) of W_t.
- We decompose W_t into different contributions corresponding to paths along a corridor of fixed width.
- For each corridor, we slightly change the measure via a tilting procedure which lowers the value of ω in the corridor.
- We use the change of measure to estimate the fractional moment of each contribution.

We start by stating a trivial lemma which will be useful for our proof and for the next section.

LEMMA 3.2. *Let $(\omega_x)_{x\in X}$ be a Gaussian field indexed by X defined on the probability space $(\Omega, \mathbb{P}, \mathcal{F})$, closed under linear combination. Define the measure $\widetilde{\mathbb{P}}$ as*

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \exp(\omega_{x_0} - \operatorname{Var}\omega_{x_0}/2).$$

Then, under $\mathbb{P}$, $(\omega_x)_{x\in X}$ are still Gaussian variables, their covariance remain unchanged and their expectation is equal to

$$\widetilde{\mathbb{P}}[\omega_x] = \mathbb{P}[\omega_x \omega_{x_0}].$$

We now proceed to the proof. Set $\gamma \in (0, 1)$ (in the sequel we will choose $\gamma = 1/2$). We note that

$$\mathbf{P}[\log W_t] = \frac{1}{\gamma}\mathbf{P}[\log W_t^\gamma] \le \frac{1}{\gamma}\log \mathbf{P}[W_t^\gamma].$$

For this reason, we have

$$p(\beta) \le \frac{1}{\gamma} \liminf_{t\to\infty} \frac{1}{t} \log \mathbf{P} W_t^\gamma. \tag{3.4}$$

Therefore, our aim is to prove that $\mathbf{P}[W_t^\gamma]$ decays exponentially. Fix $T := C_1 \beta^{-4/(2-\theta)}$. Choose $t := Tn$ (n is meant to tend to infinity). For $y = (y^1, \dots, y^d) \in \mathbb{Z}^d$, define $I_y := \prod_{i=1}^d [y^i\sqrt{T}, (y^i + 1)\sqrt{T})$ (where $\prod$ here denotes interval product).

We decompose the partition function W_t into different contributions corresponding to different families of paths. We have

$$W_t := \sum_{y_1,\dots,y_n\in\mathbb{Z}^d} W_{(y_1,\dots,y_n)},$$

where

$$W_{(y_1,\dots,y_n)} := P\left[\exp\left(\int_0^t \beta\omega(\mathrm{d}s, B_s) - \beta^2/2\,\mathrm{d}s\right)\mathbf{1}_{\{B_{kT}\in I_{y_k},\forall k=1,\dots,n\}}\right].$$

We use the inequality $(\sum a_i)^\gamma \le \sum a_i^\gamma$, which holds for an arbitrary collection of positive numbers, to get

$$\mathbf{P}[W_t^\gamma] \le \sum_{y_1,\dots,y_n\in\mathbb{Z}^d} \mathbf{P}\big[W^\gamma_{(y_1,\dots,y_n)}\big]. \tag{3.5}$$

In order to bound the right-hand side of (3.5), we use the following change-of-measure argument: given $Y = (y_1, \dots, y_n)$, and $\widetilde{\mathbf{P}}_Y$ a probability measure on ω, we have

$$\begin{aligned}\mathbf{P}W^\gamma_{(y_1,\dots,y_n)} &= \widetilde{\mathbf{P}}_Y\left[\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\widetilde{\mathbf{P}}_Y}W^\gamma_{(y_1,\dots,y_n)}\right]\\ &\le \left(\mathbf{P}\left[\left(\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\widetilde{\mathbf{P}}_Y}\right)^{\gamma/(1-\gamma)}\right]\right)^{1-\gamma}\big(\widetilde{\mathbf{P}}_Y[W_{(y_1,\dots,y_n)}]\big)^\gamma.\end{aligned} \tag{3.6}$$

One then needs to find a good change of measure to apply this inequality. Let C_2 be a (large) fixed constant. Define the blocks A_k by

$$\begin{aligned}A_k &:= [(k-1)T, kT] \times \prod_{i=1}^d [y^i_{k-1} - C_2\sqrt{T}, y^i_{k-1} + C_2\sqrt{T}],\\ \overline{A}_k &:= \prod_{i=1}^d [y^i_{k-1} - C_2\sqrt{T}, y^i_{k-1} + C_2\sqrt{T}],\\ J_Y &:= \bigcup_{k=1}^n A_k,\end{aligned}$$

with the convention that $y_0 = 0$. Moreover, we define the random variable

$$\begin{aligned}\Omega_k &:= \frac{\int_{A_k}\omega(\mathrm{d}t, x)\,\mathrm{d}x}{\sqrt{T\int_{\overline{A}_k^2} Q(x-y)\,\mathrm{d}x\,\mathrm{d}y}},\\ \Omega_Y &:= \sum_{k=1}^n \Omega_k.\end{aligned}$$

Note that, with this definition, $(\Omega_k)_{k\in\{1,\dots,n\}}$ are standard centered independent Gaussian variables. Define $\widetilde{\mathbf{P}}_Y$ by

$$\frac{\mathrm{d}\widetilde{\mathbf{P}}_Y}{\mathrm{d}\mathbf{P}} := \exp(-\Omega_Y - n/2).$$

From this definition and using the fact that $\gamma = 1/2$, we have

$$\left(\mathbf{P}\left[\left(\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\widetilde{\mathbf{P}}_Y}\right)^{\gamma/(1-\gamma)}\right]\right)^{1-\gamma} = \exp(n/2). \tag{3.7}$$

We also define the measure $\widetilde{\mathbf{P}}_1$ by

$$\frac{\mathrm{d}\widetilde{\mathbf{P}}_1}{\mathrm{d}\mathbf{P}} := \exp(-\Omega_1 - 1/2).$$

We now consider the expectation of $W_{(y_1,\dots,y_n)}$ with respect to $\widetilde{\mathbf{P}}_Y$. As the covariance structure of the Gaussian field remains the same after the change of measure (cf. Lemma 3.2), we have

$$\begin{aligned}
\widetilde{\mathbf{P}}_Y[W_{(y_1,\dots,y_n)}] &= P\exp\left(\beta\widetilde{\mathbf{P}}_Y\left[\int_0^t \omega(\mathrm{d}s, B_s)\right]\right)\mathbf{1}_{\{B_{kT}\in I_{y_k},\forall k=1,\dots,n\}}\\
&= P_O^{(0)}\left[\exp\left(\beta\widetilde{\mathbf{P}}_Y\left[\int_0^T \omega\big(\mathrm{d}s, B_s^{(0)}\big)\right]\right)\mathbf{1}_{\{B_T^{(0)}\in I_{y_1}\}}\right.\\
&\quad\times P_{B_T}^{(1)}\left[\exp\left(\beta\widetilde{\mathbf{P}}_Y\left[\int_0^T \omega\big(\mathrm{d}(s+T), B_s^{(1)}\big)\right]\right)\right.\\
&\quad\times \mathbf{1}_{\{B_T^{(1)}\in I_{y_2}\}}\cdots\mathbf{1}_{\{B_T^{(n-2)}\in I_{y_{n-1}}\}}\\
&\quad\times P_{B_T}^{(n-1)}\left[\exp\left(\beta\widetilde{\mathbf{P}}_Y\left[\int_0^T \omega\big(\mathrm{d}\big(s+(n-1)T\big), B_s^{(n-1)}\big)\right]\right)\right.\\
&\qquad\qquad\left.\left.\left.\times\mathbf{1}_{\{B_T^{(n-1)}\in I_{y_n}\}}\right]\cdots\right]\\
&\le \prod_{k=1}^n \max_{x\in I_{y_{k-1}}} P_x\left[\exp\left(\beta\widetilde{\mathbf{P}}_Y\left[\int_0^T \omega\big(\mathrm{d}(s+kT), B_s\big)\right]\right)\mathbf{1}_{\{B_T\in I_{y_k}\}}\right]\\
&= \prod_{k=1}^n \max_{x\in I_O} P_x\left[\exp\left(\beta\widetilde{\mathbf{P}}_1\left[\int_0^T \omega(\mathrm{d}s, B_s)\right]\right)\mathbf{1}_{\{B_T\in I_{y_k-y_{k-1}}\}}\right].
\end{aligned}$$

The first equality is obtained by the use of the Markov property for Brownian motion: $P^{(i)}_{B_T^{(i-1)}}$ denotes the Wiener measure for the Brownian motion conditioned to start at $B_T^{(i-1)}$ and which, conditionally on $B_T^{(i-1)}$, is independent of all the $P^{(k)}$, $k<i$; $\omega(\mathrm{d}(s+kT, B_s)$ denotes the time increment of the field at $(s+KT, B_s)$. The inequality is obtained by maximizing over $x\in I_{y_k}$ for intermediate points;

P_x is just the Wiener measure with initial condition x. The last equality just uses translation invariance. Returning to (3.5), and using (3.6) and (3.7), we get

$$\mathbf{P}W_t^\gamma \le e^{n/2}\Bigg[\sum_{y\in\mathbb{Z}^d}\Bigg(\max_{x\in I_O} P_x\Bigg[\exp\Bigg(\beta\widetilde{\mathbf{P}}_1\Bigg[\int_0^T \omega(\mathrm{d}s, B_s)\Bigg]\Bigg)\mathbf{1}_{\{B_T\in I_y\}}\Bigg)^\gamma\Bigg]\Bigg]^n. \tag{3.8}$$

We are able to prove that the right-hand side decays exponentially with n if we are able to show that

$$\sum_{y\in\mathbb{Z}^d}\Bigg(\max_{x\in I_O} P_x\Bigg[\exp\Bigg(\beta\widetilde{\mathbf{P}}_1\Bigg[\int_0^T \mathrm{d}\omega(s, B_s)\Bigg]\Bigg)\mathbf{1}_{\{B_T\in I_y\}}\Bigg]\Bigg)^\gamma \tag{3.9}$$

is small. To do so, we have to estimate the expectation of the Hamiltonian under $\widetilde{\mathbf{P}}_1$. We use Lemma 3.2 and get

$$-\widetilde{\mathbf{P}}_1\Bigg[\int_0^T \omega(\mathrm{d}s, B_s)\Bigg] = \mathbf{P}\Bigg[\Omega_1\int_0^T \omega(\mathrm{d}s, B_s)\Bigg] = \frac{\int_{A_1} Q(x-B_s)\,\mathrm{d}x\,\mathrm{d}s}{\sqrt{T\int_{A_1^2} Q(x-y)\,\mathrm{d}x\,\mathrm{d}y}}. \tag{3.10}$$

The above quantity is always positive. However, it depends on the trajectory B. One can check that, when $d \ge 2$, $\theta < 2$ or $d = 1$, $\theta < 1$, the assumption of polynomial decay for Q implies that [we leave the case $d = 1$, $Q \in \mathbb{L}_1(\mathbb{R})$ to the reader]

$$\int_{A_1^2} Q(x-y)\,\mathrm{d}x\,\mathrm{d}y \asymp T^{d-\theta/2}.$$

To control the numerator on the right-hand side of (3.10), we need an assumption on the trajectory. We control the value for trajectories that stay within A_1. For all trajectories $(s, B_s)_{s\in[0,T]}$ that stay in A_1, we trivially have

$$\begin{aligned}
&\int_0^T \int_{[-C_2\sqrt{T}, C_2\sqrt{T}]^d} Q(x-B_s)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad \ge T \min_{y\in[-C_2\sqrt{T}, C_2\sqrt{T}]^d} \int_{[-C_2\sqrt{T}, C_2\sqrt{T}]^d} Q(x-y)\,\mathrm{d}x
\end{aligned}$$

and the right-hand side is asymptotically equivalent to $T^{1+(d-\theta)/2}$.

Altogether, we get that there exists a constant C_3 (depending on C_2) such that, uniformly on trajectories staying in A_1,

$$\widetilde{\mathbf{P}}_1\Bigg[\int_0^T \omega(\mathrm{d}s, B_s)\Bigg] \le -C_3 T^{(2-\theta)/4}. \tag{3.11}$$

The distribution of the Brownian motion allows us to find, for any $\varepsilon > 0$, $R = R_\varepsilon$ such that

$$\begin{aligned}
&\sum_{\|y\|\ge R}\Bigg(\max_{x\in I_O} P_x\Bigg[\exp\Bigg(\beta\widetilde{\mathbf{P}}_1\Bigg[\int_0^T \omega(\mathrm{d}s, B_s)\Bigg]\Bigg)\mathbf{1}_{\{B_T\in I_y\}}\Bigg]\Bigg)^\gamma \\
&\quad \le \sum_{\|y\|\ge R}\Big(\max_{x\in I_O} P_x\{B_T\in I_y\}\Big)^\gamma \le \varepsilon,
\end{aligned}$$

where the first inequality simply uses the fact that $\widetilde{\mathbf{P}}_1(\cdots)$ is negative. For the terms corresponding to $\|y\| < R$, we use the rough bound

$$\sum_{\|y\|<R}\left(\max_{x\in I_O} P_x\left[\exp\left(\beta\widetilde{\mathbf{P}}_1\left[\int_0^T \omega(\mathrm{d}s, B_s)\right]\right)\mathbf{1}_{\{B_T\in I_y\}}\right]\right)^\gamma$$
$$\leq (2R)^d\left(\max_{x\in I_O} P_x\left[\exp\left(\beta\widetilde{\mathbf{P}}_1\left[\int_0^T \omega(\mathrm{d}s, B_s)\right]\right)\right]^\gamma\right).$$

Set $\delta := (\varepsilon/(2R)^d)^{1/\gamma}$. The remaining task in order to find a good bound on (3.9) is to show that

$$\max_{x\in I_O} P_x \exp\left(\beta\widetilde{\mathbf{P}}_1\left[\int_0^T \omega(\mathrm{d}s, B_s)\right]\right) \leq \delta.$$

To get the above inequality, we separate the right-hand side into two contributions: trajectories that stay within A_1 and trajectories that go out of A_1. Bounding these contributions gives

$$\max_{x\in I_O} P_x\left[\exp\left(\beta\widetilde{\mathbf{P}}_1\left[\int_0^T \omega(\mathrm{d}s, B_s)\right]\right)\right]$$
$$\leq P\left\{\max_{s\in[0,T]} |B_s| \geq |C_2-1|\sqrt{T}\right\}$$
$$+ \max_{x\in I_O} P_x\left[\exp\left(\beta\widetilde{\mathbf{P}}_1\left[\int_0^T \omega(\mathrm{d}s, B_s)\right]\right)\mathbf{1}_{\{(s,B_s):s\in[0,T]\subset A_1\}}\right],$$

where the first term in the right-hand side is an upper bound for

$$\max_{x\in I_O} P_x\left[\exp\left(\beta\widetilde{\mathbf{P}}_1\left[\int_0^T \omega(\mathrm{d}s, B_s)\right]\right)\mathbf{1}_{\{(s,B_s):s\in[0,T]\not\subset A_1\}}\right].$$

We can fix C_2 so that the first term is less that $\delta/2$. Equation (3.11) guaranties that the second term is less than

$$\exp(-\beta C_3 T^{(2-\theta)/4}) = \exp(-C_3 C_1^{(2-\theta)/4}) \leq \delta/2,$$

where the last inequality is obtained by choosing C_1 sufficiently large. We have thus shown that (3.9) is less than 2ε. Combining this result with (3.8) and (3.4) implies (for ε small enough) that

$$p(\beta) \leq -\frac{1}{T},$$

which is the desired result. $\square$

4. Fluctuation exponent and volume exponent. In this section, we prove Theorem 1.6, Proposition 1.9 and Theorem 1.11. We give a sketch of our proof for the superdiffusivity result in dimension one. The idea is to compare the energy gain and the entropy cost for going to a distance t^α away from the origin.

We look at the weight under μ_t of the trajectories $(B_s)_{s\in[0,t]}$ that stay within a box of width t^α centered on the origin for $s \in [t/2, t]$ (box $B1$) and compare this with the weight of the trajectories that spend all the time $s \in [t/2, t]$ in another box of the same width (see Figure 1, trajectories a and b). The entropic cost for one trajectory to reach the box $B2$ and stay there is equal to $\log P\{B \text{ stays in } B2\} \sim t^{2\alpha-1}$.

In order to estimate the energy variation between the two boxes, we look at Ω_i, the sum of all the increments of ω in the box Bi $(i = 1, 2)$:

$$\Omega_i := \int_{Bi} \omega(\mathrm{d}s, x)\,\mathrm{d}x.$$

The Ω_i are Gaussian variables that are identically distributed, with variance $\approx t^{\alpha(2-\theta)+1}$. Therefore, in each box, the empirical mean of the increment of ω by a unit of time in Bi $(\Omega_i/|Bi|)$ is Gaussian with variance $\approx t^{-\alpha\theta-1}$ [the typical fluctuations are of order $t^{(-\alpha\theta-1)/2}$]. Multiplying this by the length of the box $(t/2)$, we get that the empirical mean of the energy for paths in a given box has typical fluctuations of order $t^{(1-\alpha\theta)/2}$.

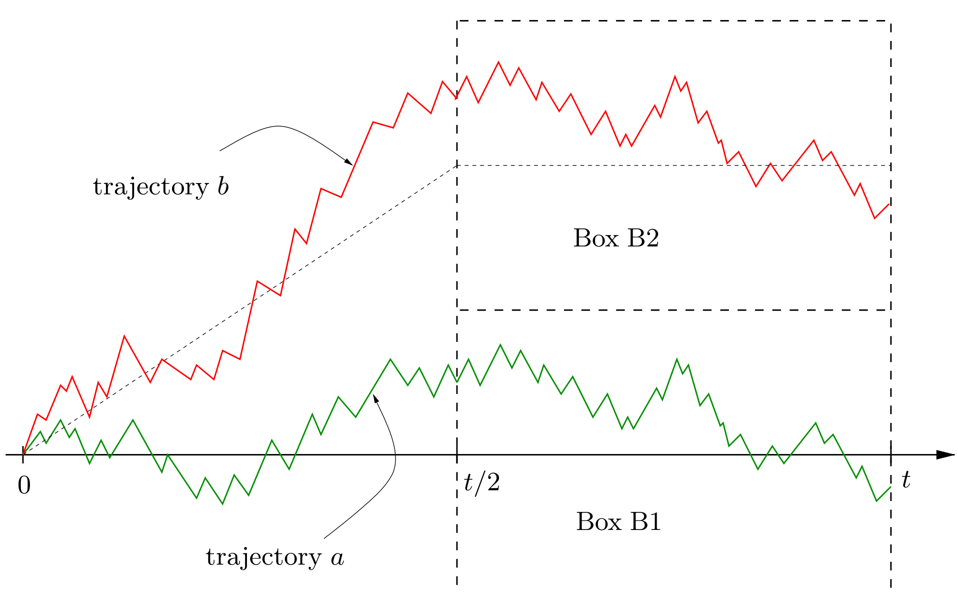

FIG. 1. *A representation of the two events for which we want to compare weights, and two typical trajectories for each event* (a *and* b).

Therefore, if $\alpha < \frac{3}{4-\theta}$, with positive probability, the energy gain for going into the box $B2$ is at least $t^{(1-\alpha)/2}$ and is bigger than the entropy cost $t^{2\alpha-1}$, so the trajectory is less likely to stay in the box $B1$ than in the box $B2$.

To make this argument rigorous, we have to:

- use Girsanov path transforms to make the argument about the entropy work;
- use a measure coupling argument to make the energy comparison rigorous;
- make the comparison with more than two boxes.

In this section, for practical reasons, we work with $\|\cdot\|_\infty$, the l_∞-norm in $\mathbb{R}^d$, and not with the Euclidean norm.

4.1. *Proof of Theorem* 1.6. Let N be some even integer, $\alpha := \frac{3}{4+\theta}$. For $k \in \mathbb{N}$, define

$$\Lambda_k := [t/2, t] \times \left[\frac{(2k-1)t^\alpha}{N^2}, \frac{(2k+1)t^\alpha}{N^2}\right] \times \left[-\frac{t^\alpha}{N^2}, \frac{t^\alpha}{N^2}\right]^{d-1}.$$

Define

$$Z_t^{(k)} := P\left[\exp\left(\beta \int_0^t \omega(\mathrm{d}s, B_s)\right)\mathbf{1}_{\{(s,B_s)\in\Lambda_k, \forall s\in[t/2,t]\}}\right].$$

The proof can be decomposed in two steps. The first step (the next lemma) has been strongly inspired by the work of Petermann [20], Lemma 2, and gives a rigorous method to bound from above the entropic cost for reaching a region t^α away from the origin.

LEMMA 4.1. *With probability greater than* $1 - 1/N$, *we have*

$$\sum_{k\in\{-N,\dots,N\}\setminus\{0\}} Z_t^{(k)} \geq \exp\left(-\frac{8}{N^2} t^{2\alpha-1}\right) Z_t^{(0)}.$$

PROOF. We use the transformation $h_k : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ which transforms a path contributing to $Z_t^{(0)}$ into a path contributing to $Z_t^{(k)}$.

$$h_k : (s, x) \mapsto x + \big((2s/t) \wedge 1\big)\frac{2k}{N^2} t^\alpha \mathbf{e}_1,$$

where $\mathbf{e}_1$ is the vector $(1, 0, \dots, 0)$ in $\mathbb{R}^d$, and we define

$$\overline{Z}_t^{(k)} := P \exp\left(\beta \int_0^t \omega(\mathrm{d}s, h_k(s, B_s))\right)\mathbf{1}_{\{(s,B_s)\in A_0, \forall s\in[t/2,t]\}}.$$

One can check that $(\overline{Z}_t^{(k)})_{k\in\{-N/2,\dots,N/2\}}$ is a family of identically distributed random variables. Moreover, elementary reasoning gives us that there exists an integer k_0, with $|k_0| \leq N/2$, such that

$$Q\left\{\overline{Z}_t^{(k_0)} = \max_{k\in\{-N/2,\dots,N/2\}} \overline{Z}_t^{(k)}\right\} \leq \frac{1}{N+1}.$$

From this, one infers, by translation invariance of the environment, that

$$Q\Big\{\overline{Z}_t^{(0)} \geq \max_{k\in\{-N/2-k_0,\dots,N/2-k_0\}} \overline{Z}_t^{(k)}\Big\} \leq \frac{1}{N+1}.$$

The final step to get the result is to compare $\overline{Z}_t^{(k)}$ with $Z_t^{(k)}$. This can be done by using a Girsanov path transform:

$$\begin{aligned} Z_t^{(k)} &= P\exp\biggl(-\frac{4k}{N^2}t^{\alpha-1}B_{t/2}^{(1)} - \frac{4k^2}{N^4}t^{2\alpha-1} + \beta\int_0^t \omega(\mathrm{d}s, h_k(s, B_s))\biggr) \\ &\quad \times \mathbf{1}_{\{(s,B_s)\in A_0, \forall s\in[t/2,t]\}} \\ &\geq \exp\biggl(-\frac{4(k^2+|k|)}{N^4}t^{2\alpha-1}\biggr)\overline{Z}_t^{(k)} \geq \exp\biggl(-\frac{8}{N^2}t^{2\alpha-1}\biggr)\overline{Z}_t^{(k)}, \end{aligned}$$

where $B^{(1)}$ is the first coordinate of the Brownian motion. □

REMARK 4.2. If one chooses $\alpha = 1/2$, this lemma shows that in any dimension, for any correlation function Q, the directed polymer is at least diffusive, that is, that the typical distance to the origin is of order at least $\sqrt{N}$.

For the rest of the proof, the idea which is used differs substantially from the one used in [20] and then adapted in [2]. Instead of using purely Gaussian tools and working with covariance matrices, we use changes of measure that are similar to those used in the previous section. This shortens the proof considerably and makes it less technical and more intuitive. Moreover, it highlights the fact that the proof could be adapted to a non-Gaussian context, for example, the model of Brownian polymer in a Poissonian environment studied by Comets and Yoshida [8].

We set $T := t^\alpha N^{-3}$ and define

$$\Omega := \frac{\int_{[-T,T]^d}\int_{t/2}^t \omega(\mathrm{d}s, x)\,\mathrm{d}x}{\sqrt{t/2\int_{[-T,T]^d\times[-T,T]^d} Q(x-y)\,\mathrm{d}x\,\mathrm{d}y}},$$

which is a standard centered Gaussian variable. We define the probability measure $\mathbf{P}_0$ on the environment by its Radon–Nikodym derivative with respect to $\mathbf{P}$:

$$\frac{\mathrm{d}\mathbf{P}_0}{\mathrm{d}\mathbf{P}}(\omega) := \exp(-\Omega - 1/2).$$

The probability $\mathbf{P}_0$ has two very important characteristics. It is not very different from $\mathbf{P}$ (see the next lemma) and it makes the environment less favorable in the box $[-t/2, t] \times [-T, T]^d$ so that, under $\mathbf{P}_0$, the trajectory will be less likely to stay in that box.

LEMMA 4.3. *Letting A be any event, we have*

$$\mathbf{P}(A) \leq \sqrt{e\mathbf{P}_0(A)}.$$

PROOF. This is a simple application of the Hölder inequality:

$$\mathbf{P}(A)=\mathbf{P}_0\left[\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{P}_0}\mathbf{1}_A\right]\le\sqrt{\mathbf{P}\left[\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{P}_0}\right]}\sqrt{\mathbf{P}_0(A)}.$$ □

Now, our aim is to show that under $\mathbf{P}_0$, the probability that the walk stays in $[-t^\alpha/N^3, t^\alpha/N^3]^d$ is small and then to use the above lemma to conclude. We use Lemma 3.2 to define, on the same space, two environments with laws $\mathbf{P}$ and $\mathbf{P}_0$. Indeed, if ω has distribution $\mathbf{P}$, then $\widehat{\omega}$ defined by

$$\widehat{\omega}(0,x):=0 \qquad \forall x\in\mathbb{R}^d,$$

$$\begin{aligned}\widehat{\omega}(\mathrm{d}s,x)&:=\omega(\mathrm{d}s,x)-\mathbf{P}[\Omega\omega(\mathrm{d}s,x)]\\&=\omega(\mathrm{d}s,x)-\frac{\mathbf{1}_{s\in[t/2,t]}\int_{[-T,T]^d}Q(x-y)\,\mathrm{d}y}{\sqrt{t/2\int_{[-T,T]^d\times[-T,T]^d}Q(x-y)\,\mathrm{d}x\,\mathrm{d}y}}\,\mathrm{d}s\end{aligned}\tag{4.1}$$

has distribution $\mathbf{P}_0$. We define

$$X_t:=P\left[\exp\left(\beta\int_0^t\omega(\mathrm{d}s,B_s)\right)\mathbf{1}_{\{B_s\in[-T,T]^d,\forall s\in[t/2,t]\}}\right],$$

$$\widehat{X}_t:=P\left[\exp\left(\beta\int_0^t\widehat{\omega}(\mathrm{d}s,B_s)\right)\mathbf{1}_{\{B_s\in[-T,T]^d,\forall s\in[t/2,t]\}}\right],$$

$$\widehat{Z}_t^{(k)}:=P\left[\exp\left(\beta\int_0^t\widehat{\omega}(\mathrm{d}s,B_s)\right)\mathbf{1}_{\{(s,B_s)\in\Lambda_k,\forall s\in[t/2,t]\}}\right].$$

From this definition

$$\mu_t^{\beta,\omega}\left\{\max_{s\in[0,t]}\|B_s\|_\infty\le t^\alpha/N^3\right\}\le\frac{X_t}{\sum_{k\in\{-N,\dots,N\}\setminus\{0\}}Z_t^{(k)}}.\tag{4.2}$$

And, for any x,

$$\begin{aligned}&\mathbf{P}_0\left[\mu_t^{\beta,\omega}\left\{\max_{s\in[0,t]}\|B_s\|_\infty\le t^\alpha/N^3\right\}\le x\right]\\&\quad=\mathbf{P}\left[\mu_t^{\beta,\widehat{\omega}}\left\{\max_{s\in[0,t]}\|B_s\|_\infty\le t^\alpha/N^3\right\}\le x\right]\\&\quad\le\mathbf{P}\left[\frac{\widehat{X}_t}{\sum_{k\in\{-N,\dots,N\}\setminus\{0\}}\widehat{Z}_t^{(k)}}\le x\right].\end{aligned}\tag{4.3}$$

We will now use measure coupling to bound the right-hand side of the above.

LEMMA 4.4. *For N large enough and $t\gg N^{3/\alpha}$, we have*

$$\frac{\widehat{X}_t}{\sum_{k\in\{-N,\dots,N\}\setminus\{0\}}\widehat{Z}_t^{(k)}}\le\frac{X_t}{\sum_{k\in\{-N,\dots,N\}\setminus\{0\}}Z_t^{(k)}}\exp(-C_4t^{1/2}T^{-\theta/2}).$$

PROOF. From the definition of X_t and $\widehat{X}_t$, we have

$$\log(\widehat{X}_t/X_t) \le \beta \sup_{\{B:\|B_s\|_\infty \le T, \forall s\in[t/2,t]\}} \int_0^t \big(\widehat{\omega}(\mathrm{d}s, B_s) - \omega(\mathrm{d}s, B_s)\big),$$

where the "sup" is to be understood as the essential supremum under the Wiener measure P. It follows from the coupling construction (4.1) that for the trajectories staying within $[-T, T]^d$ in the time interval $[t/2, t]$, we have

$$\begin{aligned}\int_0^t \big(\widehat{\omega}(\mathrm{d}s, B_s) - \omega(\mathrm{d}s, B_s)\big) &= -\mathbf{P}\Big[\int_{t/2}^t \omega(\mathrm{d}s, B_s)\Omega\Big] \\ &= -\frac{\mathbf{1}_{[-T,T]^d} \int_{t/2}^t Q(B_s - y)\,\mathrm{d}s\,\mathrm{d}y}{\sqrt{t/2 \int_{[-T,T]^d\times[-T,T]^d} Q(x-y)\,\mathrm{d}x\,\mathrm{d}y}}\,\mathrm{d}t \\ &\le -\frac{\sqrt{t/2}\,\inf_{x\in[-T,T]^d} \int_{[-T,T]^d} Q(x-y)\,\mathrm{d}y}{\sqrt{\int_{[-T,T]^d\times[-T,T]^d} Q(x-y)\,\mathrm{d}x\,\mathrm{d}y}}\end{aligned}$$

so that

$$\log(\widehat{X}_t/X_t) \le -\frac{\beta\sqrt{t/2}\,\inf_{x\in[-T,T]^d} \int_{[-T,T]^d} Q(x-y)\,\mathrm{d}y}{\int_{[-T,T]^d\times[-T,T]^d} Q(x-y)\,\mathrm{d}x\,\mathrm{d}y}. \tag{4.4}$$

Performing a similar computation, one gets that

$$\begin{aligned}&\log \frac{\sum_{k\in\{-N,\dots,N\}\setminus\{0\}} Z_t^{(k)}}{\sum_{k\in\{-N,\dots,N\}\setminus\{0\}} \widehat{Z}_t^{(k)}} \\ &\quad\le \frac{\beta\sqrt{t/2}\,\max_{\|x\|_\infty \ge t^\alpha/N^2} \int_{[-T,T]^d} Q(x-y)\,\mathrm{d}y}{\sqrt{\int_{[-T,T]^d\times[-T,T]^d} Q(x-y)\,\mathrm{d}x\,\mathrm{d}y}}.\end{aligned} \tag{4.5}$$

It remains to give an estimate for the right-hand sides of (4.4) and (4.5).

First, we remark that

$$\int_{[-T,T]^d\times[-T,T]^d} Q(x-y)\,\mathrm{d}x\,\mathrm{d}y \asymp T^{2d-\theta}. \tag{4.6}$$

For (4.4), we note that one can find a constant C_5 such that for all $x, y \in [-T, T]^d$, $Q(x-y) \ge C_5 T^{-\theta}$, so that

$$\inf_{x\in[-T,T]^d} \int_{[-T,T]^d} Q(x-y)\,\mathrm{d}y \ge C_5 T^{d-\theta}. \tag{4.7}$$

For (4.5), we note that one can find a constant C_6 such that for all $y \in [-T, T]^d$, $\|x\|_\infty \ge t^\alpha/N^2$, $Q(x-y) \le C_6(t^\alpha/N^2)^{-\theta} = C_6(NT)^{-\theta}$, so that

$$\max_{\|x\|_\infty\ge t^\alpha/N^2} \int_{[-T,T]^d} Q(x-y)\,\mathrm{d}y \le C_6 N^{-\theta} T^{d-\theta}. \tag{4.8}$$

For N large enough, the term above will be dominated by $C_5 T^{d-\theta}$ so that, combining (4.6), (4.7) and (4.8), one gets that there exists C_4 such that

$$\log \frac{\widehat{X}_t \sum_{k\in\{-N,\dots,N\}\setminus\{0\}} Z_t^{(k)}}{X_t \sum_{k\in\{-N,\dots,N\}\setminus\{0\}} \widehat{Z}_t^{(k)}} \le -C_4 t^{1/2} T^{-\theta/2}.$$

□

Now, the preceding result, together with Lemma 4.1, ensures that, with probability larger than $1 - 1/N$, we have

$$\begin{aligned}\frac{\widehat{X}_t}{\sum_{k\in\{-N,\dots,N\}\setminus\{0\}} \widehat{Z}_t^{(k)}} &\le \frac{Z_t^{(0)}}{\sum_{k\in\{-N,\dots,N\}\setminus\{0\}} Z_t^{(k)}} \exp(-t^{1/2} T^{-\theta/2}) \\ &\le \exp\biggl(\frac{8}{N^2} t^{2\alpha-1} - C_4 t^{1/2} T^{-\theta/2}\biggr).\end{aligned} \tag{4.9}$$

We can bound the term in the exponential on the right-hand side when t is large enough:

$$\begin{aligned}\frac{8}{N^2} t^{2\alpha-1} - C_4 t^{1/2} T^{-\theta/2} &= \biggl[\frac{8}{N^2} - C_4 N^{3\theta/2}\biggr] t^{(2-\theta)/(4+\theta)} \\ &\le -t^{(2-\theta)/(4+\theta)}.\end{aligned} \tag{4.10}$$

We now combine all of the elements of our reasoning. Equations (4.9) and (4.10) combined with (4.3) give us

$$\mathbf{P}_0\Bigl\{\mu_t^{\beta,\omega}\Bigl\{\max_{0\le s\le t} \|B_s\|_\infty \le t^\alpha/N^3\Bigr\} \ge \exp\bigl(-t^{(2-\theta)/(4+\theta)}\bigr)\Bigr\} \le \frac{1}{N}.$$

Lemma 4.3 allows us to get from this

$$\mathbf{P}\Bigl\{\mu_t^{\beta,\omega}\Bigl\{\max_{0\le s\le t} \|B_s\|_\infty \le t^\alpha/N^3\Bigr\} \ge \exp\bigl(-t^{(2-\theta)/(4+\theta)}\bigr)\Bigr\} \le \sqrt{e/N}$$

so that

$$\mathbf{P}\mu_t^{\beta,\omega}\{\max \|B_s\|_\infty \le t^\alpha/N^3\} \le \exp\bigl(-t^{(2-\theta)/(4+\theta)}\bigr) + \sqrt{e/N}.$$

We obtain the desired result by choosing N large enough. □

REMARK 4.5. The above proof is valid for $\theta < 2$, $d \ge 2$ and also for $\theta < 1$, $d = 1$. For the case $d = 1$, $\theta > 1$ [or, more generally, for $d = 1$, $Q \ge 0$, $\int_{\mathbb{R}} Q(x)\,\mathrm{d}x < \infty$], one has to choose $\alpha = 3/5$ and replace (4.6) by

$$\int_{[-T,T]\times[-T,T]} Q(x-y)\,\mathrm{d}x\,\mathrm{d}y \asymp T$$

and (4.7), (4.8) by

$$\inf_{x\in[-T,T]} \int_{[-T,T]} Q(x-y)\,\mathrm{d}y \ge C_5,$$

$$\max_{x\ge T^\alpha/N^2} \int_{[-T,T]} Q(x-y)\,\mathrm{d}y \le C_5/2.$$

4.2. *Proof of Theorem* 1.11. Let α be as in the assumption of the theorem. We define $T := t^\alpha$ and

$$\check{Z}_t := \mathbf{P}\big[\exp(\beta H_t(B))\mathbf{1}_{\{\max_{s\in[0,t]}\|B_s\|_\infty \le t^\alpha\}}\big].$$

We will show that the typical fluctuations of $\log \check{Z}_t$ are large and then that those of $\log Z_t$ are also large because Z_t and $\check{Z}_t$ are close to each other with large probability (in this respect, we need to do more than simply bound the variance of $\log \check{Z}_t$). Define Ω by

$$\Omega := \frac{\int_{[0,t]\times[-T,T]^d} \omega(\mathrm{d}s, x)\,\mathrm{d}x}{\sqrt{t\int_{[-T,T]^{2d}} Q(x-y)\,\mathrm{d}x\,\mathrm{d}y}}.$$

Under $\mathbf{P}$, Ω is a standard Gaussian. We define the probability $\mathbf{P}_0$ by

$$\mathrm{d}\mathbf{P}_0 := \exp(\Omega - 1/2)\,\mathrm{d}\mathbf{P}.$$

It follows from its definition that the distribution of $\log \check{Z}_t$ under $\mathbf{P}$ has no atom, so one can define x_0 (depending on β and t) such that

$$\mathbf{P}[\log \check{Z}_t \le x_0] = e^{-2}.$$

We use Lemma 3.2 to perform a measure coupling as before: if ω has distribution $\mathbf{P}$, then $\widehat{\omega}$ defined by

$$\begin{aligned}
\widehat{\omega}(0, x) &:= 0 \qquad \forall x \in \mathbb{R}^d,\\
\widehat{\omega}(\mathrm{d}s, x) &:= \omega(\mathrm{d}s, x) + \mathbf{P}[\Omega\omega(\mathrm{d}s, x)]\\
&= \omega(\mathrm{d}s, x) + \frac{\int_{[-T,T]^d} Q(x-y)\,\mathrm{d}y}{\sqrt{t\int_{[-T,T]^{2d}} Q(x-y)\,\mathrm{d}x\,\mathrm{d}y}}\,\mathrm{d}s
\end{aligned}$$

has distribution $\mathbf{P}_0$.

One defines

$$\widehat{Z}_t := P\bigg[\exp\bigg(\beta\int_0^t \widehat{\omega}(\mathrm{d}s, B_s)\bigg)\mathbf{1}_{\{\max_{s\in[0,t]}\|B_s\|_\infty \le T\}}\bigg].$$

For all paths B such that $\max_{s\in[0,t]}\|B_s\|_\infty \le T$, we have

$$\begin{aligned}
\int_0^t \widehat{\omega}(\mathrm{d}s, B_s) - \int_0^t \omega(\mathrm{d}s, B_s) &= \int_0^t \frac{\int_{[-T,T]^d} Q(B_s-y)\,\mathrm{d}y}{\sqrt{t\int_{[-T,T]^{2d}} Q(x-y)\,\mathrm{d}x\,\mathrm{d}y}}\,\mathrm{d}s\\
&\ge \sqrt{t}\frac{\min_{\|x\|_\infty\le T}\int_{[-T,T]^d} Q(x-y)\,\mathrm{d}y}{\sqrt{\int_{[-T,T]^{2d}} Q(x-y)\,\mathrm{d}x\,\mathrm{d}y}}.
\end{aligned}$$

One can check that there exists a constant c depending only on Q such that for all t,

$$\frac{\min_{\|x\|_\infty\le T}\int_{[-T,T]^d} Q(x-y)\,\mathrm{d}y}{\sqrt{\int_{[-T,T]^{2d}} Q(x-y)\,\mathrm{d}x\,\mathrm{d}y}} \ge cT^{-(\theta\wedge d)/2}.$$

Therefore, one has $\mathbf{P}$ almost surely

$$\log \widehat{Z}_t \geq \log \check{Z}_t + c\beta t^{(1-\alpha(\theta\wedge d))/2}.$$

We use Lemma 4.3 (which is still valid in our case, even if the change of measure is different) to see that

$$\mathbf{P}\{\log \check{Z}_t \leq x_0 + c\beta t^{(1-\alpha(\theta\wedge d))/2}\} \leq \sqrt{e\mathbf{P}_0\{\log \check{Z}_t \leq x_0 + c\beta t^{(1-\alpha(\theta\wedge d))/2}\}}.$$

Moreover,

$$\begin{aligned}\mathbf{P}_0\{\log \check{Z}_t \leq x_0 + c\beta t^{(1-\alpha(\theta\wedge d))/2}\} &= \mathbf{P}\{\log \widehat{Z}_t \leq x_0 + c\beta t^{(1-\alpha(\theta\wedge d))/2}\} \\ &\leq \mathbf{P}\{\log \check{Z}_t \leq x_0\} = e^{-2}.\end{aligned}$$

The inequality is given by (4.2). So, combining everything, we have

$$\begin{aligned}\mathbf{P}\{\log \check{Z}_t \geq x_0 + c\beta t^{(1-\alpha(\theta\wedge d))/2}\} &\geq 1 - e^{-1/2}, \\ \mathbf{P}\{\log \check{Z}_t \leq x_0\} &= e^{-2}.\end{aligned} \tag{4.11}$$

It is enough to prove that the variance of $\log \check{Z}_t$ diverges with the correct rate. Slightly more work is required to prove the same for $\log \check{Z}_t$. Recall that

$$\mu_t^{\beta,\omega}\Big\{\max_{s\in[0,t]} \|B_s\|_\infty \leq T\Big\} = \check{Z}_t/Z_t.$$

The assumption on α gives that, for t large enough,

$$\mathbf{P}\{(\check{Z}_t/Z_t) \leq 1/2\} \leq 1/100.$$

By a union bound, we have

$$\mathbf{P}\{\log \check{Z}_t \leq x_0\} \leq \mathbf{P}\{\log Z_t \leq x_0 + \log 2\} + \mathbf{P}\{\check{Z}_t \leq Z_t/2\}.$$

Combining this with (4.11) (and the trivial bound $Z_t \geq \check{Z}_t$) gives that for t large enough,

$$\begin{aligned}\mathbf{P}\{\log Z_t \geq x_0 + c\beta t^{(1-\alpha(\theta\wedge d))/2}\} &\geq 1 - e^{-1/2}, \\ \mathbf{P}\{\log Z_t \leq x_0 + \log 2\} &\geq e^{-2} - 1/100.\end{aligned}$$

This implies that

$$\operatorname{Var}\log Z_t \geq (e^{-2} - 1/100)(1 - e^{-1/2})\big(c\beta t^{(1-\alpha(\theta\wedge d))/2} - \log 2\big)^2. \qquad \square$$

4.3. *Proof of Proposition* 1.9. To prove this result, we will follow the method of Méjane [18]. First, we need to use a concentration result. We prove it using stochastic calculus.

LEMMA 4.6. *Let f be a nonnegative function on $\mathbb{R}^d$ and $r \in [0,t]$. We define $\widetilde{Z}_t := P[f(B_r)\exp(\beta\int_0^t \omega(\mathrm{d}s, B_s))]$. Then, for all $x > 0$,*

$$\mathbf{P}\{|\log \widetilde{Z}_t - \mathbf{P}[\log \widetilde{Z}_t]| \geq x\sqrt{t}\} \leq 2\exp\biggl(-\frac{x^2}{2\beta^2}\biggr).$$

PROOF. Let $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0} := (\sigma\{\omega(s,x), s \in [0,t], x \in \mathbb{R}^d\})_{t\geq 0}$ be the natural filtration associated with the environment. We consider the following continuous martingale with respect to the filtration $\mathcal{F}$:

$$(M_u := \mathbf{P}[\log \widetilde{Z}_t | \mathcal{F}_u] - \mathbf{P}[\log \widetilde{Z}_t])_{u\in[0,t]}. \tag{4.12}$$

We have $M_0 = 0$ and the result to prove becomes

$$\mathbf{P}\{|M_t| \geq x\sqrt{t}\} \leq 2\exp\biggl(-\frac{x^2}{2\beta^2}\biggr).$$

The proof uses a classical result on concentration for martingales.

LEMMA 4.7. *If $(M_u)_{u\geq 0}$ is a continuous martingale (with associated law $\mathbf{P}$) starting from 0 with finite exponential moments of all orders, then, for all $(x,y) \in \mathbb{R}_+^2$ and $u \geq 0$, we have*

$$\mathbf{P}\{M_u \geq x\langle M\rangle_u + y\} \leq \exp(-2xy).$$

PROOF. We have

$$\begin{aligned}\mathbf{P}\{M_u \geq x\langle M\rangle_u + y\} &= \mathbf{P}\{\exp(2xM_u - 2x^2\langle M\rangle_u - 2xy) \geq 1\}\\ &\leq \mathbf{P}[\exp(2xM_u - 2x^2\langle M\rangle_u - 2xy)] = \exp(-2xy),\end{aligned}$$

where we have just used the fact that for any given x, $\exp(xM_u - 2x^2\langle M\rangle_u)$ is a martingale. □

To use the previous lemma, we have to compute the bracket of the martingale M defined in (4.12). One can compute it explicitly, but the form of the result is rather complicated so that we have to introduce several items of notation before giving a formula. One defines the probability measure $(\widetilde{\mu}_t)$ (depending on ω, β, t and r) by giving its Radon–Nikodym derivative with respect to the Wiener measure:

$$\frac{\mathrm{d}\widetilde{\mu}_t(\mathrm{d}x)}{\mathrm{d}P}(B) := \frac{1}{\widetilde{Z}_t} f(B_r)\exp\biggl(\beta\int_0^t \omega(\mathrm{d}s, B_s)\biggr)$$

[it is the polymer measure where the paths have been reweighted by $f(B_r)$]. One defines $\nu_{t,u}$ to be the (random) probability measure on $\mathbb{R}$ defined by

$$\nu_{t,u}(\mathrm{d}x) := \mathbf{P}[\widetilde{\mu}_t(B_u \in \mathrm{d}x)|\mathcal{F}_u].$$

For the martingale M defined by (4.12), we have

$$\langle M\rangle_t = \beta^2 \int_0^t \int_{\mathbb{R}^2} \nu_{t,u}^{\otimes 2}(\mathrm{d}x\,\mathrm{d}y) Q(x-y). \tag{4.13}$$

An easy consequence of this is that

$$\langle M\rangle_t \le \beta^2 t \qquad \text{almost surely.}$$

From this, we infer that

$$\mathbf{P}\{M_t \ge x\sqrt{t}\} \le \mathbf{P}\left\{M_t \ge \frac{x}{2\beta^2\sqrt{t}}\langle M\rangle_t + \frac{\sqrt{t}x}{2}\right\} \le \exp(-x^2\beta^{-2}/2),$$

where the last inequality is obtained by applying Lemma 4.7. Carrying out the same computation for the martingale $-M$ gives the desired result. □

We now turn to the proof of the result. Let $\alpha > 3/4$ be fixed. Let $B_t^{(1)}$ be the first coordinate of B_t in $\mathbb{R}^d$. By the Markov inequality, we have, for every $\lambda > 0$,

$$\mu_t\{B_r^{(1)} \ge a\} \le e^{-\lambda a + r\lambda^2/2} \mu_t(\exp(\lambda B_r^{(1)} - r\lambda^2/2)). \tag{4.14}$$

We use Girsanov's formula:

$$\begin{aligned} \mu_t(\exp(-\lambda B_r^{(1)} - r\lambda^2/2)) &= \frac{P[\exp(\lambda B_r^{(1)} - r\lambda^2/2 + \beta \int_0^t \omega(\mathrm{d}s, B_s))]}{P[\exp(\beta\int_0^t \omega(\mathrm{d}s, B_s))]} \\ &= \frac{P[\exp(\beta\int_0^t \omega(\mathrm{d}s, h^{\lambda,r}(s, B_s)))]}{P[\exp(\beta\int_0^t \omega(\mathrm{d}s, B_s))]}, \end{aligned} \tag{4.15}$$

where $h^{\lambda,r}$ is the function from $\mathbb{R}_+ \times \mathbb{R}^d$ to $\mathbb{R}^d$ defined by

$$h^{\lambda,r}(s,x) := x + \lambda(r \wedge s)\mathbf{e}_1$$

and $\mathbf{e}_1$ is the vector $(1, 0, \dots, 0)$ in $\mathbb{R}^d$. By translation invariance, the environment $(\omega(s, h^{\lambda,r}(s,x)))_{s\in[0,t],x\in\mathbb{R}^d}$ has the same law as $(\omega(s,x))_{s\in[0,t],x\in\mathbb{R}^d}$ and, therefore, we get, from the last line of (4.15), that

$$\mathbf{P}[\log \mu_t(\exp(-\lambda B_r^{(1)} - r\lambda^2/2))] = 0.$$

Substituting this into (4.14), we get

$$\mathbf{P}\log \mu_t\{B_r^{(1)} \ge a\} \le -\lambda a + r\lambda^2/2.$$

As λ is arbitrary, we can take the minimum over λ for the right-hand side to get $-\frac{a^2}{2r}$. We use the result for $a = t^\alpha$ to get

$$\mathbf{P}\log\mu_t\{B_r^{(1)} \geq t^\alpha\} \leq -t^{2\alpha}/2r.$$

Using Lemma 4.6 with $f(y) := \mathbf{1}_{\{y\geq a\}}$ and $f \equiv 1$, for $x = t^\varepsilon$ with $\varepsilon < 4\alpha - 3$, one gets

$$\mathbf{P}\left\{\log\mu_t\{B_r^{(1)} \geq t^\alpha\} \leq -\frac{t^{2\alpha}}{2r} + 2t^{(1+2\varepsilon)/2}\right\} \leq 4\exp\left(-\frac{t^{2\varepsilon}}{2\beta^2}\right).$$

For t sufficiently large, we have, for all $r \leq t$,

$$-\frac{t^{2\alpha}}{2r} + 2t^{(1+2\varepsilon)/2} \leq -t^{1/2}.$$

We can get this inequality for $B^{(i)}$ and $-B^{(i)}$ for any $i \in \{1, \dots, d\}$. Combining all of these results, we get

$$\mathbf{P}\{\mu_t\{\|B_r\|_\infty \geq t^\alpha\} \leq 2d\exp(-t^{1/2})\} \leq 8d\exp\left(-\frac{t^{2\varepsilon}}{2\beta^2}\right).$$

Using the above inequality for all $r \in \{1, 2, \dots, \lfloor t\rfloor\}$, we obtain that

$$\begin{aligned}\mathbf{P}\Big\{\mu_t\Big\{\max_{r\in\{1,2,\dots,\lfloor t\rfloor\}}\|B_r\|_\infty \geq t^\alpha\Big\} \leq 2dt\exp(-t^{1/2})\Big\}\\ \leq 8dt\exp\left(-\frac{t^{2\varepsilon}}{2\beta^2}\right).\end{aligned} \tag{4.16}$$

To complete the proof, we need to control the term

$$\mathbf{P}\mu_t\Big\{\max_{\substack{s\in[n,n+1]\\ n\in\{0,\dots,\lfloor t\rfloor\}}}\|B_s - B_n\|_\infty \geq t^\alpha\Big\}.$$

One computes that

$$\begin{aligned}&\mathbf{P}\Big[W_t\mu_t\Big\{\max_{\substack{s\in[n,n+1]\\ n\in\{0,\dots,\lfloor t\rfloor\}}}\|B_s - B_n\|_\infty \geq t^\alpha\Big\}\Big]\\ &\quad = P\Big\{\max_{\substack{s\in[n,n+1]\\ n\in\{0,\dots,\lfloor t\rfloor\}}}\|B_s - B_n\|_\infty \geq t^\alpha\Big\} \leq 4d(t+1)\exp(-t^{2\alpha}/2).\end{aligned} \tag{4.17}$$

Lemma 4.6 applied to $f \equiv 1$ and $x = \beta\sqrt{t}$ gives us that

$$\mathbf{P}\{W_t \leq \exp(t(p(\beta) - \beta))\} \leq \exp(-t/2). \tag{4.18}$$

If A is an event and $x > 0$ are given, then

$$\begin{aligned}\mathbf{P}[\mu_t(A)] &= \mathbf{P}[\mu_t(A)\mathbf{1}_{\{W_t\leq x\}}] + \mathbf{P}[\mu_t(A)\mathbf{1}_{\{W_t>x\}}]\\ &\leq \mathbf{P}\{W_t \leq x\} + x^{-1}\mathbf{P}[W_t\mu_t(A)].\end{aligned}$$

Therefore, we have, from (4.18) and (4.17), that

$$\mathbf{P}\mu_t\Big\{\max_{\substack{s\in[n,n+1]\\ n\in\{0,\dots,\lfloor t\rfloor\}}}\|B_s-B_n\|_\infty\geq t^\alpha\Big\}\leq e^{-t/2}+4d(t+1)e^{t(\beta-p(\beta))-t^{2\alpha}/2}$$
$$\leq 2e^{-t/2}. \tag{4.19}$$

Combining (4.16) and (4.19), we get (for t large enough)

$$\mathbf{P}\mu_t\Big\{\max_{s\in[0,t]}\|B_s\|_\infty\leq 2t^\alpha\Big\}\leq 2\exp(-t/2)+8dt\exp(-t^{2\varepsilon}/\beta^2)$$
$$\leq\exp(-t^\varepsilon). \qquad\square$$

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DIPARTEMENTO DE MATEMATICA
UNIVERSIT DEGLI STUDI ROMA TRE
LARGO SAN LEONARDO MURIALDO
00146 ROME
ITALY
E-MAIL: lacoin@math.jussieu.fr